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On a closure of mass conservation equation and stability analysis in the mathematical theory of vehicular traffic flow

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Abstract

This Note deals with the development of mathematical methods for the closure of the mass conservation equation for macroscopic hydrodynamical models of traffic flow on roads. The closure is obtained by a phenomenological model, relating the local mean velocity to local density earlier in time. An evolution equation is obtained for the flux and a stability analysis is performed; this qualitatively describes some features of congested flow. ***To cite this article: V. Coscia, C. R. Mecanique 332 (2004).***

Résumé

Sur une fermeture des équations de conservation de la masse et l'analyse de la stabilité dans la théorie mathématique de la circulation véhiculaire. Cette Note est dédiée au dévelopement d'une méthodes mathématiques pour la fermeture des équations macroscopiques de la conservation de la masse, intervenant dans la modélisation du trafic des véhicules. La fermeture est obtenue en utilisant un modèle phénoménologique approprié pour relier la vitesse moyenne locale à la densité locale (avec retard en temp). Une équation pour le flux est derivée et une analyse de stabilité est conduite. ***Pour citer cet article : V. Coscia, C. R. Mecanique 332 (2004).***

Keywords: Continuum mechanics; Traffic flow; Continuum models; Nonlinear sciences

Mots-clés : Milieux continus ; Flux du traffic ; Models continues ; Sciences non-linéaires

1. Introduction

As is known [1–4], traffic flow phenomena can be described by macroscopic equations related to the conservation of mass and momentum for a flow of vehicles regarded as a continuum. This type of representation can certainly be criticized, as the mean distances between vehicles are large enough to be in contrast with the

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paradigms of continuum mechanics. On the other hand, the control and optimization of traffic flow conditions on highways and networks of roads is crucial for an improvement in the quality of life. Thus, it is useful to look for relatively simple models, suitable for providing an approximate description of the physical reality.

One of the technical difficulties in the assessment of mathematical models is the identification of the parameters involved. This suggests that, at least for some specific applications, relatively simple models may allow a relatively more careful identification. Indeed, this is the case of models obtained by the conservation of mass, closed by a phenomenological relation, suitable for linking the local mean velocity to the local mass density and density profiles. It is a problem of closure of the mass conservation equation proposed as an alternative to the use of the mass and momentum equations, with the aim of dealing with the technical difficulty of closing the momentum equation by phenomenological models suitable for describing the acceleration applied to the vehicles in the elementary volume by all surrounding vehicles.

Recent research papers have shown that first order models are useful for dealing with complex traffic flow conditions such as variable road conditions [5] or networks of roads [6–8]. Moreover, the analysis developed in [9], taking advantage of the related qualitative analysis proposed in [10], has shown that the identification of the parameters of first order models can be effectively realized. The above identification will be used in the sections which follow.

An additional aspect to be taken into account is that it is useful to deal with the evolution of the flow rather than the density. Indeed, measurements for the flow are relatively more precise than those for the density. In addition, the statement of boundary conditions in the case of networks of roads is technically immediate for the flow, which is not the case for the density.

This paper is developed on the basis of the above reasoning and proposes a new closure for the mass conservation equation based on a delay assumption of the reaction of drivers. This closure, linked with some analytical interpretation of experimental results given in [9], allows us to derive an evolution equation for the flow. A qualitative analysis of the stability properties of such a model confirms some instability features of congested traffic flow observed and interpreted by Kerner [11,12]. The content of this paper is organized thus. Section 2 describes the mathematical setting related to conservation of mass and momentum. The new model proposed in this paper is described in Section 3, together with a qualitative analysis of its stability properties. Finally, the last section deals with a critical analysis and with the indication of some research perspectives.

2. Mathematical setting

This section provides the mathematical setting related to the hydrodynamic modeling of a one-lane flow of vehicles on a road. The conservation equations will be given, following [4], in terms of dimensionless variables:

- $t = t_r/T$ is the dimensionless time variable with reference to a characteristic time T, where t_r is the real time;
- $x = x_r/\ell$ is the dimensionless space variable with reference to a characteristic length of the road ℓ, where x_r is the real dimensional space;
- $u = n/n_M$ is the dimensionless density with reference to the maximum density n_M of vehicles corresponding to bumper-to-bumper traffic jam;
- $v = v_R/v_M$ is the dimensionless velocity with reference to the maximum mean velocity v_M, where v_R is the real velocity of the single vehicle;
- q is the dimensionless linear mean flux with reference to the maximum admissible mean flux $q_M = n_M v_M$.

In what follows the characteristic time T will be assumed according to the condition $v_M T = \ell$; this means that T is the time necessary to cover the whole road length at the maximum mean velocity.

Macroscopic models are obtained by conservation equations corresponding to mass and linear momentum, referring, for each lane, to the variables $u = u(t,x) \in [0,1]$, and $v = v(t,x) \in [0,1]$.

Still referring to [4], the reference framework is that concerning conservation of mass, and linear momentum

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(uv) = 0 \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = f[u, v] \end{cases} \tag{1}$$

where f defines the acceleration of the vehicles in the elementary volume. The word *acceleration* is used, when dealing with traffic flow models, to avoid the use of the term *force* for a system where the mass cannot be properly defined.

3. On the closure of mass conservation – a new first order model

The closure of the first equation in (1) can be obtained by exploiting the interpretation of the experimental data proposed in [9]. Considering that this equation involves both u and v, a self-consistent model can be obtained if a phenomenological relation can be proposed to link v to u by a suitable analytical or functional equation. Experimental results provide the above link in steady uniform flow conditions.

Specifically, the following model:

$$\begin{cases} u \leqslant u_c: \quad v = 1 \\ u \geqslant u_c: \quad v = \exp\left\{-\alpha \dfrac{u - u_c}{1 - u}\right\}, \quad \alpha > 0 \end{cases} \tag{2}$$

was proposed in [9] referring to the so called *velocity diagram* where v is related to u in steady uniform flow, or to the *fundamental diagram*, where the flux q is related to u.

Referring to the above model, u_c is a critical density which separates the free flow from the congested flow, while α is a parameter related to the specific features of the road and environmental conditions. Comparisons with experimental data show the following ranges of variability of the above parameters:

$$u_c \in [0, .1] \qquad \alpha \in [1, 2.5] \tag{3}$$

where relatively larger values of α denote strong decay of the mean velocity with local density and hence relatively less favourable road-weather conditions. Analogously, relatively larger values of u_c denote persistence of the free flow conditions and hence favourable road-weather conditions.

The above analytic expressions cannot be used to close the mass conservation equation. Indeed, as critically analyzed in [4], one cannot use a relation which is valid in steady uniform equilibrium conditions to close an equation that should be valid far from these conditions. Due to the above reasoning, various closures have been proposed in the literature, which take into account either the delay of the driver to reach the above equilibrium conditions or the fact that the driver has a perception of the density different from the real density as it is influenced by the local density gradients [13].

In this Note we assume that the velocity at which cars travel is appropriate to the density at an earlier time:

$$v(\bar{u}) = v\big(u(x, t - \tau)\big) \tag{4}$$

where τ is a parameter, small with respect to one, that corresponds to a relatively large time, thus introducing a kind of retarded adaptation of the driver to the actual traffic conditions. In terms of flux, the closure relation is expressed as:

$$\begin{cases} \bar{u} \leqslant u_c: \quad v = 1, \quad q = u \\ \bar{u} \geqslant u_c: \quad v < 1, \quad q = \varphi(\bar{u}) \end{cases} \tag{5}$$

where

$$q = \varphi(\bar{u}) = u \exp\left\{-\alpha \frac{\bar{u} - u_c}{1 - \bar{u}}\right\} \tag{6}$$

Now, when $\bar{u} > u_c$, from (4) and (6) we get:

$$\varphi(\bar{u}) = u \exp\left\{-\alpha \frac{\bar{u} - u_c}{1 - \bar{u}}\right\} = uv\big(u(x, t - \tau)\big)$$

Considering that the retardation parameter is small, from the previous relation we find:

$$\varphi(\bar{u}) = uv(u) - \tau u v'(u)\frac{\partial u}{\partial t} = u\left(1 + \tau\alpha \frac{1 - u_c}{(1-u)^2}\frac{\partial u}{\partial t}\right)\exp\left\{-\alpha \frac{u - u_c}{1 - u}\right\}, \quad u = (\bar{u})_{\tau=0}, \quad \text{as } \tau \to 0$$

where u without a bar means the density evaluated at x at time t. Computing the time derivative of $\varphi(\bar{u})$ for small τ, we finally get the following model:

$$\begin{cases} \dfrac{\partial q}{\partial \mathbf{x}} = -\dfrac{\partial u}{\partial t} \\ \dfrac{\partial q}{\partial t} = \dfrac{\partial u}{\partial t} \end{cases} \quad \text{if } \bar{u} \leqslant u_c$$
$$\begin{cases} \dfrac{\partial q}{\partial \mathbf{x}} = -\dfrac{\partial u}{\partial t} \\ \dfrac{\partial q}{\partial t} = \bar{\Phi}(u)\dfrac{\partial q}{\partial x} + \tau\bar{\Psi}(u)\dfrac{\partial^2 q}{\partial t \partial x} + \tau\overline{\Gamma}(u)\left(\dfrac{\partial q}{\partial x}\right)^2 \end{cases} \quad \text{if } \bar{u} > u_c \tag{7}$$

where:

$$\bar{\Phi}(u) = -\exp\left\{-\alpha \frac{u - u_c}{1 - u}\right\}$$
$$\Psi(u) = -\frac{\alpha u(1 - u_c)}{(1-u)^2}\exp\left\{-\alpha \frac{u - u_c}{1 - u}\right\}$$

and:

$$\overline{\Gamma}(u) = -\frac{\alpha(1 - u_c)}{(1-u)^3}\left(2 + \frac{\alpha(1 - u_c)}{1 - u}\right)\exp\left\{-\alpha \frac{u - u_c}{1 - u}\right\}$$

Some interesting features of the latter model can be exploited using relation (4) to close the continuity equation in (1). In the case of a small retardation time we find:

$$\frac{\partial u}{\partial t} + q'(u)\frac{\partial u}{\partial x} = \tau \frac{\partial}{\partial x}\left(uv'(u)\frac{\partial u}{\partial t}\right) \tag{8}$$

where:

$$q'(u) = v(u) + uv'(u) \tag{9}$$

In the case of nearly uniform traffic flow, the density u can be considered ‘almost’ constant: $u(x,t) = U + w(x,t)$. Substituting back into (9) and retaining terms up to the first order in w, we have that the density ‘perturbations’ obey the following linearized equation:

$$\frac{\partial w}{\partial t} + q'(U)\frac{\partial w}{\partial x} = \tau U v'(U)\frac{\partial^2 w}{\partial x \partial t} \tag{10}$$

It is easy to verify that the above equations have solutions in the form of normal modes $w(x,t) = W\,\mathrm{e}^{\mathrm{i}kx + \omega t}$, with the growth-rate parameter ω depending on the perturbation wavelength as:

$$\omega = \frac{-\mathrm{i}kq'(U)}{1 - \mathrm{i}kUv'(U)\tau}$$

On the other hand, the retardation parameter τ is small, for we can approximate the function $\omega(\tau)$ as:

$$\omega = -\mathrm{i}kq'(U) + k^2 U q'(U) v'(U)\tau, \quad \text{for } \tau \ll 1 \tag{11}$$

Finally, the absolute value of the perturbations varies as:

$$\left|w(x,t)\right| = \left|W \exp\left\{\mathrm{i}k\left(x - q'(U)t\right)\right\} \mathrm{e}^{k^2 U q'(U) v'(U)\tau t}\right| \leqslant |W|\, \mathrm{e}^{k^2 U q'(U) v'(U)\tau t} \tag{12}$$

While the quantities k^2 and U are certainly positive, we must consider the sign of the product $q'(U)v'(U)$. The term $v'(U)$ is less than zero whenever $U \geqslant u_c$. On the other hand, it is pretty easy to verify, assuming the velocity diagram (2), that the quantity $q'(U) = v(U) + U\, v'(U)$ is negative for $0 < U < 2 + \alpha(1 - u_c)$, which is certainly true, since $0 < u_c \leqslant U < 1$. Then, the exponential term in Eq. (12) grows for large t. This means that, in case of heavy traffic, uniform flows are (linearly, and then also nonlinearly) exponentially unstable, that is, small density perturbations increase in time, possibly leading to one of the observed instabilities of congested traffic flow such as ghost queues and 'stop–go' phenomena. These phenomena, which are documented and analyzed in [11,12], are described by this model in terms of instability.

4. Critical analysis and perspectives

A first order macroscopic model of traffic flow has been proposed in this Note, by closing the mass conservation equation using a relation which describes the retardation of the driver to reach the uniform steady equilibrium conditions that are experimentally observed and that may be analytically represented. First order models should be regarded as a relatively simpler alternative to second order models [14,15]. First order models may have a relatively lower ability for describing traffic flow phenomena, but certainly show flexibility to be used in the description of road networks [5]. Moreover, the identification of parameters appears to be a well understood problem, following the analysis developed in [9].

The model is then effectively ready for practical applications. Indeed, it is interesting that it has the ability to describe instability properties which are experimentally observed. This positive output, however, should not hide that the macroscopic description should be, as already mentioned in Section 1, replaced by alternative modeling such as that obtained by a collective description of dynamical systems [16] or by generalized Boltzmann or mean field models, as documented in [17], based on the methods of generalized kinetic theory [18].

Bearing all the above in mind, it is worth pointing out a few problems which are left open by this Note, and that may be regarded as research perspectives for applied mathematicians:

(i) The model describes a phase transition corresponding to $u = u_c$ with a moving boundary separating the free flow from the congested flow. An interesting problem refers to the analysis of the evolution of the above boundary.

(ii) The model is based on the concept of a 'retardation' of the driver to reach the uniform steady equilibrium conditions. On the other hand, one may argue, as in [10], that the trend of the driver is to steady equilibrium conditions nonuniform in space, according to the model proposed in [13]. Thus, it may be interesting to develop a qualitative stability analysis based on the effect of the 'retardation' term.

Finally, one may observe that the term τ is a parameter which corresponds 'globally' to the driver's behaviour, while it is well understood that drivers show different behaviour corresponding to their being aggressive, experienced, inexpert or fearful. Thus, one may consider τ as a random variable with values spanning a certain interval, which is an attempt to consider all above scenario of behaviours. Hence τ is a random variable and the dynamics is stochastic, as effectively observed in real flow conditions.

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