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On the optimization of higher eigenvalues

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Abstract

A new approach is proposed for the study of the classical Lagrange problem on the optimal form of a column with clamped ends and a fixed volume. It is proved that there exists a column with the maximal possible value of the kth eigenvalue and that such a column is unique. A method for the numerical solution is proposed. The method is based on the analysis of the critical points of a non-linear functional related to the Lagrange problem. ***To cite this article: Y.V. Egorov, C. R. Mecanique 332 (2004).***

Résumé

Sur l'optimisation de valeurs propres supérieures. On propose une nouvelle approche au problème classique de Lagrange de la forme optimale de la colonne encastrée aux volume et hauteur fixés. On prouve qu'il existe une colonne avec la valeur de la k-ième valeur propre maximale et que telle colonne est unique. La méthode est basée sur l'étude des points critiques d'une fonctionnelle nonlinéaire. Une méthode pour la solution numerique du problème est proposée. ***Pour citer cet article : Y.V. Egorov, C. R. Mecanique 332 (2004).***

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Mots-clés : Mécanique des solides numérique ; Problème de Lagrange ; Optimisation de valeurs propres ; Colonne optimale

1. Introduction

The Lagrange problem on the optimal form of a column is important for applications, and interesting for Variational Calculus, since a principally new approach to its solution, different from the classical one, has to be applied. The statement of the problem and the review of results obtained before 2002 can be found in the article [1] by Seyranian.

In this article the problem of optimization of higher eigenvalues in the Lagrange problem is considered for columns of different configurations. Firstly, the existence and the uniqueness are proved for columns with the maximal possible value of the kth eigenvalue, $k \in \mathbf{N}$. This problem, in a slightly different form (for beams), was studied in the Olhoff articles [2,3], where the applications of such problems to the building theory is discussed.

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The approach proposed is based on the study of properties of a related non-linear functional. This approach was used in our papers [4–7] for the problem of optimization of the first eigenvalue.

Here the clamped columns are considered. The cases of elastic or free ends can be studied in the same way.

2. The statement of the problem and the main results

The Lagrange problem can be reduced (see [1] or [4]) to the following mathematical problem:

Problem L. To find a positive function $Q(x) \in C([0,1])$ such that

$$\int_0^1 Q(x)^\alpha \, \mathrm{d}x = 1, \quad \text{where } 0 < \alpha \leqslant 1 \tag{1}$$

and such that the minimal value λ of the functional

$$L_1[Q,y] \equiv \frac{\int_0^1 Q(x) y'(x)^2 \, \mathrm{d}x}{\int_0^1 y^2(x) \, \mathrm{d}x} \tag{2}$$

in the function class $y \in C^1(0,1)$, satisfying

$$y(0) = 0, \quad y(1) = 0, \quad \int_0^1 y(x) \, \mathrm{d}x = 0 \tag{3}$$

is maximal.

The number λ in this problem is the first eigenvalue of the following Sturm–Liouville problem:

$$\big(Q(x)y'\big)' + \lambda y = C \tag{4}$$

for $0 < x < 1$ with boundary conditions (3). The constant C here is not given; it is defined by conditions (3).

Now consider a more general problem for any fixed $k \in \mathbf{N}$:

Problem $\mathbf{L}_k$. To find a positive function $Q(x) \in C[0,1]$ such that

$$\int_0^1 Q(x)^\alpha \, \mathrm{d}x = 1, \quad \text{where } 0 < \alpha \leqslant 1$$

for which the kth eigenvalue λ_k of problem (4)–(3) is maximal.

For $\alpha \in]0,1[$ let us define the non-linear functional

$$F[u,v] \equiv \left(\int_0^1 \big[u'(x)^2 + v'(x)^2\big]^{\alpha/(\alpha-1)} \, \mathrm{d}x \right)^{(1-\alpha)/\alpha} \int_0^1 \big[u(x)^2 + v(x)^2\big] \, \mathrm{d}x$$

in the function class $u, v \in C^1(0,1)$, satisfying (3).

Definition 2.1. $\mathcal{A}$ is the set of positive continuous functions Q satisfying (1).

S is the set of the functions y from the class $C^1(0,1)$ satisfying (3).

S_k is the set of pairs u, v from the class S such that the function $\theta(x) = \arctan(v'(x)/u'(x))$ is monotone decreasing and $\theta(0) - \theta(1) \leqslant (k+1)\pi$, and the function $r(x) = u'(x)^2 + v'(x)^2$ has not more than $2k+1$ extremum points in the interval $]0, 1[$ and is monotone on the intervals between these points and the points $x = 0$ and $x = 1$.

R_k is the set of the functions y from the class S, having k zeroes in $]0, 1[$.

The main idea of our method is to find the pair (u, v), giving the minimal value of the functional $F[u, v]$ in a subclass of the set $S \times S$, and then to show that these functions u, v define the form of the column with the maximal value of λ_k with the formula

$$Q(x) = \left(u'(x)^2 + v'(x)^2\right)^{-1/(1-\alpha)}$$

Actually the functional F has an infinity of points of local minimum (u_j, v_j) in the space $S \times S$ on the unit sphere in $H^1(0, 1)$ and $F[u_j, v_j] \to 0$ as $j \to \infty$. The most interesting for the considered problem is the point (u_k, v_k) such that

$$F[u_k, v_k] = m \equiv \min_{(u,v)\in S_k} F[u, v]$$

Theorem 2.2. *Let $0 < \alpha < 1$. There exists an unique solution of Problem L_k. The optimal function Q_0 can be found from the relation*

$$Q_0(x) = \left(u'(x)^2 + v'(x)^2\right)^{-1-p}, \quad p = \frac{\alpha}{1-\alpha}$$

where $u(x), v(x)$ is the solution of the system of equations

$$\left(\frac{u'(x)}{(u'(x)^2 + v'(x)^2)^{1+p}}\right)' + Mu = C_1; \quad u(0) = u(1) = 0; \quad \int_0^1 u(x)\,\mathrm{d}x = 0 \tag{5}$$

$$\left(\frac{v'(x)}{(u'(x)^2 + v'(x)^2)^{1+p}}\right)' + Mv = C_2; \quad v(0) = v(1) = 0; \quad \int_0^1 v(x)\,\mathrm{d}x = 0 \tag{6}$$

where C_1 and C_2 are constants. Moreover, if k is odd, then $u(x) = -u(1-x)$, $v(x) = v(1-x)$ and $C_1 = 0$, and if k is even, then $u(x) = u(1-x)$, $v(x) = -v(1-x)$ and $C_2 = 0$. The function Q_0 is symmetric, $Q_0(x) = Q_0(1-x)$, and can be found also as

$$Q_0(x) = \left[\frac{pr(x)}{(2p+1)m}\right]^{1+1/p} \tag{7}$$

where r is the solution to the Cauchy problem: $r(0) = (3 + 1/p)m$,

$$r^{2+2/p} r'^2 = 4\left[(c_1 - r)\left[(2 + 1/p)m\right]^{1/p} r^{2+1/p} - c_2^2\left((2 + 1/p)m\right)^{2+2/p}\right]$$

which is not constant on any subinterval, has $k+1$ points of minimum in $]0, 1[$, *and*

$$c_1 = (3p+1)\frac{m}{p} + 4m^2 s^2 a_p^{2p+2}, \quad a_p = \left(\frac{3p+1}{2p+1}\right)^{1/p}$$

$$c_2 = 2msa_p^{p+1}\sqrt{a_p^{-1} - s^2}, \quad s = v'(0)$$

If $P(r) = (c_1 - r)r^{2+1/p} - c_2^2((2 + 1/p)m)^{2+1/p}$, and r_1, r_2 are the real roots of P, then $0 < r_1 < r_0 = (3 + 1/p)m < r_2 < c_1$ and

$$(k+1)\int_{r_1}^{r_0} \frac{r^{1+1/p}\,\mathrm{d}r}{\sqrt{P(r)}} + k\int_{r_0}^{r_2} \frac{r^{1+1/p}\,\mathrm{d}r}{\sqrt{P(r)}} = \left[\frac{(2p+1)m}{p}\right]^{1/2p} \tag{8}$$

$$(k+1)\int_{r_1}^{r_0} \frac{\mathrm{d}r}{(c_1-r)\sqrt{P(r)}} + k\int_{r_0}^{r_2} \frac{\mathrm{d}r}{(c_1-r)\sqrt{P(r)}} = \frac{2k\pi}{c_2}\left(\frac{p}{(2p+1)m}\right)^{1+1/2p} \tag{9}$$

The latter system of two equations for two unknown constants m, s can be solved numerically. The existence of unique solution is guaranteed by the first part of Theorem 2.2. Note that m is the minimal value of the functional F in this class, and s is equal to $v'(0)$. The minimal value of the kth eigenvalue is $M = 1/m$.

In the case when $\alpha = 1$ we consider the following auxiliary problem: to find

$$m = \inf \int_0^1 \left(u(x)^2 + v(x)^2\right) \mathrm{d}x$$

in the class of functions $(u, v) \in S_k$ satisfying the relation

$$u'(x)^2 + v'(x)^2 = 1$$

Theorem 2.3. *Let $\alpha = 1$. There exists a unique solution to the Problem L_k. The optimal function Q_0 can be found from the relation*

$$Q_0(x) = Mr(x), \quad r(x)^2 = \left(b - w(x)\right)^2 + \left(z(x) - a(1-2x)\right)^2$$

where M, a, b are constants and $w(x), z(x)$ is the solution of the system of equations

$$r(x)w'' = b - w(x), \qquad r(x)z'' = a(1-2x) - z(x)$$
$$w(0) = w'(0) = 0, \qquad z(0) = z'(0) = 0$$

Moreover, $w(x) = w(1-x)$, $z(x) = -z(1-x)$.

The function Q_0 is symmetric, $Q_0(x) = Q_0(1-x)$, $\int_0^1 Q_0(x)\,\mathrm{d}x = 1$, and r can be found also as the solution to the problem:

$$r^2 r'^2 = P(r); \quad P(r) = cr^2 - 2r^3 - 4a^2b^2, \quad c = 4a^2 + 2\sqrt{a^2+b^2}$$

which is not constant on any subinterval and has $k+1$ points of minimum in $]0,1[$. If $P(r) = (c-2r)r^2 - 4a^2b^2$, and r_1, r_2 are the real roots of P, then $0 < r_1 < r_0 = \sqrt{a^2+b^2} < r_2 < c$ and

$$(k+1)\int_{r_1}^{r_0} \frac{r\,\mathrm{d}r}{\sqrt{P(r)}} + k\int_{r_0}^{r_2} \frac{r\,\mathrm{d}r}{\sqrt{P(r)}} = \frac{1}{2}$$

$$(k+1)\int_{r_1}^{r_0} \frac{\mathrm{d}r}{r\sqrt{P(r)}} + k\int_{r_0}^{r_2} \frac{\mathrm{d}r}{r\sqrt{P(r)}} = \frac{k\pi + \arctan(a/b)}{2ab}$$

The latter system of two equations for two unknown constants a, b can be solved numerically. The existence of the unique solution is guaranteed by the first part of Theorem 2.3.

For the most important in applications cases when $\alpha = 1/2$ or $1/3$ or 1 we obtained (using MATLAB) the optimal values of λ_1 and λ_2 given in Table 1.

If $\alpha = 1/2$ the optimal columns for $k = 1$ and $k = 2$ are obtained by rotation around the x-axis of the curves shown in Fig. 1.

Table 1
Optimal values of λ_1 and λ_2

α	λ_1	λ_2
1/3	52.667196	123.653245
1/2	52.356254	117.880174
1	47.003050	107.984669

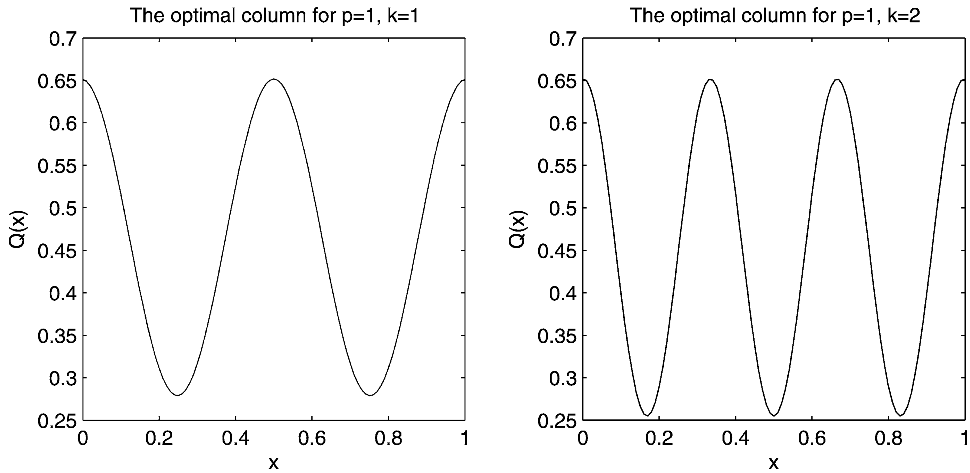

Fig. 1. Optimal columns.

3. Proof of Theorem 2.2 (for $\alpha < 1$)

Lemma 3.1. *There exist two functions $(u_0, v_0) \in S_k$ such that*

$$\sup_{(u,v)\in S_k} F[u,v] = F[u_0, v_0] = M$$

These functions are linearly independent, analytic in $]0,1[$ and satisfy the Euler–Lagrange equations (5), (6). Moreover the functions u_0 and v_0 have k and $k+1$ zeroes, respectively, in the interval $]0,1[$. Furthermore, if k is odd, then the function u_0 is odd, and v_0 is even, and if k is even, then the function u_0 is even, and v_0 is odd.

Set

$$L[Q,u,v] = \frac{\int_0^1 Q(x)(u'(x)^2 + v'(x)^2)\,\mathrm{d}x}{\int_0^1 (u(x)^2 + v(x)^2)\,\mathrm{d}x}$$

Let u_0, v_0 be the functions found in Lemma 3.1 such that $\int_0^1 (u_0'(x)^2 + v_0'(x)^2)^{-p}\,\mathrm{d}x = 1$. Let $Q_0(x) = (u_0'(x)^2 + v_0'(x)^2)^{-1-p}$. Obviously, $Q_0 \in C^\infty[0,1]$ and $Q_0(x) = Q_0(1-x)$.

We have

$$L[Q_0,u_0,v_0] = \frac{\int_0^1 Q_0(x)(u_0'(x)^2 + v_0'(x)^2)\,\mathrm{d}x}{\int_0^1 (u_0(x)^2 + v_0(x)^2)\,\mathrm{d}x} = \frac{\int_0^1 (u_0'(x)^2 + v_0'(x)^2)^{-p}\,\mathrm{d}x}{\int_0^1 (u_0(x)^2 + v_0(x)^2)\,\mathrm{d}x} = \frac{1}{m}$$

On the other hand, if Q is a function from $\mathcal{A}$, then

$$\inf_{u\in R_k,\ v\in R_k} L[Q,u,v] \leqslant \inf_{(u,v)\in S_k} L[Q,u,v] \leqslant \frac{1}{m} \equiv M$$

Therefore, $\sup_Q \inf_{y\in R_k} L_1[Q, y] \leqslant M$.

One can then check that

$$\inf_{y\in R_k} L_1[Q_0, y] = \inf_{(u,v)\in S_k} L[Q_0, u, v] = L[Q_0, u_0, v_0] = M$$

Let $\inf_{u\in R_{k,\mathrm{odd}}} L[Q_0, u, 0] = \mu$. As in [5], using the Sturm–Liouville theory one can show that $M = \mu$ and $u_1 = cu_0$. Using Lemma 2 from [4], one can check that $\inf_{v\in R_{k,\mathrm{even}}} L[Q_0, 0, v]$ is attained on a function v_1, which is proportional to v_0.

If $y \in R_k$, then $y = u + v$, where $u \in R_{k,\mathrm{odd}}$, $v \in R_{k,\mathrm{even}}$. Therefore,

$$L_1[Q_0, y] = \frac{\int_0^1 Q_0(x)(u'(x)^2 + v'(x)^2)\,\mathrm{d}x}{\int_0^1 (u(x)^2 + v(x)^2)\,\mathrm{d}x} \geqslant M$$

since

$$\int_0^1 Q_0(x)u'(x)^2\,\mathrm{d}x \geqslant M\int_0^1 u(x)^2\,\mathrm{d}x, \qquad \int_0^1 Q_0(x)v'(x)^2\,\mathrm{d}x \geqslant M\int_0^1 v(x)^2\,\mathrm{d}x$$

as it has been shown before. Adding the inequalities, we obtain that

$$\inf_{y\in R_k} L_1[Q_0, y] = \inf_{(u,v)\in S_k} L[Q_0, u, v] = L[Q_0, u_0, v_0] = L_1[Q_0, y_0] = M$$

and $y_0 = u_0 + v_0$.

Let us prove *the uniqueness*. If Q_0 is a solution to Problem L_k, then there exists a solution $(u_1, v_1) \in S_k$ of (5), (6) such that $Q_0(x) = (u_1'(x)^2 + v_1'(x)^2)^{-p-1}$. Note that

$$u_1(0) = 0, \quad v_1(0) = 0, \quad u_1'(0)^2 + v_1'(0)^2 = a_p^{-1}$$

There is an orthogonal transformation

$$\tilde{u}_0 = \alpha_1 u_1 + \alpha_2 v_1, \qquad \tilde{v}_0 = \alpha_3 u_1 + \alpha_4 v_1$$

such that the solution $(\tilde{u}_0, \tilde{v}_0)$ satisfies the equations: $\tilde{u}_0(0) = 0$, $\tilde{v}_0(0) = 0$, $\tilde{v}_0'(0) = k$, $\tilde{u}_0'(0) = \sqrt{a_p^{-1} - k^2}$, and therefore, coincides with the solution found in Lemma 3.1, as the solution of the Cauchy problem with the same initial data. Thus

$$Q_0(x) = \left(\tilde{u}_0'(x)^2 + \tilde{v}_0'(x)^2\right)^{-p-1} = \left(u_0'(x)^2 + v_0'(x)^2\right)^{-p-1}$$

Therefore, any solution of the Lagrange problem coincides with the solution found above, in Lemma 3.1.

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