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Complexification phenomenon in an example of sensitive singular perturbation

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Abstract

We consider singular perturbation problems depending on a parameter $\varepsilon \geqslant 0$ such that for $\varepsilon > 0$ the solution u^ε belongs to a Sobolev space on a domain Ω, but the limit u^0 is not a distribution on Ω. A very simple model problem, solvable by Fourier transform allows us to study the complexification process of u^ε as $\varepsilon \searrow 0$. The limit holds in the topology of a space of analytical functionals. ***To cite this article: C.A. De Souza, É. Sanchez-Palencia, C. R. Mecanique 332 (2004).***

Résumé

Phénomène de complexification dans un exemple de perturbation singulière sensitive. Nous considérons des problèmes de perturbation singulière dépendant d'un paramètre $\varepsilon \geqslant 0$ tel que pour $\varepsilon > 0$ la solution u^ε appartient à un space de Sobolev sur un domaine Ω, mais la limite u^0 n'est pas une distribution sur Ω. Un problème modèle très simple qui peut être résolu explicitement par transformation de Fourier permet d'étudier le processus de complexification de u^ε lorsque $\varepsilon \searrow 0$. La limite a lieu dans la topologie d'un espace de fonctionnelles analytiques. ***Pour citer cet article : C.A. De Souza, É. Sanchez-Palencia, C. R. Mecanique 332 (2004).***

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Mots-clés : Mécanique des solides numérique ; Complexité ; Perturbation singulière ; Sensitivité

1. Introduction

We consider singular perturbation problems of the form:

Problem $\mathbf{P}^\varepsilon$. Find $u^\varepsilon \in V$ such that $\forall v \in V$

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$$a(u^\varepsilon, v) + \varepsilon^2 b(u^\varepsilon, v) = \langle f, v\rangle \tag{1}$$

where V is a Hilbert space, $f \in V'$, a and b Hermitian continuous positive forms on V and ε is a small positive parameter. It is also assumed that $a(v,v)^{1/2}$ defines a norm on V, i.e.

$$a(v,v) = 0 \quad \Rightarrow \quad v = 0 \tag{2}$$

and that the left-hand side of (1) is coercive on V for $\varepsilon > 0$, i.e.:

$$a(v^\varepsilon, v) + \varepsilon^2 b(v^\varepsilon, v) \geqslant \varepsilon^2 c\, \|v\|_V^2 \tag{3}$$

for some $c > 0$. The existence and the uniqueness of u^ε follows from the Lax–Milgram lemma for $\varepsilon > 0$.

According to (2), we construct $V_a =$ completion of V for the norm $\|v\|_a^2 = a(v,v)$. Obviously, $V \subset V_a$ and $V_a' \subset V'$. The *limit problem*:

Problem $\mathbf{P^0}$. Find $u^0 \in V$ such that $\forall v \in V_a$

$$a(u^0, v) = \langle f, v\rangle \tag{4}$$

is well-posed for $f \in V_a'$ and V_a is the finite energy space of this problem.

There are very many examples of such problems coming from thin shell theory, where they are elliptic for $\varepsilon > 0$ but of the type of the points of the middle surface of the shell (either elliptic, parabolic or hyperbolic) for $\varepsilon = 0$. It often appears that V_a' is a very small space, and 'usual loadings' f are not in V_a'. The asymptotic behaviour of u^ε when $f \notin V_a'$ is drastically dependent on the structure of the limit equations and of the loading. Several examples may be found in [1] exhibiting, in particular, a phenomenon of accumulation of energy along the characteristics of the limit problem.

In the case of partial differential equations on a domain Ω the above problem is said to be *sensitive* when the space V_a is so large that it is not contained in the space $\mathcal{D}'(\Omega)$ of distributions on Ω. In that case, V_a' does not contain the space $\mathcal{D}(\Omega)$ of test-functions of distributions so that 'almost any loading' f is out of V_a'. This may be considered as some kind of instability: very smooth data imply 'large perturbations' going out of the finite energy space. This kind of problem appears in several situations of shell theory, for instance when the middle surface is elliptic and the shell is fixed by a part of the boundary and free by the rest [2] (see also [3]). This is a consequence of ill-posedness of the limit problem, which is elliptic but the boundary conditions on the free boundary do not satisfy the Shapiro–Lopatinskii condition.

In this Note we consider a simplified model example of such a situation. The space V is constrained so that the problem reduces essentially to a one-dimensional one, which is solved by Fourier transform even for $f \notin V_a'$. The complexification process $\varepsilon \searrow 0$, with limit in a space of analytical functionals is analysed via the numerical computation of Fourier transforms.

2. Position of the problem and Fourier image of the solution

Let $\Omega = \mathbb{R} \times (0,1)$ in the plane of the variable $x = (x_1, x_2)$. Let Γ_0 be the 'fixed boundary' $x_2 = 0$ and Γ_1 the 'free boundary' $x_2 = 1$. We consider the space

$$V = \left\{ v \in H^2(\Omega);\ \Delta v = 0,\ v|_{\Gamma_0} = 0 \right\} \tag{5}$$

where Δ denotes the Laplacian. It should be noticed that because of the constraints $\Delta v = 0$ and $v(x_1, 0) = 0$, this space may be identified with a space of functions of the only variable x_1. Indeed, according to the uniqueness theorem for the Cauchy problem for the Laplacian, $v \in V$ is defined by $\partial_2 v(x_1, 0)$. The Hermitian forms a and b are defined by

$$a(u, v) = \int_{\Gamma_0} \big(\partial_2 u(x_1, 0)\big)\big(\overline{\partial_2 v(x_1, 0)}\big)\,\mathrm{d}x_1 \quad \text{and} \quad b(u, v) = \int_{\Omega} \sum_{|\alpha|=2} (\partial_\alpha u)(\overline{\partial_\alpha v})\,\mathrm{d}x \tag{6}$$

The property (2) follows from the uniqueness of the Cauchy problem for the Laplacian, as the vanishing of $a(v, v)$ implies that v and its normal derivative vanish on Γ_0. The loading f will be the Dirac mass

$$f(x_1, x_2) = \delta(x_1)\delta(x_2 - \gamma) \tag{7}$$

for a certain $\gamma \in (0, 1]$. We note that for $\gamma \in (0, 1)$ f is a 'point force' at the interior of Ω, whereas for $\gamma = 1$ it is in fact a 'point force' on the free boundary Γ_1. In both cases $f \in V'$.

Let us perform the Fourier transform with respect to x_1, formally defined by

$$\widehat{\omega}(\xi, x_2) = \int_{-\infty}^{+\infty} \omega(x_1, x_2)\,\mathrm{e}^{-\mathrm{i}\xi x_1}\,\mathrm{d}x_1 \quad \text{and} \quad \omega(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{\omega}(\xi, x_2)\,\mathrm{e}^{\mathrm{i}\xi x_1}\,\mathrm{d}\xi \tag{8}$$

As the Fourier transform preserves (up to the factor 2π) the scalar product in $L^2(\mathbb{R})$, the Fourier image of the problem is easily obtained. Indeed, defining

$$\widehat{V} = \big\{\hat{v} \in L^2(\Omega);\ \big(\xi^2\hat{v}, \xi\partial_2\hat{v}, \partial_2^2\hat{v}\big) \in \big(L^2(\Omega)\big)^3,\ \big(-\xi^2 + \partial_2^2\big)\hat{v} = 0,\ \hat{v}(\xi, 0) = 0\big\}, \tag{9}$$

$$\hat{a}(\hat{u}, \hat{v}) = \int_{-\infty}^{+\infty} \partial_2\hat{u}(\xi, 0)\overline{\partial_2\hat{v}(\xi, 0)}\,\mathrm{d}\xi \quad \text{and} \quad \hat{b}(\hat{u}, \hat{v}) = \int_{\Omega} \big(\xi^4\hat{u}\bar{\hat{v}} + \partial_2^2\hat{u}\overline{\partial_2^2\hat{v}} + 2\xi^2\partial_2\hat{u}\overline{\partial_2\hat{v}}\big)\,\mathrm{d}\xi\,\mathrm{d}x_2 \tag{10}$$

the Fourier image of P^ε is

Problem $\widehat{\mathbf{P}}^{\boldsymbol{\varepsilon}}$. Find $\hat{u}^\varepsilon \in \widehat{V}$ such that $\forall\hat{v} \in \widehat{V}$

$$\hat{a}(\hat{u}^\varepsilon, \hat{v}) + \varepsilon^2\hat{b}(\hat{u}^\varepsilon, \hat{v}) = \int_{-\infty}^{+\infty} \hat{v}(\xi, \gamma)\,\mathrm{d}\xi \tag{11}$$

where ξ appears as a parameter.

Indeed, $\widehat{V}$ inherits from V the property that its elements may be identified with functions of the only variable ξ, for instance $\partial_2\hat{v}(\xi, 0)$. The constraint $\partial_2^2\hat{v} = \xi^2\hat{v}$ allows us to write

$$\hat{v}(\xi, x_2) = \partial_2\hat{v}(\xi, 0)\big[\xi^{-1}\sinh(\xi x_2)\big] \tag{12}$$

where it is apparent that the function $\hat{v} \in \widehat{V}$ is determined by $\partial_2\hat{v}(\xi, 0)$. Using the representation (12) for $\hat{u}^\varepsilon$ and $\hat{v}$ in (11), it becomes a problem in dimension 1, with parameter ξ. In order to discuss the dependence of the solution with respect to the position of the point of application of the force (see (7)) the solution will be denoted by $\hat{u}^\varepsilon_\gamma$. For each value of the parameter ξ, the problem can be written:

$$\big[1 + \varepsilon^2\phi^2(\xi)\big]\partial_2\hat{u}^\varepsilon_\gamma(\xi, 0) = \xi^{-1}\sinh(\gamma\xi) \tag{13}$$

where

$$\phi(\xi) = \big[\xi\sinh(2\xi)\big]^{1/2} \tag{14}$$

This function plays un important role in the following. It is even, and for large $|\xi|$ it behaves as $(|\xi|/2)^{1/2}\,\mathrm{e}^{|\xi|}$ so that its growing is exponential. The inverse function of ϕ denoted by $\Phi(\eta)$ is well defined for large η and we have:

$$\eta=\phi\big(|\xi|\big) \quad \Leftrightarrow \quad |\xi|=\Phi(\eta)\approx \log\eta \quad \text{for } \eta\ggg 1 \tag{15}$$

From (13), using using (12) for $\hat{u}_\gamma^\varepsilon$ we obtain the explicit expression of this function:

$$\hat{u}_\gamma^\varepsilon(\xi,x_2)=\big[1+\varepsilon^2\phi^2(\xi)\big]^{-1}\big[\xi^{-1}\sinh(\xi x_2)\big]\xi^{-1}\sinh(\gamma\xi) \tag{16}$$

the inverse Fourier transform of which is the searched unknown $u_\gamma^\varepsilon(x_1,x_2)$. In particular, for $\gamma=1$ and $x_2=1$ (i.e. on the free boundary)

$$\hat{u}_1^\varepsilon(\xi,1)=\big[1+\varepsilon^2\phi^2(\xi)\big]^{-1}\big[\xi^{-1}\sinh\xi\big]^2 \tag{17}$$

From (16) it is obvious that u_γ^ε converges as ε tends to zero in $\mathcal{D}'(\mathbb{R}_\xi)$ (γ and x_2 playing the role of parameters) to the limit solution

$$\hat{u}_\gamma^0(\xi,x_2)=\big[\xi^{-1}\sinh(\xi x_2)\big]\xi^{-1}\sinh(\gamma\xi) \tag{18}$$

and for $\gamma=1$ and $x_2=1$, to

$$\hat{u}_1^0(\xi,1)=\big[\xi^{-1}\sinh\xi\big]^2 \tag{19}$$

3. The limit solution $u^0(x_1,x_2)$

The inverse Fourier transform $\mathcal{F}^{-1}$ from ξ to x_1 furnishes the limit solution $u_\gamma^0(x_1,x_2)$ from (18). The variable x_2 plays here the role of a parameter. We see in (18) that $\hat{u}^0(\xi,x_2)$ has exponential growth as $|\xi|\nearrow\infty$. Consequently the Fourier transform cannot be handled in the classical framework of tempered distributions.

Concerning general distributions, it is known [4] that the (direct or inverse) Fourier transform maps the space $\mathcal{D}$ of infinitely differentiable functions with compact support onto the space $\mathcal{Z}$ of analytic functions $\hat{\theta}(\xi)$ on $\mathbb{C}$ satisfying

$$|\xi|^q\big|\hat{\theta}(\xi)\big|\leqslant C_q\,\mathrm{e}^{aq\,\mathrm{Im}(\xi)} \quad (q=0,1,2,\ldots) \tag{20}$$

with a and C_q depending on $\hat{\theta}$. Correspondingly, the space $\mathcal{D}'$ of distributions is mapped on the space $\mathcal{Z}'$ of continuous functionals on $\mathcal{Z}$ ($\mathcal{Z}'$ is the space of analytic functionals).

Obviously, the limit solution $u^0(x_1,x_2)$ which is obtained by the inverse Fourier transform of $\hat{u}^\varepsilon(\xi,x_2)$ (x_2 is always a parameter) is well defined as an element of $\mathcal{Z}'$. But $\mathcal{Z}'$ is a space of analytic functionals, not of distributions. In fact the concept of support does not make sense for elements of $\mathcal{Z}'$, as the test functions are analytic functions in $\mathcal{Z}$ which enjoy properties of analytic continuation which are incompatible with localisation. Indeed, the concept of support of a distribution is associated with the partition of the unity in $\mathcal{D}$ (see [5], Section I.3) which is impossible in $\mathcal{Z}$. The reader may refer to [6] for these questions. Nevertheless, the following elementary reasoning shows that the inverse Fourier transforms of entire functions (such as $\hat{u}^0$) have no support: indeed, if $\hat{u}(\xi)$ is an entire function of the complex variable ξ, it admits the representation

$$\hat{u}(\xi)=\sum_0^\infty c_n\xi^n \quad \text{in } \mathcal{D}' \tag{21}$$

consequently the inverse Fourier transform $u(x)$ should admit the representation

$$u(x)=\sum_0^\infty(-\mathrm{i})^n c_n\delta^{(n)}(x) \quad \text{in } \mathcal{Z}' \tag{22}$$

so that its support (if it exists!) should be at the origin (note, by the way, that (22) also shows that u is not a distribution, as distributions are locally of finite order, i.e. they are derivatives of finite order of continuous functions (see [5], Section III.8). Moreover, if $\hat{u}(\xi)$ is entire, $\hat{u}(\xi)\,\mathrm{e}^{ib\xi}$ also does for any real b, and it admits a representation of the form (21) with coefficients c_n'.

It then follows that

$$\hat{u}(\xi) = \sum_0^\infty c_n' \xi^n \,\mathrm{e}^{-\mathrm{i}b\xi} \quad \text{in } \mathcal{D}' \tag{23}$$

so that

$$u(x) = \sum_0^\infty (-\mathrm{i})^n c_n' \delta^{(n)}(x-b) \quad \text{in } \mathcal{Z}' \tag{24}$$

and the support should be at $x = b$. In fact, both representations (22) and (24) hold true: there is no contradiction between them because they may only operate on analytic functions of $\mathcal{Z}$. Nevertheless, as we shall see in the next section, (22) is a more suited representation than (24) as the approximating function $u^\varepsilon(x_1, 1)$ 'concentrate and complexify' around $x_1 = 0$.

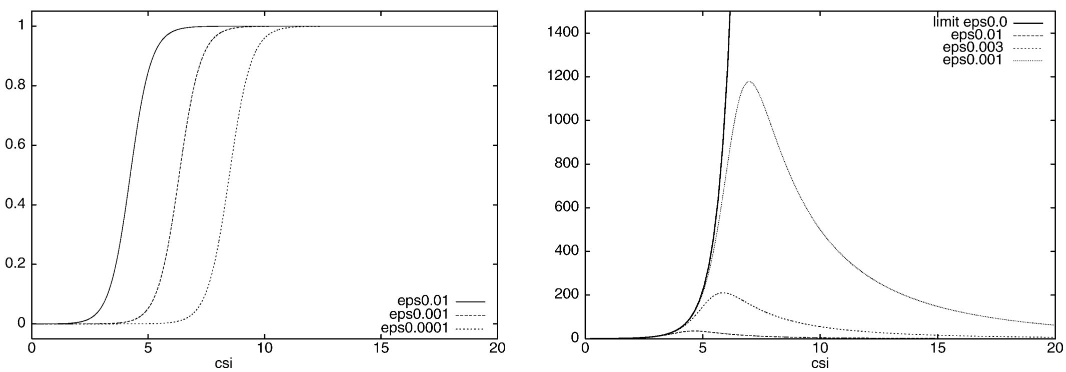
4. The complexification process

Let us focus for the sake of simplicity on the case $\gamma = 1$ and the solution on the free boundary $x_2 = 1$; i.e. we consider (17) and (19). Taking the inverse Fourier transform, we have

$$u_1^\varepsilon(x_1, 1) \to u_1^0(x_1, 1) \quad \text{in } \mathcal{Z}' \tag{25}$$

where u^ε are H^2-functions and u^0 is the analytical functional mentioned in Section 3. As the $\mathcal{Z}'$ is a 'very poor' topology we are analyzing a little better the behaviour of u^ε for small ε. For evident symmetry properties, the functions u_γ^ε and $\hat{u}_\gamma^\varepsilon$ are real and even in x_1 and ξ respectively.

A first interpretation of the complexification process follows from (17) and its graphic representation Fig. 1(b). For fixed ξ, $\hat{u}_1^\varepsilon(\xi, 1)$ is increasing for decreasing ε. Moreover, it is apparent from Fig. 1(b) that as ε decreases the 'joined portions' of $\hat{u}_1^\varepsilon(\xi, 1)$ are concerned with larger and larger values of $|\xi|$. In other words, $\varepsilon \searrow 0$ amounts to increase 'high frequency components', responsible for loss of smoothness. The limit is the exponentially growing function $\hat{u}_1^0(\xi, 1)$, also represented in Fig. 1(b).



(a) (b)

Fig. 1. (a) High-pass pseudo-filter $F(\varepsilon, \xi)$ for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$; (b) $\hat{u}_1^\varepsilon(\xi, 1)$ for $\varepsilon = 10^{-2}, 3\times 10^{-3}, 10^{-3}$.

Fig. 1. (a) Pseudo-filtre passe-haut $F(\varepsilon, \xi)$ pour $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$; (b) $\hat{u}_1^\varepsilon(\xi, 1)$ pour $\varepsilon = 10^{-2}, 3\times 10^{-3}, 10^{-3}$.

In order to clarify this process, it will prove useful to separate the 'bounded frequencies' which imply 'smooth convergence' from 'high frequencies' responsible for complexification. To this end we decompose $\hat{u}_1^\varepsilon(\xi,1)$ on the form

$$\hat{u}_1^\varepsilon(\xi,1) = \hat{v}_1^\varepsilon(\xi,1) + \widehat{w}_1^\varepsilon(\xi,1) \tag{26}$$

$$\hat{v}_1^\varepsilon(\xi,1) = \hat{u}_1^\varepsilon(\xi,1)\zeta(\xi), \qquad \widehat{w}_1^\varepsilon(\xi,1) = \hat{u}_1^\varepsilon(\xi,1)\big(1-\zeta(\xi)\big) \tag{27}$$

where $\zeta(\xi)$ is a fixed even cut-off function of $\mathcal{D}(\mathbb{R}_\xi)$. From (17) and (19) it is apparent that $\hat{u}^\varepsilon$ converges in $\mathcal{D}$. Then the corresponding $w_1^\varepsilon(x_1,1)$ converges in $\mathcal{Z}$.

Let us write $\widehat{w}_1^\varepsilon(\xi,1)$ in the form

$$\widehat{w}_1^\varepsilon(\xi,1) = \varepsilon^{-2}F(\varepsilon,\xi)\psi(\xi) \tag{28}$$

with

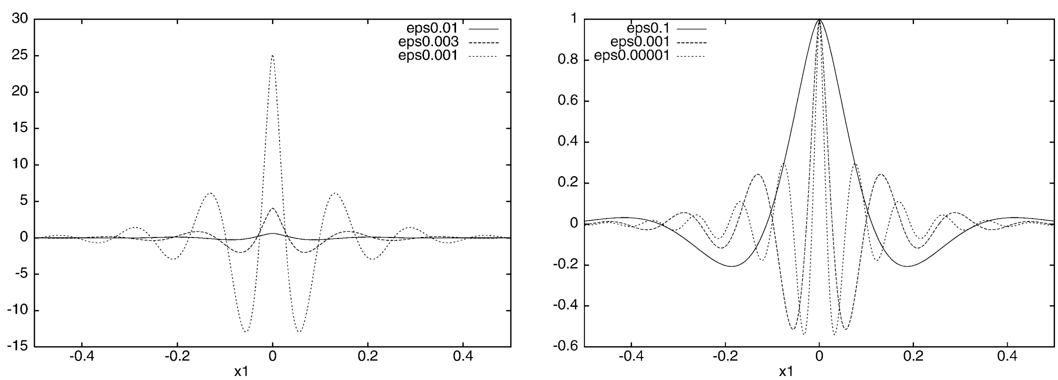
$$F(\varepsilon,\xi) = \big[\varepsilon^{-2}\phi(\xi)^{-2}+1\big]^{-1} \quad \text{and} \quad \psi(\xi) = \left(\frac{\sinh\xi}{\xi}\right)^2\phi^{-2}(\xi)\big(1-\zeta(\xi)\big) \tag{29}$$

It follows from (14) that $\psi(\xi)$ is a smooth function vanishing a neighbourhood of $\xi = 0$ and decreasing as $2^{-1}|\xi|^{-3}$ for $\xi \to \infty$.

The function $F(\varepsilon,\xi)$ may be considered as some kind of 'high-pass filter', as it takes values near to 1 for $\phi(\xi)^{-2}\varepsilon^{-2} \lll 1$ and of order $\mathcal{O}(\varepsilon^2)$ for $\phi(\xi)^{-2}\varepsilon^{-2} \ggg 1$. This corresponds to a 'cut-off frequency' $|\xi|$ such that $\phi(|\xi|) = \mathcal{O}(\varepsilon^{-1})$, i.e. nearly $|\xi| = \mathcal{O}\log(\varepsilon^{-1})$ according to (14) and (15). Nevertheless, as we saw, $F(\varepsilon,\xi)$ does not vanish for $|\xi| \lll \log\varepsilon^{-1}$ (this only happen for $\xi = 0$) but is of order $\mathcal{O}(\varepsilon^2)$. This is why we call F a 'pseudo-filter'.

The evolution with ε of $\widehat{w}_1^\varepsilon(\xi,1)$ may be interpreted as the application of the pseudo-filter $F(\varepsilon,\xi)$ to $\psi(\xi)$. Bearing in mind that this function behaves for large $|\xi|$ as $2^{-1}|\xi|^{-3}$, we see that the 'remainder' is smaller and smaller as ε tends to zero but, it is multiplied by ε^{-2} and we get a non-zero limit. Taking the inverse Fourier transform the complexification process for $u_1^\varepsilon(x,1)$ may be interpreted as a progressive 'almost suppression' of the Fourier components with frequency small with respect to $\log(\varepsilon^{-1})$ and multiplication by ε^{-2}.

Fig. 2(a) is a plot of $u_1^\varepsilon(x_1,1)$ for various values of ε. It is apparent that it is an oscillating function with a maximum at $x_1 = 0$ and decreasing oscillations. The maximum growths as ε tends to zero. In order to compare the shapes of $u_1^\varepsilon(x_1,1)$ for different values of ε, Fig. 2(b) is a plot of them normalized by their maxima, i.e. $u_1^\varepsilon(x_1,1)/u_1^\varepsilon(0,1)$. In order to 'quantify the shape' of these functions, we consider for each the relative minima which are larger (in modulus) than 10^{-2} times the maximum of the function. It is then easily seen that the number



(a) (b)

Fig. 2. (a) $u_1^\varepsilon(x_1,1)$ for $\varepsilon = 10^{-2}, 3\times10^{-3}, 10^{-3}$; (b) $u_1^\varepsilon(x_1,1)$ normalized for $\varepsilon = 10^{-1}, 10^{-3}, 10^{-5}$.

Fig. 2. (a) $u_1^\varepsilon(x_1,1)$ pour $\varepsilon = 10^{-2}, 3\times10^{-3}, 10^{-3}$; (b) $u_1^\varepsilon(x_1,1)$ normalisé pour $\varepsilon = 10^{-1}, 10^{-3}, 10^{-5}$.

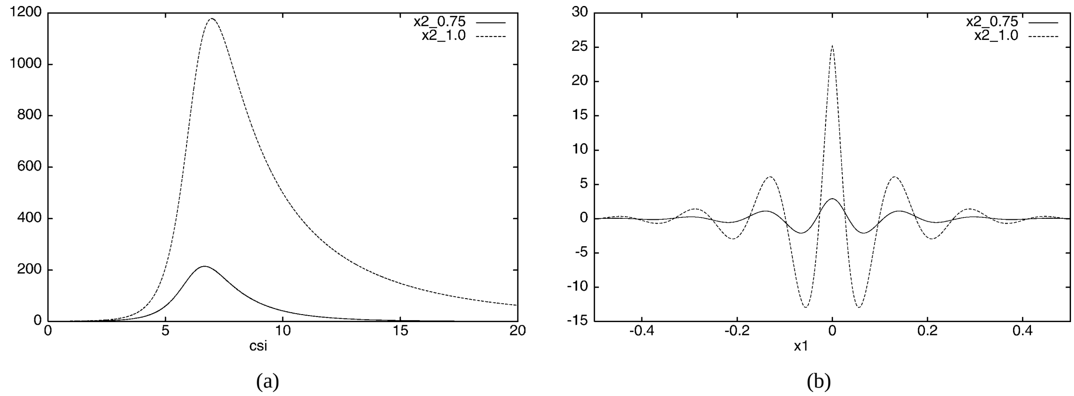


Fig. 3. (a) $\hat{u}_1^\varepsilon(\xi, 0.75)$ and $\hat{u}_1^\varepsilon(\xi, 1.0)$ for $\varepsilon = 10^{-3}$; (b) $u_1^\varepsilon(x_1, 0.75)$ and $u_1^\varepsilon(x_1, 1.0)$ for $\varepsilon = 10^{-3}$.

Fig. 3. (a) $\hat{u}_1^\varepsilon(\xi, 0{,}75)$ et $\hat{u}_1^\varepsilon(\xi, 1{,}0)$ pour $\varepsilon = 10^{-3}$; (b) $u_1^\varepsilon(x_1, 0{,}75)$ et $u_1^\varepsilon(x_1, 1{,}0)$ pour $\varepsilon = 10^{-3}$.

of such 'significant minima' is approximately proportional to $\phi^{-1}(\varepsilon) \approx \log\varepsilon^{-1}$. Moreover, they are located inside a region $|\xi| \leqslant c(\log\varepsilon^{-1})^{-1}$. This last property is consistent with the fact that $\hat{u}_1^\varepsilon(\xi, 1)$ in Fig. 1(b) is mainly significant in the region $\mathcal{O}(|\xi|) = \phi^{-1}(\varepsilon) \approx \log\varepsilon^{-1}$. As a result $u_1^\varepsilon(x_1, 1)$ behaves as an internal layer of thickness $\approx (\log\varepsilon^{-1})^{-1}$. The values of the function and the complexity of its pattern increase as ε tends to zero.

Let us consider a little the solution for $x_2 < 1$ *as well as* $\gamma < 1$ (see (7)). The limit solution $u_\gamma^0(x_1, x_2)$ is the inverse Fourier transform of $\hat{u}_\gamma^0(\xi, x_2)$, given by (18). For each fixed γ and x_2, $u_\gamma^0(x_1, x_2)$ is an entire function of ξ, exponentially growing for ξ tending to $+\infty$ or $-\infty$, so that, as in Section 3, $u_\gamma^0(x_1, x_2)$ are not distributions. For $\varepsilon \neq 0$ the Fourier transforms $u_\gamma^\varepsilon(x_1, x_2)$ are given by (16). These functions are symmetric in γ and x_2, which implies an interesting reciprocity property

$$u_\gamma^\varepsilon(x_1, x_2) = u_{x_2}^\varepsilon(x_1, \gamma) \tag{30}$$

i.e. when the force is applied at $x_2 = a$, the solution at $x_2 = b$ is the same as the solution at $x_2 = a$ when the force is applied at $x_2 = b$.

Moreover, in order to compare $\hat{u}_\gamma^\varepsilon(\xi, x_2)$ with the above considered $\hat{u}_1^\varepsilon(\xi, 1)$, we have from (16), (17):

$$\hat{u}_\gamma^\varepsilon(\xi, x_2) = \hat{u}_1^\varepsilon(\xi, 1)\frac{\sinh(\xi x_2)\sinh(\gamma\xi)}{(\sinh\xi)^2} \tag{31}$$

and the inverse Fourier transforms follow analogous trends as above, but with lower intensity. In particular, the main contribution is given by the frequencies $|\xi| = \mathcal{O}(\log\varepsilon^{-1})$. In that region (31) becomes

$$\hat{u}_\gamma^\varepsilon(\xi, x_2) \approx \hat{u}_1^\varepsilon(\xi, 1)\,\mathrm{e}^{-|\xi|[(1-x_2)+(1-\gamma)]} \tag{32}$$

We then see that the dependence with respect to x_2 is very sharp, exhibiting a layer of the thickness $\mathcal{O}((\log\varepsilon^{-1})^{-1})$ in the vicinity of the free boundary $x_2 = 1$; the intensity decreases exponentially away from it. This property, which is independent of the location of the applied force, is an evident consequence of the structure of the functions for high frequency (see (12)).

In Fig. 3(b) were represented $u_1^\varepsilon(x_1, 0.75)$ and $u_1^\varepsilon(x_1, 1.0)$ for $\varepsilon = 10^{-3}$. This illustrates the drastic decreasing of the functions, even if the very presence of a layer is not evident, as $\log\varepsilon^{-1}$ is not very small.

5. Complements and concluding remarks

(a) *On the definition of the model*. Obviously, the above-mentioned phenomena are a consequence of the peculiar structure of the space V (see (5)). The constraint $\Delta v = 0$ implies that essentially it is a problem in the variable x_1. For 'large frequency' $|\xi| \ggg 1$, the functions $\hat{v}(\xi, x_2)$ are exponentially large with respect to the genuine unknown

$\partial_2 \hat{v}(\xi, 0)$ (see (12)). This feature is associated with solving a Cauchy elliptic problem. Moreover, the force f acts upon $v(x_1, x_2)$ and this enhances its action exponentially for large $|\xi|$.

All these features are present in a more complicated, but not essentially different way, in the problem of an elliptic shell with a part of the boundary Γ_0 fixed and the remainder Γ_1 free. Indeed, this problem (with other simplifications) was addressed in [3], first example, for fixed but large values of k (analogous to our $|\xi|$). For large values of $|\xi|$ it appears a constraint which amounts to an inextensional character of the deformation (this property is called 'simplification 2' in [3]). In fact, taking in [3] the values of the parameters $a = b = \mu = L = 1$, $|\xi|$ and x_2 instead of k and x_1 (note that the axes are reversed with respect to ours), formula (12) of [3] gives

$$u_3(x_2) = 4c(1+\nu)\,\mathrm{e}^{|\xi| x_2} \tag{33}$$

which is somewhat analogous to our equation (12) and exhibits exponential growth. In fact the only difference between both problems to be pointed out is that in [3] the constraint appears in a progressive asymptotic way as $|\xi|$ tends to infinity, whereas in (5) it is prescribed everywhere. Clearly this difference has no important consequences, as the phenomena under consideration are mainly concerned with high frequencies. Otherwise, this difference implies a drastic simplification in our problem which may be explicitly solved.

(b) *The sensitive character of the problem*. It is not hard to check that the problem of Section 2 is sensitive, i.e. according to the definition in Section 1, that V_a is not contained in $\mathcal{D}'(\Omega)$. Indeed, the Fourier image of V_a is clearly formed by the functions of the form (12) with

$$\int_{-\infty}^{\infty} \left|\partial_2 \hat{v}(\xi, 0)\right|^2 \mathrm{d}\xi < \infty \tag{34}$$

Coming back to (12) it is apparent that $\hat{v}(\xi, x_2)$ with $x_2 \in (0, 1)$ may have an exponential growing for $|\xi|$ tending to infinity. The inverse Fourier transform are not distributions, for reasons analogous to that of Section 3 (see [6], Theorem 4.4 if necessary).

(c) Combining the fact that the section with $x_2 = const$ contains an internal layer of thickness $\mathcal{O}(\log \varepsilon^{-1})$ at $x_1 = 0$ and that the dependence in x_2 involves again a layer near $x_2 = 1$, with thickness $\mathcal{O}(\log \varepsilon^{-1})$ again, we see that in fact, $u_\gamma^\varepsilon(x_1, x_2)$ involves a complexification (i.e. with limit out of the distribution space) in a region of Ω with diameter $\mathcal{O}(\log \varepsilon^{-1})$ centered at $x_1 = 0$, $x_2 = 1$, i.e. at the point of the free boundary which is nearest to the point of application of the force (see (7)). The intensity is highly dependent of the position of the applied force. These properties are consequences of the structure of the functions for high frequency $|\xi|$ (see (12)).

(d) The complexification process is highly anisotropic, as it is mainly defined by the dependence in x_2, and consists in a progressive increase of the intensity and of the oscillating character in the x_1 direction, whereas the dependence in x_2 involves a somewhat classical layer in the vicinity of $x_2 = 1$.

(e) The complexification process depends on ε mainly by the Fourier components with frequency $|\xi| = \mathcal{O}(\log \varepsilon^{-1})$. In particular the thickness of the layer is $\mathcal{O}((\log \varepsilon^{-1})^{-1})$. Consequently *the convergence to the limit is very slow* and *the complexification takes place very slowly*.

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