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Matrix-fracture exchange in a fractured porous medium: stochastic upscaling

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Abstract

We present in this Note a stochastic approach to the matrix-fracture exchange in a heterogeneous fractured porous medium. We introduce an intermediate scale, called the unit-scale, between the local-scale (fracture-scale) and the large-scale characteristic of the reservoir mesh (reservoir block). This paper focuses on the problem of upscaling fluid exchange phenomena from the unit scale to the reservoir mesh or block scale. Simplifying the Darcian flow terms enables us to obtain a probabilistic solution of the dual continuum problem, in continuous time, in the case of a purely random exchange coefficient. This is then used to develop several upscaling approaches to the fluid exchange problem, and to analyze the so-called 'effective' exchange coefficient. The results are a first contribution to the more general problem of upscaling multidimensional flow-exchange processes in space and time, in randomly heterogeneous dual continua. ***To cite this article: M. Kfoury et al., C. R. Mecanique 332 (2004).***

Résumé

Coefficient d'échange matrice-fracture en milieu poreux fracturé : changement d'échelle par approche stochastique. On présente dans cette Note une approche stochastique (probabiliste) du problème d'échange matrice-fractures en milieu poreux fracturé hétérogène. On introduit l'échelle intermédiaire des sous-blocs ou « unités », lors du passage de l'échelle locale (détails des fractures) à l'échelle globale de la maille représentative du réservoir (« bloc réservoir »). Une solution probabiliste en temps continu, sans transport, avec terme d'échange purement aléatoire, est développée. Ceci permet l'homogénéisation (instantanée ou non) du problème d'échange pur. Les résultats obtenus sont une première contribution au problème plus général du passage de l'échelle des unités à l'échelle du bloc ou maille réservoir, pour un écoulement double-milieu avec échanges matrice-fracture. ***Pour citer cet article : M. Kfoury et al., C. R. Mecanique 332 (2004).***

Keywords: Porous media; Upscaling; Fractures; Dual porosity; Exchange coefficient; Homogenization; Stochastic approach

Mots-clés : Milieux poreux ; Changement d'échelle ; Fractures ; Double porosité ; Coefficient d'échange ; Homogénéisation ; Approche stochastique

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1. Introduction

This work focuses on the upscaling of flow equations for heterogeneous, fractured porous reservoirs, a problem encountered in petroleum engineering applications. Two major difficulties arise: (i) the geometrical modeling of the heterogeneous spatial structure of the oil reservoir; and (ii) the modeling of fluid displacements in the reservoir (fluid dynamics). The data available to achieve these goals are scant and highly uncertain, whence the idea of using an upscaling scheme for the flow equations, together with an appropriate model of local scale fluid dynamics and of spatial structure (e.g. a porous reservoir with an imbedded fracture network). Before stating the problem at hand, let us briefly review related work.

Fracture networks, fractured matrix, and single medium models. There have been useful developments on random media and fracture network modeling. In particular, fracture network tools relate field observations (at core scale, seismic exploration scale, etc.) to basic parameters like fracture apertures, lengths, orientations, density, and connectivity (or coordination number), and these parameters have been used subsequently to derive 'effective properties', such as fracture permeability, associated to a continuum description of the fracture network [1–3]. The description of matrix flow and its connection to fracture network flow has also led to an important literature [4–8]. We now focus exclusively on the dual continuum approach to this problem.

Dual continuum models of matrix-fracture flow. Barrenblatt et Zheltov [4] were among the first to introduce a dual continuum flow model that takes into account, not only fracture flow, but also matrix flow, and, most importantly, matrix-fracture exchange flow. For recent work on this model, see [5,6]: the latter includes an analyzis of the approximation involved in using the dual continuum equations as a homogenized model for matrix-fracture flow. In the dual continuum approach, one must evaluate a priori the macroscopic exchange coefficient associated with the homogenized description of fluid exchanges between fractures and matrix. This can be done in many different ways, as discussed in [6].

Sequential upscaling of matrix-fracture flow. Conceptually, upscaling starts with a fine representation of the reservoir (explicitly resolving fracture apertures), to end up with a coarser continuum model (called dual continuum) where fractures are no longer 'visible'. This is illustrated as 'step (I)' in Fig. 1. However, the complexity of the spatial structure of real reservoirs should be kept in mind. It may be too drastic to attempt direct upscaling from small aperture scale up to 100 m size blocks, particularly considering the complications due to percolation effects, and the computational costs involved on a single large block compared to smaller units (fracture number may be as large as 10^5, [1]). For all these reasons, it is proposed here to implement a novel, two-step upscaling approach (I_a,I_b) illustrated in Fig. 1.

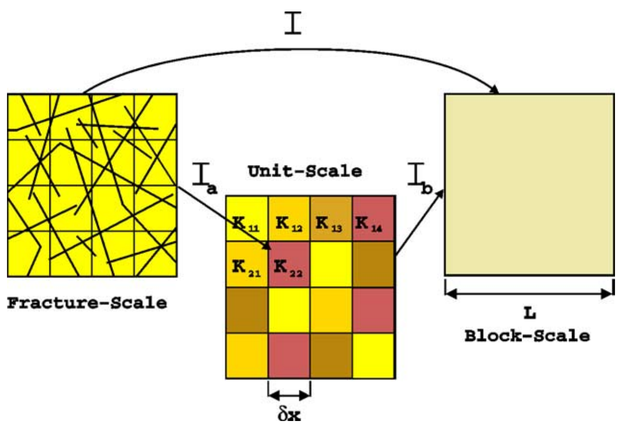

Fig. 1. Upscaling steps and transfer of information at different resolution scales: fracture scale (aperture scale); unit scale 'dx' (the block is partitioned into 'units'); block scale L (typically the mesh size for the reservoir simulator).

The first step (I_a) starts with a fine scale representation of the fractured medium and ends up with small dual continuum units (partitioning the domain). This introduces a new intermediate scale, the '*units*', which can be chosen by considering several criteria (Representative Elementary Volume, percolation effects, computational efficiency). Geostatistical analyses of the fractured reservoir can be used as in [5] to produce, at the end of step (I_a), a raster map of dual continuum properties resolved at the unit scale. Each 'unit' possesses a fracture permeability, a matrix permeability, and an exchange coefficient.

The second step (I_b) starts with the dual continuum *units* and ends up with a larger continuum *block* (reservoir mesh). The 'local scale' is described by a dual continuum model with random coefficients, and the large scale is a generalized continuum (as yet unknown). This '*second upscaling*' problem has not been addressed before. Braester et al. [9] have treated a dual continuum problem where fracture permeability is a random field, but their exchange coefficient was assumed deterministic and constant over the entire domain. Since the exchange coefficient term may be formally equivalent mathematically to a linear chemical vaction source term, there is some literature about transport, with the most relevant to our study being reaction coefficient; see the paper by Alvarado, with a small pertubation analysis of the problem [11]. In the remainder of this Note, we consider the case of a dual continuum, with randomly heterogeneous exchange coefficient, and we analyse upscaling for the 'pure' exchange problem.

2. Governing equations for the dual continuum at the unit scale

The governing equations at the unit scale are [4,6]:

$$\phi_m c_m \frac{\partial}{\partial t}\{P_m\}^m = \nabla\cdot\left(\frac{1}{\mu}\mathbf{K}_m.\nabla\{P_m\}^m\right) - \alpha\big(\{P_m\}^m - \{P_f\}^f\big) \tag{1}$$

$$\phi_f c_f \frac{\partial}{\partial t}\{P_f\}^f = \nabla\cdot\left(\frac{1}{\mu}\mathbf{K}_f\cdot\nabla\{P_f\}^f\right) - \alpha\big(\{P_f\}^f - \{P_m\}^m\big) \tag{2}$$

where $\{P_m{}^m\}$ and $\{P_f{}^f\}$ are, respectively, the pressures $[Pa]$ in the matrix and fractures (phase-averaged over a unit volume), c_m and c_f $[Pa^{-1}]$ are the compressibility coefficients in the matrix and fractures, K_m and K_f are the equivalent permeabilities of the matrix and of the fracture network at unit scale, ϕ_m and ϕ_f represent the [*dimensionless*] volumetric fractions of the matrix and fractures in the unit.[1] In addition, μ is the dynamic viscosity of the fluid $[Pa.s]$, and finally, α $[Pa^{-1}s^{-1}]$ represents the matrix-fractures exchange coefficient at unit scale.

We will now assume, in the dual continuum model at hand (1), (2), that the matrix is only weakly conductive compared to the fracture network, that is $K_m \ll K_f$, which allows us to neglect pressure gradient 'diffusive transport' terms in the matrix flow equation (1). We assume also that, since the fracture medium permeability is comparatively high, the pressure gradient in the 'fracture continuum' is negligible in each sub-bloc unit. We assume also that fluctuations of $\phi_m,\ c_m,\ \phi_f$ and c_f are negligible.

The simplified dual continuum flow problem over a block is now governed by the differential system:

$$\frac{\partial}{\partial t}\{P_m\}^m = -\frac{\alpha}{\mu\phi_m c_m}\big(\{P_m\}^m - \{P_f\}^f\big) \tag{3}$$

$$\frac{\partial}{\partial t}\{P_f\}^f = -\frac{\alpha}{\mu\phi_f c_f}\big(\{P_f\}^f - \{P_m\}^m\big) \tag{4}$$

Furthermore, we now define a mean pressure denoted $\overline{P}$, and a pressure difference called 'exchange pressure', denoted $\widetilde{P}$. This yields a more compact formulation of the system (3) and (4):

[1] Matrix porosity 'n' is implicitly taken into account, via ϕ_m or else via c_m.

$$\overline{P} = \frac{\{P_m\}^m + \{P_f\}^f}{2}; \qquad \widetilde{P} = \frac{\{P_m\}^m - \{P_f\}^f}{2} \tag{5}$$

$$\frac{\partial \widetilde{P}}{\partial t} = -\frac{\alpha}{\bar{c}} \widetilde{P} \quad \text{and} \quad \frac{\partial \overline{P}}{\partial t} = -\frac{\alpha}{\tilde{c}} \widetilde{P} \tag{6}$$

$$\frac{1}{\bar{c}} = \frac{1}{2\mu}\left(\frac{1}{\phi_m c_m} + \frac{1}{\phi_f c_f}\right) \quad \text{and} \quad \frac{1}{\tilde{c}} = \frac{1}{2\mu}\left(\frac{1}{\phi_m c_m} - \frac{1}{\phi_f c_f}\right) \tag{7}$$

Before considering solution of this problem in a random medium, note that the first equation in (6) governs the 'exchange' pressure alone. It does *not* depend on the 'mean' pressure, which is governed by the second equation. Therefore, the exchange pressure equation can be analyzed and solved independently.

3. Exact probabilistic solution of the dual continuum exchange problem

We develop here the exact probabilistic solution, in continuous time, of the dual continuum matrix-fracture exchange problem as formulated above (5) and (6). The exchange coefficient is taken as a random variable, or equivalently, a *purely random* spatial field. The exchange pressure is therefore also purely random at any given time 't'. We compute exactly the time-dependent probability distribution and moments of exchange pressure, which leads to an exact closure of the upscaling problem(s).[2] The effective macroscopic exchange equation and coefficients are thus calculated either instantaneously or globally over the entire history of the exchange process.

3.1. Probability distribution of exchange pressure

We now focus on the exchange equation (first equation in (6)) governing 'exchange pressure' $\widetilde{P}$. We normalize this equation using the constant deterministic parameters P_0, t_0, and the mean coefficient $\bar{\alpha}$:

$$p(x,\tau) = \frac{\widetilde{P}(x,t)}{\widetilde{P}_0(x,0)}; \quad \alpha_0 = \bar{\alpha}; \quad a = \frac{\alpha}{\alpha_0}; \quad \tau = \frac{t}{t_0}; \quad t_0 = \frac{\bar{c}}{\alpha_0} \tag{8}$$

where $\bar{\alpha}$ is the mean[3] of the mass exchange coefficient α. The random exchange problem is now described by the dimensionless equation:

$$\frac{\partial p}{\partial \tau} = -ap; \quad p(x,0) = p_0(x) \tag{9}$$

where $p(t)$ is the random exchange pressure and p_0 is the deterministic initial condition. Its stochastic solution is :

$$p = g(a,\tau) = p_0\,\mathrm{e}^{-a\tau} \tag{10}$$

Let us denote $f_p(p)$ and $f_A(a)$ the *P*robability *D*ensity *F*unctions (*PDF*'s) of the purely random variables p and a respectively. We note that g is a monotonically decreasing function of a. This helps establish the exact relation between the PDF of dimensionless pressure 'p' and the PDF of exchange coefficient 'a':

$$f_p(p) = \frac{f_A(a)}{|g\prime(a)|} \tag{11}$$

After calculation, we obtain the probability law of $p(\tau)$, that is, respectively, the *PDF* and the cumulative distribution function (*CDF*) of p at time τ:

$$f_p\big(p(\tau)\big) = \frac{1}{p} \times \frac{1}{\tau} \times f_A\big(a(p)\big); \qquad F_p\big(p(\tau)\big) = 1 - F_A\left(-\frac{1}{\tau}\ln\left(\frac{p}{p_0}\right)\right) \tag{12}$$

[2] As will be seen, there are several ways to define the upscaling problem.

[3] The 'mean' is the mathematical expectation, which is also the arithmetic mean taken through the ensemble space of realizations. Furthermore, if the ergodicity hypothesis is satisfied, this is also equivalent to a spatial mean.

3.2. Mean exchange pressure

In order to treat the upscaling problem, we first need to determine the mean (or mathematical expectation) of the time-dependent exchange pressure $p(\tau)$, in terms of the assumed *PDF* of the random exchange coefficient a. Based on earlier results, we obtain:

$$\bar{p}(\tau) = \int_{\Re^+} p f_P(p)\,\mathrm{d}p = \frac{1}{\tau}\int_{\Re^+} f_A\left(-\frac{1}{\tau}\ln\left(\frac{p}{p_0}\right)\right)\mathrm{d}p = p_0 \int_{-\infty}^{+\infty} f_A(b)\,\mathrm{e}^{-b\tau}\,\mathrm{d}b \tag{13}$$

where we transformed the variable of integration using $b = -\frac{1}{\tau}\ln(\frac{p}{p_0})$, which can be interpreted as another normalized version of the random exchange coefficient.

3.2.1. Case of a uniform distribution of the exchange coefficient a

Here the random variable a has a uniform *PDF* over the interval $[a_{\min}, a_{\max}]$ ($a_{\min} \approx 0$):

$$f_A(a) = \frac{1}{a_{\max} - a_{\min}} \tag{14}$$

Let us define for convenience the following quantities (all related to the uniform *PDF*):

$$\bar{a} = \frac{a_{\max} + a_{\min}}{2}; \qquad \Delta a = \frac{a_{\max} - a_{\min}}{2} \tag{15}$$

Note that $\bar{a} = 1$ by previous normalization, and since $a \geqslant 0$. We must have here, by construction $a_{\min} \approx 0$ and $a_{\max} \leqslant 2$. Using the general expression (13) for the mean exchange pressure, we obtain:

$$\bar{p} = p_0 \times \frac{\mathrm{e}^{-a_{\min}\tau} - \mathrm{e}^{-a_{\max}\tau}}{(a_{\max} - a_{\min}) \times \tau} = p_0 \times \frac{\mathrm{e}^{-\bar{a}\tau}\sinh(\Delta a \times \tau)}{\Delta a \times \tau} \tag{16}$$

3.2.2. Case of an exponential distribution of the exchange coefficient

Here we assume that the random variable a has an exponential *PDF* $f_A(a)$ over the semi-infinite interval $]0, +\infty[$. Using again the general expression (13) for the mean exchange pressure, we obtain:

$$f_A(a) = \frac{1}{\bar{a}} \times \mathrm{e}^{-a/\bar{a}}; \qquad \bar{p} = p_0 \times \frac{1}{1 + \bar{a} \times \tau} \tag{17}$$

Note that a random variable a with exponential distribution always satisfies $\sigma_a = \bar{a}$. From (8), we have $\alpha_0 = \bar{\alpha}$, so that $\bar{a} = 1$ by construction. Regrouping all results, we obtain finally the time-evolution of the mean exchange pressure, both graphically in Fig. 2, and analytically in the previous equations (16) and (17). These are compared to the so-called *deterministic or 'naïve mean' exchange pressure*, i.e., $\bar{p} = p_0 \times \mathrm{e}^{-\tau}$. This corresponds to the decay of pressure in a homogeneous medium with constant coefficient $\alpha_0 = \bar{\alpha}$, which is a kind of 'naïve' homogenization of the random problem. Fig. 2 shows examples of averaged pressure evolution for the different PDF, and for the different upscaling laws.

4. Upscaling of matrix-fracture exchange process

4.1. Instantaneous upscaling (*local in time, global in space*)

One possible upscaling approach (amongst others) is to determine the instantaneous, time-dependent, homogenized exchange coefficient $A_1(\tau)$ such that $A_1(\tau)$ verifies exactly a 'global' differential equation that has the same form as the local equation, but with the local random exchange coefficient a or A being replaced by the upscaled time-dependent exchange coefficient $A_1(\tau)$. Thus, we define $A_1(\tau)$ by imposing:

$$\frac{\partial \bar{p}}{\partial \tau} = -A_1(\tau)\bar{p} \tag{18}$$

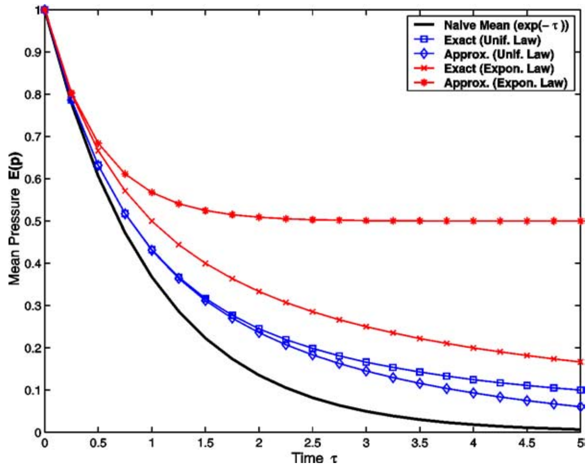

Fig. 2. Temporal relaxation of mean exchange pressure $\bar{p}(t)$ for different types of averages: 'naïve' mean, exact mean for uniformly and exponentially distributed α (see this section), and comparison with mean pressure from approximate perturbation solutions (not detailed here), [10].

This gives immediately:

$$A_1(\tau) = -\frac{1}{\bar{p}(\tau)}\frac{\mathrm{d}\bar{p}}{\mathrm{d}\tau} = -\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln \bar{p}) \quad \Leftrightarrow \quad A_1(\tau) = \frac{E(a \times p)}{E(p)} \tag{19}$$

The analytical relation given by (19) allows us to calculate $A_1(\tau)$ by exploiting earlier probabilistic relations (*PDF* and expectation of $p(\tau)$). This is done in particular for the case where a is uniformly distributed in the $[a_{\min}, a_{\max}]$ interval,[4] and in the case of an exponential distribution.[5] The time-dependent instantaneous upscaled coefficient is finally given analytically below at least for the two types of probability laws considered in this work.[6]

$$\text{Upscaling for uniform distribution } (\bar{a} = 1,\ \Delta a \approx 2)\text{:} \quad A_1(\tau) = \bar{a} - \Delta a\left(\frac{\mathrm{ch}(\Delta a\ \tau)}{\mathrm{sh}(\Delta a\ \tau)} - \frac{1}{\Delta a\ \tau}\right) \tag{20}$$

$$\text{Upscaling for exponential distribution } (\sigma_a = \bar{a} = 1)\text{:} \quad A_1(\tau) = \frac{\bar{a}}{1 + \bar{a}\tau} \tag{21}$$

These results indicate that the 'exact' upscaled coefficient is far from constant. The upscaled coefficient $A_1(\tau)$ is initially equal to the *arithmetic mean* ($t = 0$), but decreases monotonically with time thereafter, to reach a minimum asymptotic value $A_1(\infty)$ as $\tau \to \infty$. However, we can state a more general result, which holds for all the distributions examined so far:

$$\lim_{\tau \to \infty} A_1(\tau) = A_1(\infty) \tag{22}$$

where $A_1(\infty)$ is the harmonic mean of the local coefficient 'a' when $\mu\alpha(\delta x)^2/k_f \gg 1$ and will be equal to the minimum of the local coefficient 'a' for $\mu\alpha(\delta x)^2/k_f \ll 1$. These results are currently being used to assess the physical significance of each type of probability distribution in a more complete model of the exchange process involving also variable porosities and compressibilities.

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[4] In this first case we have by construction $0 \approx a_{\min} \leqslant \bar{a} = 1 \leqslant a_{\max} \leqslant 2$.

[5] In this second case, by construction, $\sigma_a = \bar{a} = 1$, the minimum value of a is the *harmonic mean* of a.

[6] For comparison, the 'naïve' upscaling would be the expected value (*arithmetic mean*) of the random exchange coefficient.

4.2. Non-instantaneous upscaling (global in space and time)

We now modify the previous upscaling criterion to obtain a single global coefficient $A^\star$. We want the constant $A^\star$ to be representative, not only of the spatial scale of the block, but also of the entire history of the matrix-fracture exchanges from initial time up to infinite time. To achieve this, we use an optimization approach. First, let us define a class of objective functions $(J_\omega,\ I_\omega)$ parametrized by the exponent $\omega \geqslant 1$:[7]

$$J_\omega = \left(\int_0^{+\infty} \left|\frac{p^\star(\tau)}{p_0} - \frac{\bar{p}(\tau)}{p_0}\right|^\omega \mathrm{d}\tau\right)^{1/\omega}; \qquad I_\omega = \left(\int_0^{+\infty} \left|\frac{A^\star p^\star(\tau)}{p_0} - \frac{\overline{Ap}(\tau)}{p_0}\right|^\omega \mathrm{d}\tau\right)^{1/\omega} \tag{23}$$

$$p^\star(\tau) = p_0 \times \mathrm{e}^{-A^\star\tau} \tag{24}$$

$$\bar{p}(\tau) = E\big(p(\tau)\big) = E\big(p_0\,\mathrm{e}^{-a\tau}\big) \tag{25}$$

We introduce the notation 'y' for normalized pressure p/p_0 and we study two case for ω $(\omega = 1, 2)$.

First case, $\omega = 2$. We now determine the global coefficient $A^\star$ by minimizing J_2 (or its square):

$$\mathrm{Min}(J_2)^2 \quad \Leftrightarrow \quad \mathrm{Min} \int_0^{+\infty} \big(\mathrm{e}^{-A^\star\tau} - \bar{y}(\tau)\big)^2\,\mathrm{d}\tau \tag{26}$$

After some manipulations, and taking into account that $A^\star$ is independent of time, we find the general solution of this problem quasi-analytically. Namely, $A^\star$ is the solution of the equation:

$$\frac{1}{2A^\star} = \int_0^{+\infty} \mathrm{e}^{-A^\star\tau}\,\bar{y}(\tau)\,\mathrm{d}\tau = L\big\{\bar{y}(\tau), A^\star\big\} \tag{27}$$

with the notation $L\{y(t), s\}$ representing the *Laplace Transform* of function $y(t)$, where s is the (generally complex) parameter of the Laplace Transform. Note that $\bar{y}(\tau)$ stands for $E\{\exp(-a\tau)\}$. For exponentially distributed 'a', this yields an equation involving the exponential integral $Ei(x)$:

$$1 - 2A^\star \exp A^\star Ei\big(A^\star\big) = 0 \quad \Rightarrow \quad A^\star = 0.610 \tag{28}$$

In order to see the practical implications of these upscaling results (Table 1), we consider briefly the case of a multidimensional domain (or reservoir block) made up of N units (or sub-blocks). The discrete space equivalent of the previous results is then obtained by replacing mathematical expectations by discrete sums over the sub-blocks for large N.

Second case, $\omega = 1$. We propose now to minimize the objective function J_1 (Eq. (23)), i.e., with the L_1 *norm* as minimisation criterion. This yields:

$$\mathrm{Min}\,J_1 = \int_0^{+\infty} \big|y^\star(\tau) - \bar{y}(\tau)\big|\,\mathrm{d}\tau = \int_0^{+\infty} \big|\exp(-A^\star\tau) - E\big\{\exp(-a\tau)\big\}\big|\,\mathrm{d}\tau \tag{29}$$

In the case of the exponential distribution, using previous results and the first-order optimality condition, we obtain that $A^\star$ must satisfy exactly the integral equation:

$$0 = \int_0^{+\infty} \tau \times \exp^{-A^\star\times\tau} \times \mathrm{sign}\big\{\mathrm{e}^{-A^\star\tau} - E\big\{\mathrm{e}^{-a\tau}\big\}\big\}\,\mathrm{d}\tau \quad \text{with } E\big\{\mathrm{e}^{-a\tau}\big\} = \frac{1}{1+\bar{a}\tau} \tag{30}$$

In this integral, the two curves decrease monotonically with time. We assume that they cross at a unique time $\tau^\star$,

[7] The functional J_ω (I_ω) has the form of an L_ω *norm* in the space of functions $p(\tau)$ $(Ap(\tau))$.

Table 1
The global exchange coefficient A^* upscaled over the entire time history

A^*, PDF	Uniform $]0,1]$	Exponential ($\sigma_a = \bar{a} = 1$)	Gaussian ($\bar{a} = 0.5, \sigma_a = 0.167$)
$A^*_{\text{Arithmetic}}$	0.503165	0.425914	0.501000
A^*_{Harmonic}	0.043877	0.074843	0.450123
$A^*_{\text{Min } J_2}$	0.330297	0.247759	0.453652
$A^*_{\text{Min } I_2}$	0.468319	0.385606	0.488491

which depends on the unknown $A^\star$. This is a consistant assumption for our case which can be verified ex post. This allows us to solve the problem and obtain the following analytical results:

$$A^\star \times \tau^\star \approx 1.678 \quad \text{and} \quad \mathrm{e}^{-A^\star \times \tau^\star} = \frac{1}{1 + \bar{a} \times \tau^\star} \quad \Rightarrow \quad A^\star \approx 0.385 \tag{31}$$

It is interesting to note that the upscaled coefficient $A^\star$ of equations (28), (31) satisfy the inequalities:

$$0 \approx E\{a^{-1}\}^{-1} \leqslant A^\star \leqslant E\{a\} \tag{32}$$

That is, the global exchange coefficient (upscaled over time history) is between the harmonic and arithmetic means of the local coefficient. In comparison, recall that the instantaneous upscaled coefficient A_1 goes from arithmetic mean at early times, to harmonic mean or minimum value of a at large times.

5. Conclusion and outlook

In conclusion, we have analyzed a novel upscaling problem for matrix-fracture exchange flow in a saturated random medium governed by dual continuum equations. Neglecting fracture pressure gradients, we developed exact random pressure solutions for different probability distributions of the exchange coefficient. We used these results to upscale the exchange coefficient analytically, based on different upscaling criteria (instantaneous, time history, etc.). This work will be continued in other papers, including comparisons with perturbation solutions (see theory in [10] and some results in Fig. 2) and with numerical simulations.

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