Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

C. R. Mecanique 332 (2004) 487–492

Comptes Rendus
Mecanique

An explicit asymptotic model for the Bleustein–Gulyaev wave

Julius Kaplunov [a,*], Leonid Kossovich [b], Alexis Zakharov [a]

[a] *Department of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, UK*
[b] *Saratov State University, Astrakhanskaya 83, Saratov, 410026, Russian Federation*

Received 9 February 2004; accepted after revision 9 March 2004

Available online 14 May 2004

Presented by Évariste Sanchez-Palencia

Abstract

The antiplane motion of a transversely isotropic piezoelectric half-space is considered. An explicit asymptotic model is derived for the far field of the surface wave. It involves, in particular, a 1D hyperbolic equation for surface shear deformation propagating with the finite wave speed predicted for the first time by J.L. Bleustein and Yu.V. Gulyaev. Neumann and Dirichlet problems are formulated to restore interior mechanical and electric fields. The derivation utilizes asymptotic arguments combined with Lourier symbolic integration. Comparison with the exact solution is presented for surface impact loading. ***To cite this article: J. Kaplunov et al., C. R. Mecanique 332 (2004).***

Résumé

Un modèle explicite pour l'onde de Bleustein–Gulyaev. Le mouvement antiplane d'un demi-espace piézoélectrique transversalement isotrope est considéré. Un modèle asymptotique explicite est dérivé pour le champ lointain de l'onde de surface. Il implique, en particulier, une équation hyperbolique de dimension un pour la déformation de la surface de cisaillement se propageant avec vitesse d'onde finie predite pour la première fois par J.L. Bleustein et Yu.V. Gulyaev. Des problèmes de Neumann et de Dirichlet sont formulés pour reconstituer les champs mécaniques et électriques intérieurs. La dérivation utilise des arguments asymptotiques combinés avec l'intégration symbolique de Lourier. La comparaison avec la solution exacte est présentée pour le chargement d'impact de surface. ***Pour citer cet article : J. Kaplunov et al., C. R. Mecanique 332 (2004).***

Keywords: Waves; Asymptotic; Model; Far field; Piezoelectric; Surface wave; Bleustein–Gulyaev wave

Mots-clés : Ondes ; Asymptotique ; Modèle ; Champ lointain ; Piézoélectrique ; L'onde de surface ; L'onde de Bleustein–Gulyaev

* Corresponding author.
E-mail addresses: kaplunov@maths.man.ac.uk (J. Kaplunov), kossovichlu@yandex.ru (L. Kossovich), zakharov@maths.man.ac.uk (A. Zakharov).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2004.03.007

1. Introduction

Surface waves seem to be hidden in the general formulations of continuum mechanics. In particular, the speed of the classical Rayleigh wave on an elastic half-plane is not a feature of the equations of motion in linear elasticity, but can only be calculated from the related dispersion equation. The absence of explicit mathematical models (say, in the form of approximate equations and boundary conditions) oriented directly to surface waves complicates dynamic analysis and also leads to certain methodical difficulties in the exposition of the general theory.

There was an attempt to derive an aproximate equation for free harmonic Rayleigh waves [1]. A recent asymptotic model for the Rayleigh wave has been proposed in the paper [2]. The considerations in [2] are based on the so-called 'Lourier symbolic method' which treats a part of partial derivatives as integration parameters. In the present context the symbolic technique refers to the asymptotic nature of the problem underlying the far field assumption.

In this Note we extend the methodology of [2] to the Bleustein–Gulyaev (B–G) surface wave [3,4], which is characteristic of the antiplane motion of a piezoelectric half-space. A 1D hyperbolic equation is derived for the surface shear deformation, propagating with finite speed, predicted in the above mentioned papers. In this case full 2D mechanical and electric fields can be defined from Neumann and Dirichlet boundary value problems specified for a half-plane. It is essential that the latter do not allow the propagation of surface dicontinuities into the interior domain. As an example, the B–G wave far field is evaluated for a surface impact load with the Gaussian distribution in space. Comparison with the exact solution is presented.

2. Statement of the problem

Let us study the antiplane motion of a transversely isotropic (e.g., crystal class $C_{6\,\mathrm{mm}}$) piezoelectric half-space with $y \geqslant 0$. Let z axis be oriented in the direction of sixfold axis for a crystal in class $C_{6\,\mathrm{mm}}$. The governing equations can be written as (e.g., see [3])

$$\nabla^2 u - \frac{\rho}{\bar{c}_{44}} \frac{\partial^2 u}{\partial t^2} = 0, \qquad \nabla^2 \psi = 0 \tag{1}$$

where u denotes z displacement component, ρ is mass density, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplace operator, $\bar{c}_{44} = c_{44} + e_{15}^2/\epsilon_{11}$ is piezoelectrically stiffened elastic constant (c_{44}, e_{15}, and ϵ_{11} are elasic, piezoelectric, and dielectric constants, respectively), and the function ψ is defined in terms of the electric potential ϕ as $\psi = \phi - (e_{15}/\epsilon_{11})u$.

Consider the boundary conditions corresponding to the surface $y = 0$ completely coated with an infinitesimally thin perfectly conducting electrode which is grounded. They are

$$\sigma_{23} = \bar{c}_{44} \frac{\partial u}{\partial y} + e_{15} \frac{\partial \psi}{\partial y} = -P, \qquad \phi = \frac{e_{15}}{\epsilon_{11}} u + \psi = 0 \tag{2}$$

where $P = P(x,t)$ is impact mechanical load, and σ_{23} is the relevant component of the stress tensor.

For the sake of definiteness we focus on the B–G wave far-field (see Fig. 1). To this end we define the characteristic length L for the mechanical load P. In particular, for the Gaussian distribution along x-axis

$$P(x,t) = f(t) \frac{1}{\sqrt{2\pi} L} \exp\left(-\frac{x^2}{2L^2}\right)$$

where $f(t)$ is a prescribed function of large variability.

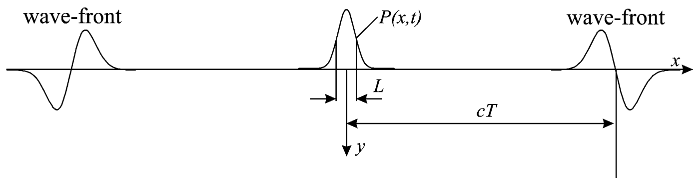

Fig. 1. The Bleustein–Gulyaev wave far-field.

3. Asymptotic analysis

First, we introduce scaling (e.g., see [5])

$$\xi = \frac{x - ct}{\varepsilon L}, \qquad \zeta = \frac{y}{\varepsilon L}, \qquad \tau = \frac{ct}{L} \tag{3}$$

where c is the B–G surface wave speed explicitly defined below, ε is a small parameter which may be presented as

$$\varepsilon = L/cT \ll 1$$

where T is typical time scale.

Eqs. (1) in terms of variables (3) become

$$\begin{aligned} &\frac{\partial^2 u}{\partial \zeta^2} + \left(\left(1 - \frac{\rho c^2}{\bar{c}_{44}}\right)\partial_\xi^2 + 2\varepsilon \frac{\rho c^2}{\bar{c}_{44}}\partial_\xi \partial_\tau - \varepsilon^2 \frac{\rho c^2}{\bar{c}_{44}}\partial_\tau^2\right)u = 0 \\ &\frac{\partial^2 \psi}{\partial \zeta^2} + \partial_\xi^2 \psi = 0 \end{aligned} \tag{4}$$

where $\partial_\xi = \partial/\partial\xi$, $\partial_\tau = \partial/\partial\tau$.

Next, we apply the Lourier symbolic method (e.g., see [6,7]) which deals with (4) as ODE's with respect to ζ, treating the operators ∂_ξ, ∂_τ as integration parameters. Then we have

$$\begin{aligned} &u = \exp\left(-\mathrm{i}\sqrt{\left(1 - \frac{\rho c^2}{\bar{c}_{44}}\right)\partial_\xi^2 + 2\varepsilon \frac{\rho c^2}{\bar{c}_{44}}\partial_\xi \partial_\tau - \varepsilon^2 \frac{\rho c^2}{\bar{c}_{44}}\partial_\tau^2}\,\zeta\right)U(\xi, \tau) \\ &\psi = \exp(-\mathrm{i}\partial_\xi \zeta)\Psi(\xi, \tau) \end{aligned} \tag{5}$$

where $U(\xi, \tau)$, $\Psi(\xi, \tau)$ are sought for functions, and the negative exponent ensures the decay of propagating surface disturbances as ζ tends to infinity.

Boundary conditions (2) in terms of scaled variables (3) can be rewritten as ($\zeta = 0$)

$$\begin{aligned} &\bar{c}_{44}\frac{\partial u}{\partial \zeta} + e_{15}\frac{\partial \psi}{\partial \zeta} = -\varepsilon L P \\ &\frac{e_{15}}{\epsilon_{11}}u + \psi = 0 \end{aligned}$$

Now we insert (5) into the latter and eliminate Ψ. Ignoring terms $\mathrm{O}(\varepsilon^2)$ we get

$$\mathrm{i}\left(\bar{c}_{44}\sqrt{1 - \frac{\rho c^2}{\bar{c}_{44}}}\left(\partial_\xi + \varepsilon\left(\frac{\bar{c}_{44}}{\rho c^2} - 1\right)^{-1}\partial_\tau\right) - \frac{e_{15}^2}{\epsilon_{11}}\partial_\xi\right)U = \varepsilon L P \tag{6}$$

Setting the leading order (l.o.) term to zero we immediately obtain the expression for the G–B wave speed (as in [3])

$$c = \sqrt{\frac{\bar{c}_{44}}{\rho}\left(1 - k^4\right)} \tag{7}$$

where $k=\sqrt{e_{15}^2/(\epsilon_{11}\bar{c}_{44})}$ is piezoelectric coupling factor. This iteration step is similar to solving the relevant dispersial equation.

With zero l.o. term, taking into account (7), we can rewrite (6) in a dimensionless form as

$$\mathrm{i}\bar{c}_{44}k^2\big(k^{-4}-1\big)\partial_\tau U = LP \tag{8}$$

To within $\mathrm{O}(\varepsilon)$ terms in (5) we get $\mathrm{d}u/\mathrm{d}\zeta = -\mathrm{i}k^2\partial_\xi U$. Hence, the symbolic equation (8) may be transformed to

$$\bar{c}_{44}\big(k^{-4}-1\big)\partial_\xi\partial_\tau\frac{\mathrm{d}u}{\mathrm{d}\zeta} = -L\partial_\xi^2 P \tag{9}$$

We can also show that in leading order

$$\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{2}{\varepsilon L^2}\partial_\xi\partial_\tau \tag{10}$$

Finally, we return to original variables in (9), utilizing the symbolic relation (10). We obtain

$$\frac{\partial^2\chi}{\partial t^2}-c^2\frac{\partial^2\chi}{\partial x^2} = \frac{2k^4}{\rho}\frac{\partial^2 P}{\partial x^2}, \quad \text{with } \chi = \frac{\partial u}{\partial y}\bigg|_{y=0} \tag{11}$$

Thus, we derived a 1D hyperbolic equation describing explicitly surface shear deformation χ in the B–G wave far field.

Now we discuss in brief a half-space contacting vacuum. In this case we should take into consideration

$$\nabla^2\hat{\phi} = 0 \quad \text{with } y \leqslant 0$$

along with governing equations (1) and impose the following boundary conditions

$$\sigma_{23} = -P, \quad \phi = \hat{\phi}, \quad D_2 = \widehat{D}_2 \quad \text{on } y = 0 \tag{12}$$

where $\hat{\phi}$ and $\widehat{D}_2$ are electric potential and normal component of electric displacement in vacuum, respectively.

Proceeding in the same manner as before we get for the G–B wave speed (see [3])

$$c = \sqrt{\frac{\bar{c}_{44}}{\rho}\left(1-k^4(1+\epsilon_{11})^2\right)}$$

and then an approximate wave equation becomes

$$\frac{\partial^2\chi}{\partial t^2}-c^2\frac{\partial^2\chi}{\partial x^2} = \frac{2k^4(1+\epsilon_{11})^2}{\rho}\frac{\partial^2 P}{\partial x^2}$$

For both types of boundary conditions (2) and (12) on the surface, by neglecting $\mathrm{O}(\varepsilon)$ terms in (4) we obtain the elliptic equation

$$k^4\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0, \quad \text{accompanied with the boundary condition } \frac{\partial u}{\partial y}\bigg|_{y=0} = \chi \tag{13}$$

By solving the Neumann problem (13) we can restore the displacement field for the interior half-space. It should be emphasized that this elliptic problem does not allow discontinuities to come through the interior which is in complete agreement with the general idea of a surface wave.

At the next stage the electric field can be determined starting from classical boundary value problems. For example, in the case of an electrode coated surface we should consider the Dirichlet problem for Laplace equation in (1) with the boundary condition ($y = 0$)

$$\psi = -\frac{e_{15}}{\epsilon_{11}}u$$

where the surface displacements u are assumed to be known from the solution of the Neumann problem (13).

4. Example

As an example, consider the effect of the mechanical impact load

$$P(x,t)=\delta(t)\frac{1}{\sqrt{2\pi}L}\exp\left(-\frac{x^2}{2L^2}\right) \tag{14}$$

For this, (11) takes the form

$$\frac{\partial^2\chi}{\partial t^2}-c^2\frac{\partial^2\chi}{\partial x^2}=\frac{2k^4}{\rho\sqrt{2\pi}L^5}\left(x^2-L^2\right)\exp\left(-\frac{x^2}{2L^2}\right)\delta(t)$$

which may be reduced to the homogeneous equation

$$\frac{\partial^2\chi}{\partial t^2}-c^2\frac{\partial^2\chi}{\partial x^2}=0 \tag{15}$$

with the initial conditions

$$\chi(x,0)=0,\qquad \frac{\partial\chi(x,0)}{\partial t}=\frac{2k^4}{\rho\sqrt{2\pi}L^5}\left(x^2-L^2\right)\exp\left(-\frac{x^2}{2L^2}\right) \tag{16}$$

The classical D'Alembert's solution for (15) (see [8], p. 776) yields

$$\chi=\frac{k^4}{\rho c\sqrt{2\pi}L^5}\int_{x-ct}^{x+ct}\left(\alpha^2-L^2\right)\exp\left(-\frac{\alpha^2}{2L^2}\right)\mathrm{d}\alpha$$

or

$$\chi=\frac{k^4}{\rho c\sqrt{2\pi}L^3}\left((x-ct)\exp\left(-\frac{(x-ct)^2}{2L^2}\right)-(x+ct)\exp\left(-\frac{(x+ct)^2}{2L^2}\right)\right) \tag{17}$$

The sketch of this function is shown in Fig. 1.

For comparison, we also obtain the exact solution. By applying the Fourier–Laplace transform with the parameters p, s, respectively, to the original equations (1) we get

$$\frac{\partial^2 U^{\mathrm{FL}}}{\partial y^2}=\left(p^2+\frac{\rho}{\bar{c}_{44}}s^2\right)U^{\mathrm{FL}}$$

$$\frac{\partial^2 \Psi^{\mathrm{FL}}}{\partial y^2}=p^2\Psi^{\mathrm{FL}}$$

where the superscript 'FL' defines Fourier–Laplace transforms.

Next, we have

$$U^{\mathrm{FL}}=F(p,s)\exp\left(-\sqrt{p^2+\frac{\rho}{\bar{c}_{44}}s^2}\,y\right) \tag{18}$$

Inserting (14), (18) into the transformed boundary conditions (2) leads to

$$F(p,s)=\frac{1}{\bar{c}_{44}\sqrt{2\pi}}\frac{\exp(-L^2p^2/2)}{\sqrt{p^2+(\rho/\bar{c}_{44})s^2}-k^2p}$$

The transformed surface shear deformation becomes

$$\chi^{\mathrm{FL}}=\frac{1}{\rho\sqrt{2\pi}}\frac{\exp(-L^2p^2/2)(p^2-(\rho/\bar{c}_{44})\omega^2+k^2p\sqrt{p^2-(\rho/\bar{c}_{44})\omega^2})}{(\omega+cp)(\omega-cp)} \tag{19}$$

where $\omega=-\mathrm{i}s$.

The contribution of the B–G wave field in (19) is represented by the sum of residues

$$\chi^{\mathrm{F}} = \mathrm{i}\frac{k^4}{\rho c\sqrt{2\pi}}\, p \exp\left(-\frac{L^2 p^2}{2}\right)\left(\mathrm{e}^{\mathrm{i}cpt} - \mathrm{e}^{-\mathrm{i}cpt}\right) \tag{20}$$

corresponding to the poles $\omega = \pm cp$.

By evaluating the inverse Fourier transform for (20) we arrive at the approximate solution (17).

Acknowledgements

The work of Alexis Zakharov was supported by Overseas Research Students (ORS) Awards and Overseas Scholarship Scheme (OSS).

References

[1] J.K. Knowles, A note on elastic surface waves, J. Geophys. Res. 71 (1966) 5480–5481.
[2] J.D. Kaplunov, L.Yu. Kossovitch, An asymptotic model for the Rayleigh wave far-field in the case of an elastic half-plane, Russian Phys. Dokl., in press.
[3] J.L. Bleustein, A new surface wave in piezoelectric materials, Appl. Phys. Lett. 13 (1968) 412–413.
[4] Yu.V. Gulyaev, Surface electroelastic waves in solids, JETP Lett. 9 (1969) 63–64.
[5] J.D. Cole, Perturbation Methods in Applied Mathematics, Blaisdell, Waltham, MA, 1968.
[6] A.I. Lourier, Spatial Problems in Elastic Theory, Gostechizdat, Moscow, 1955.
[7] J.D. Kaplunov, L.Yu. Kossovitch, E.V. Nolde, Dynamics of Thin Walled Elastic Bodies, Academic Press, San-Diego, 1998.
[8] D.A. McQuarrie, Mathematical Methods for Scientists and Engineers, University Science, Sausalito, CA, 2003.