Available online at www.sciencedirect.com

SCIENCE DIRECT®

C. R. Mecanique 332 (2004) 1013–1018

COMPTES RENDUS
MECANIQUE

http://france.elsevier.com/direct/CRAS2B/

Variance of the virtual displacement

Claude Vallée [a], Jean-Marie Souriau [b], Géry de Saxcé [c], Danielle Fortuné [a]

[a] *L.M.S., université de Poitiers, S.P.2M.I., téléport 2, boulevard Marie et Pierre Curie, B.P. 30179, 86962 Futuroscope-Chasseneuil cedex, France*
[b] *10, rue Mazarine, 13100 Aix-en-Provence, France*
[c] *L.M.L., université des sciences et des technologies de Lille, département de mécanique fondamentale, cité scientifique, 59655 Villeneuve d'Ascq cedex, France*

Received 15 January 2003; accepted after revision 8 September 2004

Available online 2 November 2004

Presented by Charles-Michel Marle

Abstract

We show that the affine structure of the 3-dimensional space is deeply enmeshed with the static laws expression. The relevance of tensorial rules for calculus in mechanics is thus enhanced. The virtual work principle is stated precisely but a little twist is given to the usual statements: the tensorial nature of the so-called virtual displacement vector is asserted to be covariant. ***To cite this article: C. Vallée et al., C. R. Mecanique 332 (2004).***

Résumé

Variance du déplacement virtuel. Nous montrons que la structure affine de l'espace tridimensionnel est fortement imbriquée avec l'expression des lois de la statique. La pertinence de l'application des règles du calcul tensoriel à la mécanique est ainsi renforcée. Le principe des travaux virtuels est établi précisément avec une seule entorse aux exposés classiques : la nature tensorielle de l'habituel vecteur déplacement virtuel est affirmée covariante. ***Pour citer cet article : C. Vallée et al., C. R. Mecanique 332 (2004).***

Keywords: Continuum mechanics; Static; Virtual displacement; Virtual work

Mots-clés : Milieux continus ; Statique ; Déplacement virtuel ; Travail virtuel

E-mail addresses: vallee@lms.univ-poitiers.fr (C. Vallée), gery.desaxce@univ-lille1.fr (G. de Saxcé), fortune@lms.univ-poitiers.fr (D. Fortuné).

doi:10.1016/j.crme.2004.09.009

Version française abrégée

Considérons N forces f_i (indexées de 1 à N) appliquées en des points M_i. Les points M_i sont repérés par leurs trois coordonnées (x_i^1, x_i^2, x_i^3), les forces f_i sont évaluées par leurs trois composantes (f_i^1, f_i^2, f_i^3). Elle sont en équilibre si :

$$\sum_{i=1}^{N} f_i^1 = 0 \qquad \sum_{i=1}^{N} f_i^2 = 0 \qquad \sum_{i=1}^{N} f_i^3 = 0$$
$$\sum_{i=1}^{N}\bigl(x_i^2 f_i^3 - f_i^2 x_i^3\bigr) = 0 \qquad \sum_{i=1}^{N}\bigl(x_i^3 f_i^1 - f_i^3 x_i^1\bigr) = 0 \qquad \sum_{i=1}^{N}\bigl(x_i^1 f_i^2 - f_i^1 x_i^2\bigr) = 0$$

Les 3 premières conditions expriment que la résultante des forces est nulle et les 3 dernières que leur moment résultant est aussi nul. Si on introduit des champs z_j du type :

$$z_j\bigl(x^1, x^2, x^3\bigr) = t_j + \sum_{k=1}^{3} \omega_{jk} x^k$$

(où les t_j et les ω_{jk} sont des scalaires, $\omega_{kj} = -\omega_{jk}$), alors les 6 lois de la statique se résument en

$$\sum_{i=1}^{N}\sum_{j=1}^{3} z_j\bigl(x_i^1, x_i^2, x_i^3\bigr) f_i^j = 0$$

Il se trouve que les champs z_j sont caractérisés par le système linéaire d'équations aux dérivées partielles

$$\frac{\partial z_j}{\partial x^k} + \frac{\partial z_k}{\partial x^j} = 0$$

Un nouvel énoncé du principe de la statique est donc :

$$\frac{\partial z_j}{\partial x^k} + \frac{\partial z_k}{\partial x^j} = 0 \quad \Rightarrow \quad \sum_{i=1}^{N}\sum_{j=1}^{3} z_j\bigl(x_i^1, x_i^2, x_i^3\bigr) f_i^j = 0$$

On reconnaît le principe des travaux virtuels en interprétant les champs z_j comme les composantes d'un déplacement virtuel et la somme

$$\sum_{i=1}^{N}\sum_{j=1}^{3} z_j\bigl(x_i^1, x_i^2, x_i^3\bigr) f_i^j$$

comme le travail virtuel des forces f_i. Nous montrons que cet énoncé du principe des travaux virtuels n'est indépendant du choix des coordonnées que si les changements de variables autorisés sont affines. Ainsi se trouvent fortement imbriquées la structure de l'espace et l'invariance par changement de coordonnées des lois de la statique. La seule entorse aux exposés classiques du principe des travaux virtuels est la nature tensorielle du déplacement virtuel qui est affirmée covariante plutôt que contravariante.

1. Balance of forces and moments

Let us consider N forces f_i (indexed by i from 1 to N) applied in N points M_i. On the mathematical level, we only will know at this beginning, as in the remote time of Archimedes [1] and Euclid, but with notations of today [2], that:

- a point M_i can be located by 3 coordinates x_i^1, x_i^2, x_i^3,
- a force f_i applied in M_i can be evaluated by 3 components f_i^1, f_i^2, f_i^3.

On a mechanical level, what Archimedes said, and after him Simon Stevin [3], was the conditions for what N forces f_i are in equilibrium. These conditions can be expressed by 6 formulae readable for an engineer knowing only what are the coordinates of a point and what are the components of a force:

$$\sum_{i=1}^{N} f_i^1 = 0 \qquad \sum_{i=1}^{N} f_i^2 = 0 \qquad \sum_{i=1}^{N} f_i^3 = 0 \tag{1}$$

$$\sum_{i=1}^{N} \left(x_i^2 f_i^3 - f_i^2 x_i^3\right) = 0 \qquad \sum_{i=1}^{N} \left(x_i^3 f_i^1 - f_i^3 x_i^1\right) = 0 \qquad \sum_{i=1}^{N} \left(x_i^1 f_i^2 - f_i^1 x_i^2\right) = 0 \tag{2}$$

The 3 first conditions constitute the so-called resultant force law of static, and the 3 last the even so-called resultant moment law. We can already notice that these equations are written without needing any notion of dot product, or cross product, or angle. Up to this point, the mathematical structure of the space is intentionally poor, what we want precisely is to discover its fair geometry for the sake of static modeling. Better than invoking a posteriori the Ockam razor principle [4], we will seek the minimal properties satisfied by the changes of coordinates insuring the permanence of the 6 static laws expression.

2. First static laws summary

Let us multiply respectively, the 3 first conditions by 1-indexed scalars t_1, t_2, t_3, the 3 last ones by 2-indexed scalars ω_{32}, ω_{13}, ω_{21}, understood to be skew-symmetrical, and let us add all of them. Then the 6 conditions ((1), (2)) synthesize in

$$\sum_{i=1}^{N} \left(\sum_{j=1}^{3} \left(t_j + \sum_{k=1}^{3} \omega_{jk} x_i^k \right) f_i^j \right) = 0$$

Also, we can summarize all static laws by: “N forces f_i, applied in N points M_i, are in equilibrium if

$$\sum_{i=1}^{N} \sum_{j=1}^{3} z_j\left(x_i^1, x_i^2, x_i^3\right) f_i^j = 0$$

for any testing field z_j of the type

$$z_j\left(x^1, x^2, x^3\right) = t_j + \sum_{k=1}^{3} \omega_{jk} x^k \tag{3}$$

with $\omega_{kj} = -\omega_{jk}$”.

3. Characterization of testing fields

We can remark [5] the characterization:

$$z_j\left(x^1, x^2, x^3\right) = t_j + \sum_{k=1}^{3} \omega_{jk} x^k \quad \Leftrightarrow \quad \frac{\partial z_j}{\partial x^k} + \frac{\partial z_k}{\partial x^j} = 0$$

3.1. *Direct property*

The direct property is obvious because in this case the partial derivative $\frac{\partial z_j}{\partial x^k} = \omega_{jk}$ is skew-symmetrical, and then its symmetrical part is null. What happens is that the converse is true.

3.2. *Converse property*

Let us assume $\frac{\partial z_j}{\partial x^k}$ to be skew-symmetrical, therefore its first derivative $\xi_{jkl} = \frac{\partial}{\partial x^l}\frac{\partial z_j}{\partial x^k}$ is symmetrical with respect to the 2 last indexes as any good second derivative and remains skew-symmetrical with respect to the 2 first ones. However, the braids lemma [6] afterward mentioned says that this is possible only when ξ_{jkl} is null. Then, it is very straightforward to achieve the proof of the converse property: each derivative $\omega_{jk} = \frac{\partial z_j}{\partial x^k}$ does not depend of the coordinates x^1, x^2, x^3; and it is easy to integrate in (3).

3.3. *Braids lemma*

"If ξ_{jkl} is symmetrical with respect to the 2 last indexes and skew-symmetrical with respect to the 2 first ones, then ξ_{jkl} is null".

Proof in 6 strokes. The scalar ξ_{jkl} is opposite to itself $\xi_{jkl} = -\xi_{kjl} = -\xi_{klj} = \xi_{lkj} = \xi_{ljk} = -\xi_{jlk} = -\xi_{jkl}$ and so is null.

4. Second static laws summary

"N forces f_i, applied in N points M_i, are in equilibrium if

$$\sum_{i=1}^{N}\sum_{j=1}^{3} z_j\left(x_i^1, x_i^2, x_i^3\right) f_i^j = 0$$

for any testing field z_j solution of

$$\frac{\partial z_j}{\partial x^k} + \frac{\partial z_k}{\partial x^j} = 0\text{''}$$

5. Consistency

One question arises: is this second summary, coordinate free? More precisely, if new coordinates y^1, y^2, y^3 are chosen, does this formulation perpetuate? To answer to this, we may have an idea about how change the components f^j and z_j of respectively a force and a testing field. We only will remember [2] of what precedes that the space is a 3-dimensional manifold V and we will assume that:

- the components f^j of a force f change locally as the coordinates, they become $F^j = \sum_{k=1}^{3} \frac{\partial y^j}{\partial x^k} f^k$ using the partial derivatives $\frac{\partial y^j}{\partial x^k}$ of the new coordinates with respect to the old ones,
- the old components z_j of the testing field are replaced by new ones Z_j in accordance with the rule $z_j = \sum_{k=1}^{3} \frac{\partial y^k}{\partial x^j} Z_k$ insuring a common meaning to $\sum_{j=1}^{3} Z_j F^j$ and $\sum_{j=1}^{3} z_j f^j$.

The question now is: do the new components Z_j remain solutions of $\frac{\partial Z_j}{\partial y^k} + \frac{\partial Z_k}{\partial y^j} = 0$?

Thus we come to our main result.

6. Allowable changes of coordinates

"The changes of coordinates consistent with the static laws expressed by our second summary are the affine ones

$$y^j = \sum_{k=1}^{3} A_k^j x^k + b^j \tag{4}$$

where the A_k^j and the b^j are respectively the coefficients of an invertible 3×3 matrix and a 3×1 matrix".

Proof. The 2 assumptions are

$$(1) \quad z_j = \sum_{k=1}^{3} \frac{\partial y^k}{\partial x^j} Z_k \qquad (2) \quad \frac{\partial z_j}{\partial x^k} + \frac{\partial z_k}{\partial x^j} = 0 \Leftrightarrow \frac{\partial Z_j}{\partial y^k} + \frac{\partial Z_k}{\partial y^j} = 0$$

Thanks to the first assumption, the symmetrical parts of the first derivatives of z_j and Z_j are related by

$$\frac{\partial z_j}{\partial x^k} + \frac{\partial z_k}{\partial x^j} = \sum_{l=1}^{3} \left(\frac{\partial}{\partial x^k} \frac{\partial y^l}{\partial x^j} + \frac{\partial}{\partial x^j} \frac{\partial y^l}{\partial x^k} \right) Z_l + \sum_{l=1}^{3} \sum_{m=1}^{3} \frac{\partial y^l}{\partial x^j} \left(\frac{\partial Z_l}{\partial y^m} + \frac{\partial Z_m}{\partial y^l} \right) \frac{\partial y^m}{\partial x^k}$$

Thanks to the second assumption

$$\sum_{l=1}^{3} \left(\frac{\partial}{\partial x^k} \frac{\partial y^l}{\partial x^j} \right) Z_l = 0$$

may hold, in particular, for constants Z_l. As a consequence, all the second derivatives $\frac{\partial}{\partial x^k} \frac{\partial y^l}{\partial x^j}$ are null, and it is easy to integrate in (4).

7. Affine space

The allowable changes of coordinates are correlated with the action of an affine group on the space. We do believe that the geometry is given by the acting group as Felix Klein claimed in his Erlangen Program [7], therefore the manifold V is no more than an affine space. In this space, coming from sake of Archimedes static, the relevant notion is the parallelism. We emphasized on coordinates and changes of coordinates rather than on basis or changes of basis, nevertheless all axiomatic definitions of affine and linear spaces can be rebuilt.

8. Virtual work principle

Our second summary of static is nothing else than the virtual work principle [8]. The only twist we gave to the classical statements is that we have evaluated a force f by its effect when submitted to a linear form z. This effect $\sum_{j=1}^{3} z_j f^j$ is the virtual work. Following the Ockam economy principle [4], we will prefer such an evaluation to the usual one by a dot product with a so-called virtual displacement vector, because we did not encounter any natural argument for a metric in our modeling of static.

The type $t_j + \sum_{k=1}^{3} \omega_{jk} x^k$ of that kind of 'covariant virtual displacement' z_j recalls the type of classical rigid body virtual displacement vectors:

- the part of small translations is played by t_j,
- the part of small rotations is played by ω_{jk}.

But ω_{jk} has not any interpretation through a small rotation because there is no metric in the affine space. In such a space, nobody is able to define what is a rigid body.

9. Conclusion

Archimedes and Euclid were contemporary. It seems that they established simultaneously the foundations of static and geometry with common words [9,10]. Our result tells that the affine structure of the 3-dimensional space is self-consistent with the expression of static laws.

The linear group associated to the affine group acts on the linear space E associated to the affine space. The good rules for calculating in E are those of tensorial calculus introduced at the beginning of the XXth century by Gregorio Ricci-Curbastro and Tullio Levi-Civita. Rules for calculating in the affine space are not so well-known, the understanding of the mechanical situation will be improved by considering specially the geometry generated by the affine group and by introducing 'affine' tensors [11] in competition with the classical 'linear' tensors.

There is no need for a metric when speaking about the equilibrium of a system of forces. This lack of natural metric in the affine space leads to give up to rotations. Therefore, the notion of rigid body displacement vector is lost and so is the characterization by the vanishing of the self-adjoint part of its derivative. However, the 'covariant virtual displacements' z_j satisfying $\frac{\partial z_j}{\partial x^k} + \frac{\partial z_k}{\partial x^j} = 0$ reveal themselves to be as much fruitful [12,13,7] for generalizations to continuum media and applications to strength of materials.

Moreover, extending the covariant virtual displacements z_j to the four dimensional affine space-time provides dynamics of rods and shells as well as dynamics of three dimensional media [14,11].

Acknowledgements

The first author would like to gratefully acknowledge Jean-Claude Darchen from the University of Rennes for productive discussions and for being source of encouragement.

References

[1] P. Sandori, Petite Logique des Forces, Seuil, Paris, 1983. Translated from: The Logic of Machines and Structures, John Wiley & Sons, New York, 1982.
[2] G.-G. Granger, La Pensée de l'Espace, Editions Odile Jacob, Paris, 1999.
[3] F. Frey, Statique Appliquée, Traité de Génie Civil de l'École Polytechnique Fédérale de Lausanne, vol. 1, Analyse des Structures et Milieux Continus, Presses Polytechniques et Universitaires Romandes, Lausanne, 1990.
[4] P. Duhem, L'Aube du Savoir, Hermann, Paris, 1997.
[5] J.-M. Souriau, Mécanique des états condensés de la matière, in : 1er Séminaire International de la Fédération Mécanique de Grenoble, held in Grenoble, France, May 19–21, 1992.
[6] M. Berger, Géométrie, Nathan, Paris, 1990.
[7] J.-M. Souriau, Les groupes comme universaux, in : La Structure du Monde : Objets, Propriétés, États de Choses, International Colloquium held in Grenoble, France, December 9–13, 1999.
[8] P. Duhem, L'Evolution de la Mécanique, Vrin, Paris, 1992.
[9] M. Serres, Gnomon : les débuts de la géométrie en Grèce, in : M. Serres (Ed.), Eléments d'Histoire des Sciences, Bordas, Paris, 1989.
[10] M. Serres, Statues, Editions François Bourin, Paris, 1987.
[11] G. de Saxcé, Calcul tensoriel affine et dynamique des coques, in : Colloque International de Théories Variationnelles, held in Saint-Agnan-en-Morvan, France, August 28–31, 2001.
[12] H.-H. Fliche, Private communication, 2000.
[13] G. de Saxcé, C. Vallée, Affine tensors in shell theory, in: 7th Conference Shell Structures, Theory and Applications, held in Gdansk-Jurata, Poland, August 9–11, 2002. Extended version published in: J. Theoret. Appl. Mech. 3 (41) (2003) 593–621.
[14] J.-M. Souriau, Milieux continus de dimension 1, 2 ou 3 : statique et dynamique, in : Proceedings of 13ème Congrès Français de Mécanique, held in Poitiers-Futuroscope, France, September 1–5, 1997.