Available online at www.sciencedirect.com
SCIENCE DIRECT®
C. R. Mecanique 332 (2004) 949–954
COMPTES RENDUS MECANIQUE
http://france.elsevier.com/direct/CRAS2B/

On the vibration of a partially fastened membrane with many 'light' concentrated masses on the boundary

Gregory A. Chechkin

Department of Differential Equations, Faculty of Mechanics and Mathematics, Moscow State University, Moscow 119992, Russia

Received 26 May 2003; accepted after revision 8 September 2004

Available online 5 November 2004

Presented by Évariste Sanchez-Palencia

Abstract

We consider a partially fastened membrane with many concentrated masses near the boundary. Masses have the diameter $(a\varepsilon)$; the density is $O(1)$ outside the masses and $O((a\varepsilon)^{-m})$, $0 < m < 2$, in the masses. We assume that the distance between masses is $O(\varepsilon)$ and a is fixed. We obtain the leading terms of the asymptotic expansion of eigenvalues and eigenfunctions of the respective spectral problems for the Laplacian in such a domain. ***To cite this article: G.A. Chechkin, C. R. Mecanique 332 (2004).***
© 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

Vibration d'une membrane partiellement attachée avec plusieurs masses « légères » concentrées sur la frontiere. Nous considérons une membrane partiellement attachée avec plusieurs masses concentrées prés de la frontière. Le diamètre des masses est ègal à $(a\varepsilon)$; la densité est d'ordre $O(1)$ en dehors des masses et la densité des masses d'ordre $O((a\varepsilon)^{-m})$, $0 < m < 2$. Nous supposons que la distance entre les masses est d'ordre $O(\varepsilon)$ et que a est fixé. Nous obtenons les termes principaux du développement asymptotique des valeurs propres et des fonctions propres du Laplacian dans un domaine de ce type. ***Pour citer cet article : G.A. Chechkin, C. R. Mecanique 332 (2004).***
© 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Keywords: Vibrations; Homogenization; Small parameter; Eigenvalues; Asymptotics

Mots-clés : Vibrations ; Homogénéisation ; Petit paramètre ; Valeur propre ; Asymptotique

1. Introduction

The study of the behavior of bodies with nonhomogeneous density (with concentrated masses) has attracted the attention of mathematicians since the beginning of XXth century (see, for instance, [1]). In [2] the author studied

E-mail address: chechkin@mech.math.msu.su (G.A. Chechkin).

1631-0721/$ – see front matter © 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crme.2004.08.002

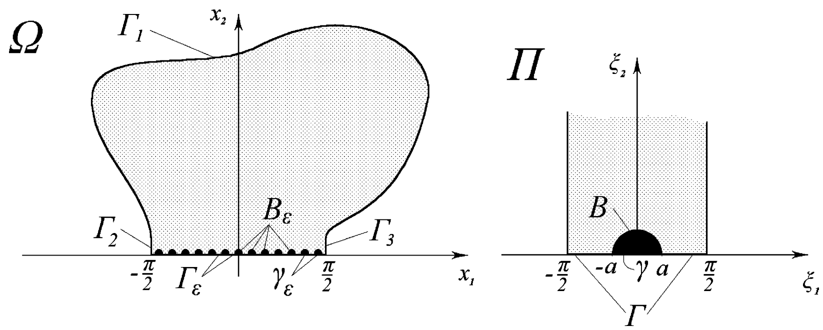

Fig. 1. The model.

the problem with concentrated masses on the basis of the spectral perturbation theory and proved the convergence theorem. For the first time global and local vibrations were introduced and investigated. It should be noted also papers such as [3–10], where different problems in domains with concentrated masses were studied.

In this note we consider a two-dimensional domain with many concentrated masses on the boundary situated in a periodic way. The distance between masses and the diameter of the mass have the same order. We assume the masses to be 'light'. Using the method of matching asymptotic expansions [11] (see also [12,13]), we construct the leading terms of the asymptotics of eigenelements to a problem for the Laplacian in such a domain.

Denote by Ω a domain in $\mathbb{R}^2$, which lies in the upper semi-plane, with a piecewise smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where Γ_4 is a segment $(-\pi/2, \pi/2)$ of the abscissa axis, Γ_2 and Γ_3 belongs to the straight lines $x_1 = -\pi/2$ and $x_1 = \pi/2$, respectively (see Fig. 1).

Let us describe in detail the fine-grained structure of Γ_4. Denote $\gamma = \{\xi\colon\ -a < \xi_1 < a,\ \xi_2 = 0\}$, $\Gamma = \{\xi\colon\ -\pi/2 < \xi_1 < -a,\ a < \xi_1 < \pi/2,\ \xi_2 = 0\}$, $a < \pi/2$, for natural $N \gg 1$ we define $\varepsilon = 1/(2N+1)$. Let $\gamma_\varepsilon = \{x \in \Gamma_4\colon\ \varepsilon^{-1}(x_1 - j, 0) \in \gamma,\ j \in \mathbb{Z}\}$ and $\Gamma_\varepsilon = \Gamma_4 \backslash \gamma_\varepsilon$. Also we use the following notation $\Pi = \{\xi\colon\ -\pi/2 < \xi_1 < \pi/2,\ \xi_2 > 0\}$, $B = \{\xi\colon\ \xi_1^2 + \xi_2^2 < a^2,\ \xi_2 > 0\}$ and $B_\varepsilon = \{x \in \Omega\colon\ \varepsilon^{-1}(x_1 - j, x_2) \in B,\ j \in \mathbb{Z}\}$.

We construct an asymptotics as $\varepsilon \to 0$ of eigenelements to the following spectral problem:

$$\begin{cases} -\Delta u_\varepsilon = \lambda_\varepsilon \rho_\varepsilon u_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \gamma_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \Gamma_\varepsilon \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases} \qquad \rho_\varepsilon = \begin{cases} 1 & \text{in } \Omega \backslash \overline{B}_\varepsilon, \\ (a\varepsilon)^{-m} & \text{in } B_\varepsilon \end{cases} \tag{1}$$

where ν is the unit outward normal to $\partial\Omega$. We assume the constants to be $0 < m < 2$ and $0 < a < \frac{\pi}{2}$.

2. Construction of leading terms of asymptotics

Assume that λ_0 is a simple eigenvalue of the problem

$$\begin{cases} -\Delta u_0 = \lambda_0 u_0 & \text{in } \Omega, \\ u_0 = 0 \quad \text{on } \Gamma_4, & \frac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \end{cases} \tag{2}$$

Following [14,15] one can prove that (2) is the limit problem for (1), i.e. for each simple eigenvalue λ_0 of problem (2), and sufficiently small ε, there exists only one and simple eigenvalue λ_ε of problem (1), that converges to λ_0 and the corresponding normalized eigenfunction u_ε converges weakly in $H^1(\Omega)$ to normalized eigenfunction u_0 as $\varepsilon \to 0$.

Consider the case $1 < m < 2$. The function $u_0(x)$ does not satisfy either the equation, or the boundary condition of problem (1) in a neighborhood of Γ_4. We construct the *external* asymptotic expansion in Ω (outside small neighborhood of Γ_4), the expansion of eigenvalues in the form

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^{3-m} u_{3-m}(x) + \cdots, \qquad \lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^{3-m}\lambda_{3-m} + \cdots \tag{3}$$

where we assume u_j to be smooth, and the *internal* asymptotic expansion in a neighborhood of Γ_4 as follows:

$$u_\varepsilon(x) = \varepsilon v_1(\xi; x_1) + \varepsilon^{3-m} v_{3-m}(\xi; x_1) + \cdots, \quad \xi = x/\varepsilon \tag{4}$$

Remark 1. Moreover, we construct the coefficients of the internal expansion (4) in the form of π-periodic in ξ_1 functions. It is easy to see that in this case due to the geometry of Γ_2 and Γ_3 the conditions $\frac{\partial v_i}{\partial \xi_1} = 0$ as $\xi_1 = \pm\frac{\pi}{2}$ and the periodicity of v_i lead to the condition $\frac{\partial v_i}{\partial \xi_1} = 0$ on Γ_2 and Γ_3.

By virtue of problem (2) and the geometry of Ω the asymptotics of $u_0 \in C^\infty(\overline{\Omega})$ as $x_2 \to 0$ can be expressed as follows:

$$u_0(x) = \alpha_0(x_1)x_2 + O\big(x_2^3\big), \quad \alpha_0(x_1) = \frac{\partial u_0}{\partial x_2}\Big|_{x_2=0} \quad \text{and} \quad \alpha_0^{(2j+1)}\big(\pm\tfrac{\pi}{2}\big) = 0 \tag{5}$$

Let us rewrite the asymptotics (5) in the variables (x_1, ξ_2), $\xi_2 = \frac{x_2}{\varepsilon}$:

$$u_0(x_1, \varepsilon\xi_2) = \varepsilon\alpha_0(x_1)\xi_2 + O\big(\varepsilon^3\xi_2^3\big) \tag{6}$$

In fact, bearing in mind (6), we have $v_1(\xi; x_1) \sim \alpha_0(x_1)\xi_2$ as $\xi_2 \to +\infty$, $\xi = \frac{x}{\varepsilon}$. Substituting (4) and (3) in (1) and keeping in mind Remark 1, we obtain the boundary-value problem for v_1:

$$\begin{cases} \Delta_\xi v_1 = 0 \quad \text{in } \Pi, \qquad \frac{\partial v_1}{\partial \xi_1} = 0 \quad \text{as } \xi_1 = \pm\frac{\pi}{2}, \\ v_1(\xi_1, 0; x_1) = 0 \quad \text{as } \xi_1 \in (-a, a), \qquad \frac{\partial v_1}{\partial \xi_2}(\xi_1, 0; x_1) = 0 \quad \text{as } \xi_1 \in \big(-\frac{\pi}{2}, -a\big) \cup \big(a, \frac{\pi}{2}\big), \\ v_1 \sim \alpha_0(x_1)\xi_2 \quad \text{as } \xi_2 \to +\infty \end{cases}$$

The π-periodic solution of the problem does exist and can be calculated directly (see [12]):

$$v_1(\xi; x_1) = \alpha_0(x_1)\Big(\mathbf{Re}\ln\big(\sin z + \sqrt{\sin^2 z - \sin^2 a}\,\big) - \ln\sin a\Big) \tag{7}$$

where $z = \xi_1 + \mathrm{i}\xi_2$ is complex.

Remark 2. Note that due to the last relation in (5) we have $\frac{\partial v_1}{\partial x_1}(\xi; x_1) = 0$ as $x_1 = \pm\frac{\pi}{2}$. Hence, the boundary condition $\frac{\partial v_1}{\partial \xi_1}(\xi; x_1) = 0$ as $\xi_1 = \pm\frac{\pi}{2}$ and the periodicity of v_1 leads to $\frac{\partial v_1}{\partial \nu}(\frac{x}{\varepsilon}; x_1) = 0$ on $\Gamma_2 \cup \Gamma_3$.

The asymptotics of the function (7) as $\xi_2 \to +\infty$ have the form:

$$v_1(\xi; x_1) = \alpha_0(x_1)\big((\xi_2 - \ln\sin a) + O\big(e^{-2\xi_2}\big)\big) \tag{8}$$

Rewriting the asymptotics (4), (8) in x, we obtain the asymptotics of u_1 in the following form $u_1(x) \sim -\alpha_0(x_1)\ln\sin a$ as $x_2 \to 0$. It means that $u_1(x) = -\alpha_0(x_1)\ln\sin a$ as $x \in \Gamma_4$ because of the smoothness of u_1. Substituting the series (3) in problem (1) and keeping in mind the last remark, we obtain the boundary-value problem for u_1:

$$\begin{cases} -\Delta u_1 = \lambda_0 u_1 + \lambda_1 u_0 \quad \text{in } \Omega, \\ u_1 = -\alpha_0 \ln\sin a \quad \text{on } \Gamma_4, \qquad \frac{\partial u_1}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \end{cases} \tag{9}$$

Writing down the solvability condition and applying the Green's formula we obtain

$$\lambda_1 = \ln\sin a \int_{\Gamma_4} \left(\frac{\partial u_0}{\partial \nu}\right)^2 \mathrm{d}x_1 \quad \text{or} \quad \lambda_1 = \ln\sin a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \alpha_0^2(x_1)\,\mathrm{d}x_1 \tag{10}$$

To determine uniquely the solution we assume that $\int_\Omega u_0(x)u_1(x)\,\mathrm{d}x = 0$. Thus, we constructed the leading terms in the external expansion and moreover the asymptotics of $u_1 \in C^\infty(\overline{\Omega})$ as $x_2 \to 0$ reads

$$u_1(x) = -\alpha_0(x_1)\ln\sin a + \alpha_1(x_1)x_2 + O\big(x_2^2\big), \quad \alpha_1(x_1) = \frac{\partial u_1}{\partial x_2}\Big|_{x_2=0} \quad \text{and} \quad \alpha_1^{(2j+1)}\big(\pm\tfrac{\pi}{2}\big) = 0 \tag{11}$$

Substituting (4) and (3) in the equation from (1) and keeping in mind Remark 1, we obtain the problem for v_{3-m}:

$$\begin{cases} -\Delta_\xi v_{3-m} = \begin{cases} 0 & \text{in } \Pi\setminus B, \\ a^{-m}\lambda_0 v_1 & \text{in } B, \end{cases} & \frac{\partial v_{3-m}}{\partial \xi_1} = 0 \quad \text{as } \xi_1 = \pm\frac{\pi}{2}, \\ v_{3-m}(\xi_1, 0; x_1) = 0 \quad \text{as } \xi_1 \in (-a, a), & \frac{\partial v_{3-m}}{\partial \xi_2}(\xi_1, 0; x_1) = 0 \quad \text{as } \xi_1 \in \big(-\frac{\pi}{2}, -a\big) \cup \big(a, \frac{\pi}{2}\big) \end{cases} \tag{12}$$

In [13] it is shown that this problem has the π-periodic in ξ_1 solution $v_{3-m}(\xi; x_1) = C_{3-m}(x_1)V_{3-m}(\xi)$ with the asymptotics

$$v_{3-m}(\xi; x_1) = C_{3-m}(x_1)\big(1 + O\big(e^{-2\xi_2}\big)\big) \quad \text{as } \xi_2 \to \infty \tag{13}$$

where $C_{3-m}(x_1)$ can be calculated directly. Multiplying the equation in (12) by the function $\mathbf{Re}\ln(\sin z + \sqrt{\sin^2 z - \sin^2 a}) - \ln\sin a$, integrating over $\Pi_R = \{\xi \in \Pi, \xi_2 < R\}$, using the Green's formula and passing to the limit as $R \to \infty$, we obtain

$$C_{3-m}(x_1) = \frac{\lambda_0\alpha_0(x_1)}{\pi a^m}\int_B \big(\mathbf{Re}\ln\big(\sin z + \sqrt{\sin^2 z - \sin^2 a}\big) - \ln\sin a\big)^2\,\mathrm{d}\xi \tag{14}$$

It should be noted that due to (14), the last relation in (5) and the boundary conditions in (12) as $\xi_1 = \pm\frac{\pi}{2}$, the function $v_{3-m}(\frac{x}{\varepsilon}; x_1)$ satisfies the Neumann boundary conditions on $\Gamma_2 \cup \Gamma_3$.

The obtained discrepancy in (13) is compensated by the term $\varepsilon^{3-m}u_{3-m}$ in the external expansion. Hence, the asymptotics of u_{3-m} has the form $u_{3-m}(x) \sim C_{3-m}(x_1)$ as $x_2 \to 0$. Because of the smoothness of u_{3-m} it means that $u_{3-m}(x) = C_{3-m}(x_1)$ as $x \in \Gamma_4$. Consequently, substituting (3) in (1), we obtain the problem for u_{3-m}:

$$\begin{cases} -\Delta u_{3-m} = \lambda_0 u_{3-m} + \lambda_{3-m}u_0 \quad \text{in } \Omega, \\ u_{3-m} = C_{3-m} \quad \text{on } \Gamma_4, \qquad \frac{\partial u_{3-m}}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \end{cases} \tag{15}$$

Writing down the solvability condition for this problem, using the Green's formula and bearing in mind (14), we deduce

$$\lambda_{3-m} = -\frac{\lambda_0}{\pi a^m}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \alpha_0^2(x_1)\,\mathrm{d}x_1 \int_B \big(\mathbf{Re}\ln\big(\sin z + \sqrt{\sin^2 z - \sin^2 a}\big) - \ln\sin a\big)^2\,\mathrm{d}\xi \tag{16}$$

Thus, in the case $1 < m < 2$ the leading terms of the asymptotics of eigenelements have the form (3), (4), where the coefficients are determined by (7), (10) and (16) and by the solutions to problems (9), (12) and (15).

Consider the case $0 < m \leqslant 1$. The external expansion and the series of eigenvalue has the form

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \cdots, \qquad \lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots \tag{17}$$

and the internal series can be expressed as follows:

$$u_\varepsilon(x) = \varepsilon v_1(\xi; x_1) + \varepsilon^2 v_2(\xi; x_1) + \cdots \tag{18}$$

where λ_1, u_1 and v_1 has the same form as in the case $m > 1$. Keeping in mind (11) and (6) we conclude that the asymptotics of v_2 is $v_2(\xi; x_1) \sim \alpha_1(x_1)\xi_2$ as $\xi_2 \to +\infty$. Substituting (18) and (3) in (1), we deduce the problem for v_2:

$$\begin{cases} -\Delta_\xi v_2 = 2\frac{\partial^2 v_1}{\partial x_1 \partial \xi_1} + \delta_m^1 \begin{cases} 0 & \text{in } \Pi\setminus B, \\ a^{-1}\lambda_0 v_1 & \text{in } B, \end{cases} & \frac{\partial v_2}{\partial \xi_1}\big(\xi; \pm\frac{\pi}{2}\big) = 0 \quad \text{as } \xi_1 = \pm\frac{\pi}{2}, \\ v_2(\xi_1, 0; x_1) = 0 \quad \text{as } \xi_1 \in (-a, a), & \frac{\partial v_2}{\partial \xi_2}(\xi_1, 0; x_1) = 0 \quad \text{as } \xi_1 \in \big(-\frac{\pi}{2}, -a\big) \cup \big(a, \frac{\pi}{2}\big), \\ v_2 \sim \alpha_1(x_1)\xi_2 \quad \text{as } \xi_2 \to \infty \end{cases} \tag{19}$$

Here δ^i_j is the Kroneker symbol. Using the technique in [13], from the constructions we get that there exists π-periodic in ξ_1 solution to problem (19), which has the structure

$$v_2(\xi;x_1)=\alpha_1(x_1)\bigl(\mathbf{Re}\ln\bigl(\sin z+\sqrt{\sin^2 z-\sin^2 a}\bigr)-\ln\sin a\bigr)+\delta^1_m v_{3-m}(\xi;x_1)+\alpha_0'(x_1)\widetilde{X}(\xi) \tag{20}$$

where v_{3-m} is a solution of (12) and $\widetilde{X}(\xi)$ is odd in ξ_1, $\widetilde{X}(\pm\frac{\pi}{2},\xi_2)=0$ and exponentially decays as $\xi_2\to\infty$. Hence, the asymptotics of v_2 reads $v_2(\xi;x_1)\sim\alpha_1(x_1)(\xi_2-\ln\sin a)+\delta^1_m C_{3-m}(x_1)$ as $\xi_2\to+\infty$. Note that the derivative $\frac{\partial}{\partial\xi_1}$ of the first and the second term in (20) is equal to zero for $\xi_1=\pm\frac{\pi}{2}$ and any x_1; finally, due to $\widetilde{X}(\pm\frac{\pi}{2},\xi_2)=0$, the last relations in (5) and (11), respectively, we have $\frac{\partial v_2}{\partial\nu}(\frac{x}{\varepsilon};x_1)=0$ on $\Gamma_2\cup\Gamma_3$.

The boundary condition for u_2 on Γ_4 follows from the asymptotics as $x_2\to 0$ of the form $u_2(x)\sim -\alpha_1(x_1)\ln\sin a+\delta^1_m C_{3-m}(x_1)$. Substituting (17) in (1), we obtain the problem for u_2:

$$\begin{cases}-\Delta u_2=\lambda_0u_2+\lambda_1u_1+\lambda_2u_0 & \text{in }\Omega,\\ u_2=-\alpha_1\ln\sin a+\delta^1_m C_{3-m}\quad \text{on }\Gamma_4, & \frac{\partial u_2}{\partial\nu}=0\quad\text{on }\Gamma_1\cup\Gamma_2\cup\Gamma_3\end{cases} \tag{21}$$

The solvability condition gives

$$\lambda_2=\lambda_2^{(1)}+\delta^1_m\lambda_{3-m},\quad \lambda_2^{(1)}=\ln\sin a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\alpha_1(x_1)\alpha_0(x_1)\,\mathrm{d}x_1=\ln\sin a\int_{\Gamma_4}\frac{\partial u_0}{\partial\nu}\frac{\partial u_1}{\partial\nu}\,\mathrm{d}x_1 \tag{22}$$

and λ_{3-m} is defined in (16).

Thus, in the case $0<m\leqslant 1$ the leading terms of the asymptotics of eigenelements have the form (17), (18), where the coefficients are determined by (7), (10), (20) and (22) and by the solutions to problems (9), (3) and (21).

3. Remarks on the complete asymptotic expansion

We construct the external asymptotic expansion of u_ε, the series for the eigenvalues λ_ε and the respective internal asymptotic expansion of u_ε in the following form:

$$\begin{aligned}&u_\varepsilon=u_0+\varepsilon\sum_{i,j=0}^{\infty}\varepsilon^{i+(2-m)j}u_{i,j}(x),\qquad \lambda_\varepsilon=\lambda_0+\varepsilon\sum_{i,j=0}^{\infty}\varepsilon^{i+(2-m)j}\lambda_{i,j},\\ &u_\varepsilon=\varepsilon\sum_{i,j=0}^{\infty}\varepsilon^{i+(2-m)j}v_{i,j}\bigl(\tfrac{x}{\varepsilon};x_1\bigr)\end{aligned} \tag{23}$$

where the terms of the external expansion $u_{i,j}\in C^\infty(\overline{\Omega})$ and the series of eigenvalues satisfy

$$\begin{cases}-\Delta u_{k,l}=\displaystyle\sum_{p+i+(2-m)(q+j)=k+(2-m)l-2}\lambda_{p,q}u_{i,j}, & \text{in }\Omega,\\ u_{k,l}=C_{k,l}\quad\text{on }\Gamma_4, & \frac{\partial u_{k,l}}{\partial\nu}=0\quad\text{on }\Gamma_1\cup\Gamma_2\cup\Gamma_3\end{cases} \tag{24}$$

$C^{(2j+1)}_{k,l}(\pm\frac{\pi}{2})=0$, $v_{k,l}(\xi;x_1)=\beta^{\mathrm{odd}}_{k,l}(x_1)V^{\mathrm{odd}}_{k,l}(\xi)+\beta^{\mathrm{even}}_{k,l}(x_1)V^{\mathrm{even}}_{k,l}(\xi)$, where $\beta^{\mathrm{odd}}_{k,l}(\pm\frac{\pi}{2})=0$, $(\beta^{\mathrm{even}}_{k,l})'(\pm\frac{\pi}{2})=0$, $V^{\mathrm{odd}}_{k,l}(\pm\frac{\pi}{2})=0$ and $\frac{\partial V^{\mathrm{even}}_{k,l}}{\partial\xi_1}(\pm\frac{\pi}{2})=0$. Hence, the terms of the internal series satisfy

$$\begin{cases}-\Delta_\xi v_{k,l}=2\displaystyle\sum_{\substack{i+(2-m)j+1\\=k+(2-m)l}}\frac{\partial^2 v_{i,j}}{\partial x_1\partial\xi_1}+\sum_{\substack{i+(2-m)j+2\\=k+(2-m)l}}\frac{\partial^2 v_{i,j}}{\partial x_1^2}+\begin{cases}\displaystyle\sum_{\substack{p+i+(2-m)(q+j)\\=k+(2-m)l-2}}\lambda_{p,q}v_{i,j} & \text{in }\Pi\backslash B,\\ \displaystyle\sum_{\substack{p+i+(2-m)(q+j)+2\\=k+(2-m)l+m}}a^{-m}\lambda_{p,q}v_{i,j} & \text{in }B,\end{cases}\\ v_{k,l}=0\quad\text{on }\gamma,\qquad \frac{\partial v_{k,l}}{\partial\xi_2}=0\quad\text{on }\Gamma,\qquad \frac{\partial v_{k,l}}{\partial\xi_1}\bigl(\xi;\pm\frac{\pi}{2}\bigr)=0\quad\text{as }\xi_1=\pm\frac{\pi}{2}\end{cases} \tag{25}$$

We assume the summation on the indexes satisfying the respective relations. In the new notation we have $\lambda_1 = \lambda_{0,0}$; $\lambda_{3-m} = \lambda_{0,1}$; $\lambda_2 = \lambda_{1,0}$; $u_1 \equiv u_{0,0}$; $u_{3-m} \equiv u_{0,1}$; $u_2 \equiv u_{1,0}$; $v_1 \equiv v_{0,0}$; $v_{3-m} \equiv v_{0,1}$; $v_2 \equiv v_{1,0}$; $-\alpha_0 \ln \sin a \equiv C_{0,0}$; $C_{3-m} \equiv C_{0,1}$; $-\alpha_1 \ln \sin a + \delta_m^1 C_{3-m} \equiv C_{1,0}$. The functions $C_{k,l}(x_1)$ are chosen to match the asymptotic expansions and $\lambda_{k,l}$ is defined by the solvability condition for (24).

Acknowledgements

This work was supported in part by RFBR grant 02-01-00693. The paper was written during the stay of the author in Narvik University College (Norway) in February 2003. I also want to express thanks to R.R. Gadyl'shin for useful discussions and remarks.

References

[1] A.N. Krylov, On some differential equations of mathematical physics, having applications in technical questions, T. Nikolay Maritime Academy 2 (1913) 325–348.
[2] É. Sanchez-Palencia, Perturbation of eigenvalues in thermoelasticity and vibration of system with concentrated masses, in: Trends Appl. Pure Math. Mech., in: Lecture Notes in Phys., vol. 195, Springer, Berlin, 1984, pp. 346–368.
[3] O.A. Oleinik, Homogenization problems in elasticity. Spectrum of singularly perturbed operators, in: Nonclassical Continuum Mechanics, in: Lecture Notes Ser., vol. 122, Cambridge University Press, 1987, pp. 188–205.
[4] Yu.D. Golovatyĭ, Natural frequencies of a fastened plate with additional mass, Uspekhi Mat. Nauk 263 (5) (1988) 185–186. English translation in: Russian Math. Surveys 43 (5) (1988) 227–228.
[5] S.A. Nazarov, Concentrated masses problems for a spatial elastic body, C. R. Acad. Sci. Paris, Ser. I 316 (6) (1993) 627–632.
[6] É. Sanchez-Palencia, H. Tchatat, Vibration de systèmes elastiques avec masses concentrées, Rend. Sem. Mat. Univ. Politec. Torino 42 (3) (1984) 43–63.
[7] M. Lobo, M.E. Pérez, Asymptotic behavior of the vibrations of a body having many concentrated masses near the boundary, C. R. Acad. Sci. Paris, Ser. II 314 (1992) 13–18.
[8] O.A. Oleinik, J. Sanchez-Hubert, G.A. Yosifian, On vibration of membrane with concentrated masses, Bull. Sci. Math. 115 (1) (1991) 1–27.
[9] G.A. Chechkin, M.E. Pérez, E.I. Yablokova, On eigenvibrations of a body with many "light" concentrated masses located nonperiodically along the boundary, Preprint of Universidad de Cantabria, Num. 1/2002, Santander, Abril 2002, 31 p.; Indiana Univ. Math. J., in press.
[10] G.A. Chechkin, Splitting a multiple eigenvalue in the problem on concentrated masses, Russian Math. Surveys (2004). Translated from: Uspekhi Mat. Nauk 59 (4) (2004) 205–206.
[11] A.M. Il'in, Matching of Asymptotic Expansions of Solutions of Boundary Value Problems, Transl. Math. Monogr., vol. 102, American Mathematical Society, Providence, RI, 1992.
[12] R.R. Gadyl'shin, Asymptotics of the minimum eigenvalue for a circle with fast oscillating boundary conditions, C. R. Acad. Sci. Paris, Ser. I. 323 (3) (1996) 319–323.
[13] R.R. Gadyl'shin, On the eigenvalue asymptotics for periodically clamped membranes, Algebra i Anal. 10 (1) (1998) 3–19. English translation in: St. Petersburg Math. J. 10 (1) (1999) 1–14.
[14] G.A. Chechkin, Spectral properties of an elliptic problem with rapidly oscillating boundary conditions, in: Boundary Value Problems for Nonclassical Equations in Mathematical Physics, Novosibirsk, 1989, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1989, pp. 197–200 (in Russian).
[15] R.R. Gadyl'shin, Ramification of a multiple eigenvalue of the Dirichlet problem for the Laplacian under singular perturbation of the boundary condition, Mat. Zametki 52 (4) (1992) 42–55. English translation in: Math. Notes 52 (4) (1992) 1020–1029.