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A simple multiaxial fatigue criterion for metals

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Abstract

A new simple multiaxial high-cycle fatigue endurance criterion, suitable for situations where the convex hull associated with the stress path approximates well an ellipsoid, is proposed. It considers, as measures of fatigue solicitation: (i) a new definition for the equivalent shear stress amplitude; and (ii) the maximum principal stress along the stress history. Assessment of the resulting criterion for a wide range of in-phase and out-phase cyclic loads shows that it compares very well with experimental data published in the literature. ***To cite this article: C.A. Gonçalves et al., C. R. Mecanique 332 (2004).***

Résumé

Un critère simple d'éndurance à la fatigue multiaxiale des métaux. On propose un critère simple d'endurance à la fatigue polycyclique, applicable à des situations pour lesquelles l'envelope convexe associé à l'histoire des contraintes s'approche bien d'une ellipse. Le critère considère, comme mesures de solicitation à la fatigue : (i) une nouvelle définition de l'amplitude de contrainte de cisaillement ; et (ii) la contrainte principale maximale au cours de l'histoire de chargement. ***Pour citer cet article : C.A. Gonçalves et al., C. R. Mecanique 332 (2004).***

Keywords: Fatigue; Fatigue strength; Multiaxial high-cycle fatigue; Nonproportional loading

Mots-clés : Fatigue ; Éndurance à la fatigue ; Fatigue multiaxiale polycyclique ; Chargement non-proportionel

1. Introduction

This note presents a new, simple multiaxial high-cycle fatigue endurance criterion, suitable for situations where the convex hull associated with the stress path approximates well an ellipsoid. The model considers the shear stresses as one of the driving forces of the fatigue process – since plasticity plays an important role in crack

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initiation – and the normal stress acting upon embryo-cracks – which has been shown by Sines [1] to affect the fatigue resistance. Models based on such assumptions are usually written as:

$$f(\mathbf{S}) + g(p) \leqslant \lambda \tag{1}$$

where $f(\mathbf{S})$ is a measure of the shear stress amplitude, $\mathbf{S}$ is the history of the deviatoric stress tensor, $g(p)$ is a function of the hydrostatic stress p observed along the stress history and λ is a material parameter. For instance, in the criterion proposed by Crossland [3], $f(\mathbf{S})$ is the $\sqrt{J_2}$ radius of the sphere circumscribing the deviatoric stress path. Notice that proportional and nonproportional paths can be circumscribed by the same sphere although a more severe solicitation is expected when the nonproportional stress history is considered. As an alternative, one could consider a quantity associated with the minimum ellipsoid circumscribing the stress path, as previously suggested by Deperrois [4], and later by Bin Li et al. [5]: the basic idea is to consider shear stress amplitudes in several orthogonal directions, summing up their effects to provide a measure of the shear fatigue solicitation. The criterion proposed by Papadopoulos relies on the argument that the accumulated plastic deformations at mesoscopic level, at each slip plane, are proportional to the resolved shear stress amplitude.

The influence of the normal stress has been taken into account by many authors through its average acting upon all the planes passing through the material point. As remarked by Papadopoulos [2], such an average is equal to the hydrostatic stress.

Here we present an alternative fatigue endurance criterion based on new definitions for functions $f(\mathbf{S})$ and $g(p)$ in expression (1).

2. The equivalent shear stress amplitude

We acknowledge the concept of the minimum circumscribing ellipsoid as an appropriate measure of the equivalent shear stress. In this setting, the term $f(\mathbf{S})$ has been defined [4,5] as:

$$f(\mathbf{S}) := \sqrt{\sum_{i=1}^{5} \lambda_i^2} \tag{2}$$

where λ_i, $i = 1, \ldots, 5$ are the semi-axes of the ellipsoid circumscribing the stress path. In general, however, such ellipsoid and hence its semi-axes are difficult to determine.

For the specific case where the ellipsoid is a good approximation for the convex hull associated with the deviatoric part of the stress path (see Fig. 1), the result presented in what follows allows us to compute $f(\mathbf{S})$ in an almost trivially way.

Proposition 2.1. *Given an ellipsoid $\mathcal{E}$ in $\mathbb{R}^m$ with centre located at the origin and an arbitrary orthonormal basis $\{\mathbf{n}_i, i = 1, \ldots, m\}$ of $\mathbb{R}^m$, let $\mathcal{P}$ be a rectangular prism circumscribing $\mathcal{E}$ such that its faces are orthogonal to each one of the basis elements. If λ_i, $i = 1, \ldots, m$ are the magnitudes of the principal semi-axes of $\mathcal{E}$ and a_i, $i = 1, \ldots, m$ denote the distances of the centre of the ellipsoid to the faces of the rectangular prism, then*:

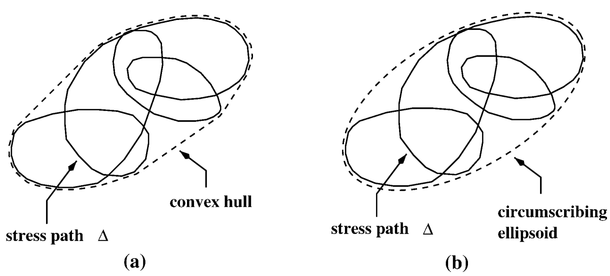

Fig. 1. (a) Convex hull of the stress path; (b) ellipsoid circumscribing the stress path.

Fig. 1. (a) Enveloppe convexe de l'histoire des contraintes ; (b) ellipsoïde circonscrivant l'histoire des contraintes.

$$\sum_{i=1}^{5} \lambda_i^2 = \sum_{i=1}^{5} a_i^2 \tag{3}$$

Proof. Let $\mathbb{S}_1^m$ be the unit sphere in $\mathbb{R}^m$:

$$\mathbb{S}_1^m := \left\{\mathbf{y} \in \mathbb{R}^m;\ \|\mathbf{y}\| = 1\right\} \tag{4}$$

where $\|\mathbf{y}\| := (y_1^2 + y_2^2 + \cdots + y_m^2)^{1/2}$ is the classical Euclidean norm in $\mathbb{R}^m$. The ellipsoid $\mathcal{E}$ can be characterized as the set of points:

$$\mathcal{E} := \left\{\mathbf{x} \in \mathbb{R}^m;\ \mathbf{x} = \mathbf{L}\,\mathbf{y},\ \mathbf{y} \in \mathbb{S}_1^m\right\} \tag{5}$$

where $\mathbf{L}: \mathbb{R}^m \to \mathbb{R}^m$ is a symmetric, positive semi-definite matrix with eigenvalues given by the magnitudes λ_i, $i = 1, \ldots, m$ of the semi-axes of $\mathcal{E}$. On the other hand, the distance a_i, from the faces of the rectangular prism orthogonal to a basis element $\mathbf{n}_i$ to the centre of the ellipsoid, can be expressed as:

$$a_i = \sup_{\mathbf{x} \in \mathcal{E}} (\mathbf{x}, \mathbf{n}_i), \quad i = 1, \ldots, m \tag{6}$$

where $(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} x_i\, y_i$ denotes the classical Euclidean inner product in $\mathbb{R}^m$. By considering the fact that the points $\mathbf{x}$ from the ellipsoid $\mathcal{E}$ satisfy (5), we can develop (6) as:

$$a_i = \sup_{\mathbf{x} \in \mathcal{E}} (\mathbf{x}, \mathbf{n}_i) = \sup_{\mathbf{y} \in \mathbb{S}_1^m} (\mathbf{L}\,\mathbf{y}, \mathbf{n}_i) = \sup_{\mathbf{y} \in \mathbb{S}_1^m} (\mathbf{y}, \mathbf{L}\,\mathbf{n}_i) = \|\mathbf{L}\,\mathbf{n}_i\| \tag{7}$$

since the supremum of $(\mathbf{y}, \mathbf{L}\,\mathbf{n}_i)$ among the points $\mathbf{y}$ from $\mathbb{S}_1^m$ is attained for $\mathbf{y}$ parallel to $\mathbf{L}\,\mathbf{n}_i$. Now, let us represent the identity operator on $\mathbb{R}^m$ as:

$$\mathbf{I} = \sum_{i-1}^{m} \mathbf{n}_i \otimes \mathbf{n}_i \tag{8}$$

where $\otimes$ denotes the tensor product operator such that $(\mathbf{a} \otimes \mathbf{b})\,\mathbf{u} = (\mathbf{a}, \mathbf{u})\,\mathbf{b}$. It follows that:

$$\mathbf{L}^2 = \mathbf{L}\left(\sum_{i-1}^{m} \mathbf{n}_i \otimes \mathbf{n}_i\right)\mathbf{L} = \sum_{i-1}^{m} \mathbf{L}\,\mathbf{n}_i \otimes \mathbf{L}\,\mathbf{n}_i \tag{9}$$

Finally, since the Frobenius norm of the linear operator $\mathbf{L}$ is given by $\|\mathbf{L}\|_F = (\sum_{i=1}^{m} \lambda_i^2)^{1/2}$, from (7) and (9) we obtain:

$$\sum_{i=1}^{m} \lambda_i^2 = \|\mathbf{L}\|_F^2 = \mathrm{tr}\big(\mathbf{L}^2\big) = \mathrm{tr}\left(\sum_{i-1}^{m} \mathbf{L}\,\mathbf{n}_i \otimes \mathbf{L}\,\mathbf{n}_i\right) = \sum_{i-1}^{m} (\mathbf{L}\,\mathbf{n}_i, \mathbf{L}\,\mathbf{n}_i) = \sum_{i-1}^{m} \|\mathbf{L}\,\mathbf{n}_i\|^2 = \sum_{i-1}^{m} a_i^2 \qquad \square \tag{10}$$

This statement precludes the need to determine the principal semi-axes of the ellipsoid when computing the measure $f(\mathbf{S})$ of shear solicitation to fatigue. Let Dev^3 denote the space of symmetric deviatoric tensors from $\mathbb{R}^3$ to $\mathbb{R}^3$ and let $\{\mathbf{N}_i,\ i = 1, \ldots, 5\}$ be an arbitrarily chosen orthonormal basis for such space. Any deviatoric stress state $\mathbf{S}(t)$ at a time instant t can be written as:

$$\mathbf{S}(t) = \sum_{i-1}^{5} s_i(t)\,\mathbf{N}_i \tag{11}$$

For an appropriate basis of Dev^3, the components $s_i(t)$ of $\mathbf{S}(t)$ in this basis can be expressed as:

$$\begin{aligned} &s_1(t) = \sqrt{\tfrac{3}{2}}\,S_{xx}(t), \quad s_2(t) = \tfrac{1}{\sqrt{2}}\big(S_{yy}(t) - S_{zz}(t)\big) \\ &s_3(t) = \sqrt{2}\,S_{xy}(t), \quad s_4(t) = \sqrt{2}\,S_{xz}(t), \quad s_5(t) = \sqrt{2}\,S_{yz}(t) \end{aligned} \tag{12}$$

From (11), it is possible to describe the stress path in terms of a curve in $\mathbb{R}^5$, where each point $\mathbf{s}(t) \in \mathbb{R}^5$ can be expressed as:

$$\mathbf{s}(t) := \big(s_1(t)\ s_2(t) \ldots s_5(t)\big)^T \tag{13}$$

Now, according to the aforementioned proposition, whenever the loading conditions are such that the convex hull of the generated stress path can be well approximated by an ellipsoid, the shear stress amplitude $f(\mathbf{S})$ can be simply computed as:

$$f(\mathbf{S}) := \sqrt{\textstyle\sum_{i=1}^{5} a_i^2} \tag{14}$$

where, in the context of the present study, a_i, $i = 1, \ldots, 5$ are the *amplitudes of the components* $s_i(t)$ *of the deviatoric stresses* defined as:

$$a_i := \frac{1}{2}\Big\{\max_t\big[s_i(t)\big] - \min_t\big[s_i(t)\big]\Big\}, \quad i = 1, \ldots, 5 \tag{15}$$

3. The normal stress

Tensile normal stresses contribute to the fatigue degradation by acting (essentially in mode 1) upon eventually existing embryo-cracks in the material. Many fatigue endurance criteria consider the hydrostatic stress as the measure of the solicitation to fatigue produced by the normal stresses since, as remarked by Papadopoulos [2], the hydrostatic stress is basically the quantity obtained by averaging the normal stresses over all the planes passing through a given material point. In this note, we claim that the worst situation – which corresponds to considering the existence of an embryo-crack oriented orthogonally to the maximum principal stress (among the three eigenvalues of the stress tensor and along all the stress path) – should be considered rather than the average solicitation given by the maximum hydrostatic stress.

4. The resulting endurance criterion

Based on the considerations developed along Sections 2.1 and 2.2, we propose the following multiaxial high cycle fatigue endurance criterion:

$$\sqrt{\textstyle\sum_{i=1}^{5} a_i^2} + \kappa\, \sigma_{\mathrm{pmax}} \leqslant \lambda \tag{16}$$

where a_i, $i = 1, \ldots, 5$ are defined as in (15) and σ_{pmax} is the maximum principal stress acting upon the material point along the loading history, while κ and λ are material parameters. If f_{-1} and t_{-1} are the fatigue endurance limits under alternate bending and alternate torsion solicitation, respectively, then the parameters κ and λ can be computed as:

$$\kappa = \sqrt{2}\,\frac{f_{-1}}{f_{-1} - t_{-1}}\left(\frac{t_{-1}}{f_{-1}} - \frac{1}{\sqrt{3}}\right) \quad \text{and} \quad \lambda = \sqrt{2}\,\frac{t_{-1} f_{-1}}{f_{-1} - t_{-1}}\left(1 - \frac{1}{\sqrt{3}}\right) \tag{17}$$

5. Assessment of the criterion

Proportional and nonproportional multiaxial fatigue experiments for a number of different materials were considered to assess the proposed criterion. Experimental data presented in Tables 1 to 3 – obtained by Nishihara and Kawamoto [6] (Table 1), Heidenreich et al. [7] (Table 2) and Froustey and Lassere [8] (Table 3) for hard metals ($1.3 \leqslant f_{-1}/t_{-1} < \sqrt{3}$) – describe biaxial normal/shear stress histories of the form:

$$\sigma(t) = \sigma_m + \sigma_a \sin(\omega t), \qquad \tau(t) = \tau_m + \tau_a \sin(\omega t - \beta) \tag{18}$$

corresponding to the maximum combination of stresses that the specimen can stand without failing, up to a limit of 10^6 cycles. In expressions (18), σ and τ describe respectively the normal and the shear stress histories, the

Table 1
Fatigue strength of hard steel ($t_{-1} = 196.2$ MPa, $f_{-1} = 313.9$ MPa): experimental data (Nishihara and Kawamoto [6]) and predictions

Tableau 1
L'intensité de fatigue d'acier dur ($t_{-1} = 196{,}2$ MPa, $f_{-1} = 313{,}9$ MPa) : données expérimentales (Nishihara et Kawamoto [6]) et prédictions

	σ_a (MPa)	σ_m (MPa)	τ_a (MPa)	τ_m (MPa)	β (°)	I^a (%)	I^b (%)	I^c (%)	I^d (%)
1-1	138.1	0	167.1	0	0	−2.27	−2.3	−2.28	−1.91
1-2	140.4	0	169.9	0	30	−2.60	−0.6	−0.64	−0.27
1-3	145.7	0	176.3	0	60	−3.61	3.1	3.10	3.49
1-4	150.2	0	181.7	0	90	−3.74	6.3	6.27	6.66
1-5	245.3	0	122.6	0	0	1.44	1.5	1.44	1.73
1-6	249.7	0	124.8	0	30	0.01	3.3	3.26	3.55
1-7	252.4	0	126.2	0	60	−8.35	4.4	4.39	4.69
1-8	258.0	0	129.0	0	90	−17.81	6.5	6.70	7.01
1-9	299.1	0	62.8	0	0	0.92	0.9	0.92	1.02
1-10	304.5	0	63.9	0	90	−2.99	2.7	2.74	2.83

[a]Crossland, [b]Papadopoulos, [c]Mamiya and Araújo, [d]Current model.

Table 2
Fatigue strength of 34Cr4 ($t_{-1} = 256$ MPa, $f_{-1} = 410$ MPa): experimental data (Heidenreich et al. [7]) and predictions

Tableau 2
L'intensité de fatigue 34Cr4 ($t_{-1} = 256$ MPa, $f_{-1} = 410$ MPa) : données expérimentales (Heidenreich et al. [7]) et prédictions

	σ_a (MPa)	σ_m (MPa)	τ_a (MPa)	τ_m (MPa)	β (°)	I^a (%)	I^b (%)	I^c (%)	I^d (%)
2-1	314.0	0	157.0	0	0	−0.55	−0.6	−0.55	−0.27
2-2	315.0	0	158.0	0	60	−12.33	−0.1	−0.11	0.18
2-3	316.0	0	158.0	0	90	−22.93	0.1	0.08	0.37
2-4	315.0	0	158.0	0	120	−12.33	−0.1	−0.11	0.18
2-5	224.0	0	224.0	0	90	−8.38	5.2	5.15	5.55
2-6	380.0	0	95.0	0	90	−7.32	0.4	0.37	0.49
2-7	316.0	0	158.0	158.0	0	0.08	0.1	0.08	6.01
2-8	314.0	0	157.0	157.0	60	−12.69	−0.6	−0.54	5.34
2-9	315.0	0	158.0	158.0	90	−23.17	−0.1	−0.11	5.83
2-10	279.0	279.0	140.0	0	0	−6.38	−6.4	−6.38	−0.21
2-11	284.0	284.0	142.0	0	90	−25.5	−4.8	−4.83	1.45
2-12	212.0	212.0	212.0	0	90	−9.39	3.4	3.41	7.23

[a]Crossland, [b]Papadopoulos, [c]Mamiya and Araújo, [d]Current model.

Table 3
Fatigue strength of 30NCD16 ($t_{-1} = 410$ MPa, $f_{-1} = 660$ MPa): experimental data (Froustey and Lassere [8]) and predictions

Tableau 3
L'intensité de fatigue 30NCD16 ($t_{-1} = 410$ MPa, $f_{-1} = 660$ MPa) : données expérimentales (Froustey et Lassere [8]) et prédictions

	σ_a (MPa)	σ_m (MPa)	τ_a (MPa)	τ_m (MPa)	β (°)	I^a (%)	I^b (%)	I^c (%)	I^d (%)
3-1	485.0	0	280.0	0	0	1.77	1.8	1.77	2.07
3-2	480.0	0	277.0	0	90	−27.27	0.7	0.70	1.00
3-3	480.0	300.0	277.0	0	0	3.91	3.9	3.91	7.63
3-4	480.0	300.0	277.0	0	45	−3.36	3.9	3.91	7.63
3-5	470.0	300.0	270.0	0	60	−10.93	1.6	1.60	5.32
3-6	473.0	300.0	273.0	0	90	−25.12	2.5	2.45	6.17
3-7	590.0	300.0	148.0	0	0	0.11	0.1	0.11	4.32
3-8	565.0	300.0	141.0	0	45	−7.23	−4.1	−4.07	0.14
3-9	540.0	300.0	135.0	0	90	−14.97	−8.1	−8.15	−3.94
3-10	211.0	300.0	365.0	0	0	−0.68	−0.7	−0.68	1.86

[a]Crossland, [b]Papadopoulos, [c]Mamiya and Araújo, [d]Current model.

subscripts a and m stand respectively for the amplitude and the mean value of stresses and β accounts for the phase angle. To assess the quality of the results provided by our model, an error index I is defined as:

$$I = \frac{1}{\lambda}\left[\left(\sum_{i=1}^{5} a_i^2\right)^{1/2} + \kappa\,\sigma_{\text{pmax}} - \lambda\right] \times 100 \quad (\%) \tag{19}$$

which gives a measure of how close the prediction of the criterion is with respect to the experimental data. A negative I yields a non-conservative fatigue strength prediction since it indicates that the stress solicitation has not attained a critical value while the experimental data are representative of limiting situations. On the other hand, a positive I provides a conservative estimate. Application of our model to the experimental data provided an error index which varied in the worst cases between -3.94% and 7.63% for all materials and loading conditions analyzed. The results provided by both Papadopoulos [2] and by Mamiya and Araújo [9] were essentially the same, varying between -8.15% and 6.7% while the Crossland criterion provided significantly poorer predictions. In our model, a shift of the error index towards the conservative region can be clearly observed whenever a mean stress is present in the loading history.

6. Discussion and conclusion

A new multiaxial fatigue criterion has been proposed. Application of this criterion to a broad range of in-phase and out-of-phase loading conditions involving three different materials under multiaxial, in-phase and out-of-phase states of stress yielded very good predictions of fatigue endurance. The proposed criterion always provided more conservative endurance estimates than all the other criteria considered in the present study, whenever shear or normal mean stresses were present in the loading history. When such mean stresses were absent, the predictions were essentially the same for all criteria with exception of Crossland. A very interesting feature of the proposed model which should be stressed is the great simplicity of its implementation. On the other hand, application of our criterion is restricted to cases where the shape of the convex hull circumscribing the microscopic loading path in Dev^3 approximates well an ellipsoid. Although this is a clear limitation, in practice there is a wide range of loading cases which fall within this condition, such as components under dynamic loadings caused by a single source. Studies on a criterion suitable for more general loading situations are in course.

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