Available online at www.sciencedirect.com
SCIENCE DIRECT®

C. R. Mecanique 333 (2005) 117–122

COMPTES RENDUS
MECANIQUE

http://france.elsevier.com/direct/CRAS2B/

Small vibrations of a linearly elastic body surrounded by heavy, incompressible, non-Newtonian fluids with free surfaces

Christian Licht [a], Tran Thu Ha [b]

[a] *Laboratoire de mécanique et génie civil, UMR 5508 CNRS, université Montpellier 2, case courrier 048, place Eugène Bataillon, 34095 Montpellier cedex 05, France*
[b] *Institute of Mechanics of Hanoi, 264 Doi Can St., Hanoi, Viet Nam*

Received 26 July 2004; accepted after revision 21 September 2004

Available online 13 November 2004

Presented by Évariste Sanchez-Palencia

Abstract

We consider the small transient motions of a coupled system constituted by a linearly elastic body and two heavy, incompressible, non-Newtonian fluids.Through a formulation in terms of non-linear evolution equations in Hilbert spaces of possible states with finite mechanical energy, we obtain existence and uniqueness results and study the influence of gravity. ***To cite this article: C. Licht, Tran Thu Ha, C. R. Mecanique 333 (2005).***

Résumé

Petites vibrations d'un corps linéairement élastique baigné par des fluides non-Newtoniens, incompressibles, pesants, à surfaces libres. On considère les petits mouvements instationnaires d'un système couplé constitué d'un solide linéairement élastique et de deux fluides non-Newtoniens, incompressibles, pesants à surfaces libres. Une formulation en termes d'équations d'évolution non-linéaires dans des espaces de Hilbert d'états possibles d'énergie mécanique finie permet d'obtenir des résultats d'existence et d'unicité et d'étudier l'influence de la gravité. ***Pour citer cet article : C. Licht, Tran Thu Ha, C. R. Mecanique 333 (2005).***

Keywords: Continuum Mechanics; Fluid–structure interactions; Non-linear semi-groups

Mots-clés : Milieux continus ; Couplages fluide–structure ; Semi-groupes non-linéaires

* Corresponding author.
E-mail addresses: licht@lmgc.univ-montp2.fr (C. Licht), ttha@im01.ac.vn (Tran Thu Ha).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2004.09.011

1. Introduction

Some problems of offshore engineering lead us to consider the *small* transient vibrations of a coupled fluids–structure system around an equilibrium configuration which can be described as follows. A linearly elastic body occupies the closure $\overline{B}$ of a domain B of $\mathbb{R}^3$ with a Lipschitz-continuous boundary ∂B. The heavy, incompressible and not necessarily Newtonian viscous fluids fill up the reunion Ω of two disjoint connected open sets of $\mathbb{R}^3$ with Lipschitz boundaries Ω^i (the inner fluid) and Ω^e (the outer fluid); Ω^i is bounded and Ω^e may be unbounded, they respectively lie in the half space $\{x_3 \leqslant h^i\}$ and in the strip $\{d^e \leqslant x_3 \leqslant h^e\}$. Parts of the boundaries $\partial\Omega^i$ and $\partial\Omega^e$ of Ω^i and Ω^e, the so-called free surfaces S_F^i and S_F^e, are included in the planes $\{x_3 = h^i\}$ and $\{x_3 = h^e\}$, let $S_F = S_F^i \cup S_F^e$. The body domain and the fluid region share a boundary S_w, the wet surfaces, let $S_w^i = S_w \cap \partial\Omega^i$ and $S_w^e = S_w \cap \partial\Omega^e$. Whereas the fluid particles adhere perfectly to a rigid 'bottom' $S_B = \partial\Omega^e \setminus (S_w^e \cup S_e^F)$, we have $\partial\Omega^i = S_w^i \cup S_i^F$. The body is respectively clamped and free of surface loading on the remaining parts Γ_0 and Γ_1 of ∂B. The state of the coupled system is determined by the quadruplet $u_g = (v_f, \eta, s, v_b)$, v_f is the fluid velocity, η the free surface elevation, s and v_b the fields of displacement and velocity in the body; we will use the index g to specify that gravity effects are taken into account (see Section 3.2 also). Let σ_f and σ_b the (fluctuations of) stresses in the fluid and in the body, the equations of the small motions may be expressed as:

$$\rho_f \frac{\partial v_f}{\partial t} - \operatorname{div}\sigma_f = f_f \quad \text{in } \Omega \tag{1}$$

$$\sigma_f n = -\rho_f g \eta n \quad \text{on } S_F \tag{2}$$

$$\frac{\partial \eta}{\partial t} = v_f \cdot n \quad \text{on } S_F \tag{3}$$

$$v_f = 0 \quad \text{on } S_B \tag{4}$$

$$\sigma_f n = \sigma_b n \quad \text{on } S_w \tag{5}$$

$$v_f = v_b \quad \text{on } S_w \tag{6}$$

$$\rho_b \frac{\partial v_b}{\partial t} - \operatorname{div}\sigma_b = f_b \quad \text{in } B \tag{7}$$

$$v_b = \frac{\partial s}{\partial t} \quad \text{in } B \tag{8}$$

$$s = 0 \quad \text{on } \Gamma_0 \tag{9}$$

$$\sigma_b \nu = 0 \quad \text{on } \Gamma_1 \tag{10}$$

Here, ρ_f, ρ_b are the densities in the fluids and in the body, g is the gravity acceleration, f_f, f_b are the densities of the applied body forces excluding the own-weights, n, ν are the unit normals outward Ω and B, and t denotes the time. The density is constant in each fluid while ρ_b and $1/\rho_b$ are positive elements of $L^\infty(B)$. The constitutive equations are:

$$\operatorname{div} v_f = 0, \quad (\sigma_f)_{\text{dev}} \in \partial d\big(x, e(v_f)\big) \quad \text{in } \Omega \tag{11}$$

$$\sigma_b = a e(s) \quad \text{in } B \tag{12}$$

where x is the space variable, e stands for the symmetrized gradient, $(\sigma_f)_{\text{dev}}$ is the deviatoric part of σ_f, a is the linearized tensor of elasticity and ∂d denotes the sub-differential of the convex density function of dissipation potential $d(x,\cdot)$. We assume that a enjoys the usual uniform properties of symmetry, boundedness and ellipticity, while $d(\cdot, e)$ is constant in each fluid and $d(x,\cdot)$ is a convex lower semi-continuous function in the set S^3_{dev} of 3×3 symmetric matrices with vanishing traces, such that:

$$d(x,0) = 0,\ \forall x \in \Omega \quad \exists c > 0,\ p > 1;\ d(x,q) \geqslant c|q|^p,\ \forall q \in S^3_{\text{dev}},\ \forall x \in \Omega \tag{13}$$

An initial state u_g^0 being given, the problem is to find $u_g(t)$, $t \in [0,T]$, satisfying (1)–(12). In [1,2], similar simplified situations were considered with Newtonian fluids, here the dissipation potential is not necessarily quadratic.

2. An existence and uniqueness result

We choose to formulate the previous problem in terms of a nonlinear evolution equation governed by a maximal-monotone operator acting on a Hilbert space of possible states with finite mechanical energy. First, for every open set G of $\mathbb{R}^3$ we denote by $H^1_\Gamma(G)$ the subset of the Sobolev space $H^1(G)$ whose elements vanish on $\Gamma \subset \partial G$ and let

$$L^2_{\mathrm{div},S_B}(\Omega) := \left\{\varphi \in L^2(\Omega)^3;\ \int_\Omega \varphi \cdot \nabla w \,\mathrm{d}x = 0 \quad \forall w \in H^1_{S_w \cup S_F}(\Omega)\right\}$$

then twice the total mechanical energy of the system defines the square of a Hilbert-norm

$$|u_g|_g^2 := \int_\Omega \rho_f |v_f|^2 \,\mathrm{d}x + g\int_{S_F} \rho_f \eta^2 \,\mathrm{d}s + \int_B ae(s)\cdot e(s)\,\mathrm{d}x + \int_B \rho_b |v_b|^2 \,\mathrm{d}x \tag{14}$$

on

$$H_g := L^2_{\mathrm{div},S_B}(\Omega) \times L^2(S_F) \times H^1_{\Gamma_0}(B)^3 \times L^2(B)^3 \tag{15}$$

Furthermore, the total dissipation

$$D(\varphi) := \int_\Omega d\big(x, e(\varphi)(x)\big)\,\mathrm{d}x \tag{16}$$

defines a convex, lower semi-continuous functional in $X_p(\Omega) := \{\varphi \in L^2_{\mathrm{div},S_B}(\Omega);\ e(\varphi) \in L^p(\Omega)^9\}$.

Next, letting $V_p := \{(\varphi,\psi) \in X_p(\Omega) \times H^1_{\Gamma_0}(B)^3;\ \varphi\cdot n \in L^2(S_F), \varphi = 0 \text{ on } S_B, \varphi = \psi \text{ on } S_w\}$, we introduce the following multivoque operator:

$$\mathrm{dom}(A_g) = \left\{\begin{array}{l} u_g = (v_f,\eta,s,v_b) \in H_g;\ (v_f,v_b) \in V_p,\ v_f \in \mathrm{dom}(D) \\ \exists (w,z) \in L^2_{\mathrm{div},S_B}(\Omega) \times L^2(B)^3 \text{ such that} \\ \left\{\begin{array}{ll} \forall(\varphi,\psi)\in V_p, & D(v_f+\varphi) \geqslant D(v_f) + \int_\Omega \rho_f w\cdot\varphi\,\mathrm{d}x - \int_{S_F}\rho_f g\eta\varphi\cdot n\,\mathrm{d}s \\ & \qquad - \int_B ae(s)\cdot e(\psi)\,\mathrm{d}x + \int_B \rho_b z\cdot\psi\,\mathrm{d}s \end{array}\right. \quad (\star) \end{array}\right\} \tag{17}$$

$$A_g u_g := \big\{(w, -v_f\cdot n, -v_b, z) \in H_g \text{ satisfying } (\star)\big\} \tag{18}$$

Eventually, we assume that the data satisfy:

$$u_g^0 \in \mathrm{dom}(A_g), \quad (f_f, f_b) \in BV\big(0,T; L^2(\Omega)^3 \times L^2(B)^3\big) \tag{19}$$

where the last functional space corresponds to functions with bounded time-variations [3]. Thereafter, straightforward integrations by parts imply that the genuine problem is formally equivalent to:

$$\frac{\mathrm{d}u_g}{\mathrm{d}t} + A_g u_g \ni (f_f^\star/\rho_f, 0, 0, f_b/\rho_b), \quad u_g(0) = u_g^0 \tag{20}$$

$f_f^\star$ being the $L^2(\Omega)^3$-projection of f_f on $L^2_{\mathrm{div},S_B}(\Omega)$. Actually, the definition of $\mathrm{dom}(A_g)$ involves the mechanical constraints not entering the definition of H_g. The differential inclusion in (20) corresponds to the kinematics conditions (3), (8) and to a virtual power formulation of the problem taking into account the constitutive equations (11), (12). Then we have the following:

Theorem 2.1. *There exists a unique* u_g *in* $W^{1,\infty}(0,T;H_g)$ *such that* $u_g(0) = u_g^0$, $u_g(t) \in \mathrm{dom}(A_g)\forall t \in [0,T]$, $\mathrm{d}u_g/\mathrm{d}t + A_g u_g \ni (f_f^\star(t)/\rho_f, 0, 0, f_b(t)/\rho_b)$ *almost everywhere in* $(0,T)$.

Proof. Because of the assumption (19), it suffices (see [3]) to prove that A_g is maximal-monotone. Actually, for all $u^1=(u_1^1,u_2^1,u_3^1,u_4^1)$, $u^2=(u_1^2,u_2^2,u_3^2,u_4^2)$ in $\mathrm{dom}(A_g)$ and (ξ^1,ξ^2) in $A_g u^1\times A_g u^2$, the very definition of $A_g u^i$, $i=1,2$, implies:

$$0=D(u_1^2)-D(u_1^1)+\big(D(u_1^1)-D(u_1^2)\big)\geqslant -(\xi^2-\xi^1,u^2-u^1)_g$$

where $(\cdot,\cdot)_g$ stands for the scalar product in H_g associated with the norm (14).

Next, let $f=(f_1,f_2,f_3,f_4)$ arbitrary in H_g. If there exists $u=(u_1,u_2,u_3,u_4)$ in $\mathrm{dom}(A_g)$ such that $u+A_g u\ni f$, necessarily $u_1+w=f_1$, $u_2-u_1\cdot n=f_2$, $u_3-u_4=f_3$, $u_4+z=f_4$ with (w,z) satisfying the inequality ($\star$) involved in the definition of $A_g u$. Therefore, (u_1,u_4) does minimize the functional:

$$(\varphi,\psi)\in V_p\mapsto J_g(\varphi,\psi)=D(\varphi)+1/2\big|(\varphi,\varphi\cdot n,\psi,\psi)\big|_g^2-\big(f,(\varphi,-\varphi\cdot n,-\psi,\psi)\big)_g\in\mathbb{R}\cup\{+\infty\}$$

This functional, being strictly convex, lower semi-continuous and coercive on V_p, admits a unique minimizer $(\bar u_1,\bar u_4)$ in V_p, so that $\bar u=(\bar u_1,\bar u_1\cdot n+f_2,\bar u_4+f_3,\bar u_4)$ satisfies $\bar u+A_g\bar u\ni f$, which ends the proof. □

3. Comments and additional results

3.1. 'Rigid-plastic' fluids

The same method and result apply when the growth exponent p, occurring in (13), equals 1, but with the boundary condition (4) relaxed by:

$$v_f\cdot n=0,\quad -(\sigma_f)_T\in\partial j(v_f)\quad \text{on } S_B \tag{4'}$$

here $(\sigma_f)_T$ is the tangent component of $\sigma_f n$ and $j(v_f)=d^\infty(v_f\otimes_{\mathrm{sym}} n)$, d^∞ is the recession function of d and $\otimes_{\mathrm{sym}}$ stands for the symmetrized tensor product (cf. [4]). In this case, $X_1(\Omega):=\{\varphi\in L^2_{\mathrm{div},S_B}(\Omega);\, e(\varphi)$ is a bounded measure on $\Omega\}$, $V_1:=\{(\varphi,\psi)\in X_1(\Omega)\times H^1_{\Gamma_0}(B)^3;\ \varphi\cdot n\in L^2(S_F),\ \varphi=\psi$ on $S_w\}$ and the total dissipation entering the definition of A_g is the sum of $\int_{S_B} j(v_f)\,\mathrm{d}s$ and a term like (16), but to be understood in the sense of convex functions of measure [4].

3.2. Neglecting gravity effects

Such a simplified modelling describes the mechanical state by the triplet $u_0=(v_f,s,v_b)$ only and considers Eqs. (1), (2) and (4)–(12) with $g=0$ and an initial state u_0^0. Let $W_p:=\{(\varphi,\psi)\in X_p(\Omega)\times H^1_{\Gamma_0}(B)^3;\ \varphi=0$ on S_B, $\varphi=\psi$ on $S_w\}$, then in the Hilbert space

$$H_0:=L^2_{\mathrm{div},S_B}(\Omega)\times H^1_{\Gamma_0}(B)^3\times L^2(B)^3 \tag{21}$$

equipped with the norm:

$$|u_0|_0^2:=\int_\Omega \rho_f|v_f|^2\,\mathrm{d}x+\int_B ae(s)\cdot e(s)\,\mathrm{d}x+\int_B \rho_b|v_b|^2\,\mathrm{d}x \tag{22}$$

we can define an operator A_0 by:

$$\mathrm{dom}(A_0)=\left\{\begin{array}{l} u_0=(v_f,s,v_b)\in H_0;\ (v_f,v_b)\in W_p,\ v_f\in\mathrm{dom}(D)\\ \exists(w,z)\in L^2_{\mathrm{div},S_B}(\Omega)\times L^2(B)^3 \text{ such that}\\ \left\{\begin{array}{ll}\forall(\varphi,\psi)\in W_p, & D(v_f+\varphi)\geqslant D(v_f)+\int_\Omega \rho_f w\cdot\varphi\,\mathrm{d}x\\ & \quad -\int_B ae(s)\cdot e(\psi)\,\mathrm{d}x+\int_B\rho_b z\cdot\psi\,\mathrm{d}s \qquad (\star\star)\end{array}\right.\end{array}\right\} \tag{23}$$

$$A_0u_0:=\big\{(w,-v_b,z)\in H_0 \text{ satisfying } (\star\star)\big\} \tag{24}$$

As previously, it can be shown that the problem is formally equivalent to

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} + A_0 u_0 \ni (f_f^\star/\rho_f, 0, f_b/\rho_b), \quad u_0(0) = u_0^0 \tag{25}$$

and that A_0 is maximal-monotone, thus (25) admits a unique solution enjoying regularity properties analogous to those of u_g in (20). Under the additional assumption on d:

$$\exists C > 0; \quad d(x,q) \leqslant C(1+|q|)^p, \quad \forall q \in S^3_{\mathrm{dev}},\ \forall x \in \Omega \tag{26}$$

we can precise the status of this simplified modelling with respect to the modelling of Section 2. Introducing a family of linear isometric operators P_g:

$$H_0 \ni u_0 = (v_f, s, v_b) \longmapsto P_g u_0 = (v_f, 0, s, v_b) \in H_g \tag{27}$$

we have the following approximation result:

Theorem 3.1. *If* $\lim_{g\to 0} |P_g u_0^0 - u_g^0|_g = 0$*, then, when* g *goes to* 0*, the unique solution* u_g *of* (20) *converges to the unique solution* u_0 *of* (25)*, in the sense that* $\lim_{g\to 0} |P_g u_0(t) - u_g(t)|_g = 0$ *uniformly on* $[0, T]$.

From an easy non-linear extension of the Trotter theory [5] of approximation of semi-groups of operators acting on variable Hilbert spaces, the proof of Theorem 3.1 reduces to the proof of

$$\lim_{g\to 0} \left| P_g (I + \lambda A_0)^{-1} f - (I + \lambda A_g)^{-1} P_g f \right|_g = 0, \quad \forall (\lambda, f) \in (0,\infty) \times H_0$$

which is a straightforward consequence of the obvious Γ-convergence of J_g to J_0 with respect to the sequential weak convergence on W_p.

Of course, the gravity acceleration does not go to zero, but a preliminary non-dimensional setting of the problem will involve a coefficient which increases from 0 with g. Thus, the practical interest of the previous theorem is to describe what is happening when this coefficient goes to zero: convergences in energy norms of the displacement and of the velocities, convergence to zero of the gravity potential energy of the fluid.

It is interesting to note that the problem at $g = 0$ may also concern the small motions of a deformable composite whose constituents, perfectly stuck together, are either linearly elastic or viscoplastic.

3.3. Taking into account fluctuations of atmospheric pressure and surface forces on the body

This realistic situation can be handled also by the tool of evolution equations governed by time-independent maximal-monotone operators in the case of Newtonian fluids with, for instance, dissipation density functions like $d(x,e) = \mu(x)|e|^2$, where μ is constant and positive in each fluid. More precisely, Eqs. (2) and (10) are replaced by:

$$\sigma_f n = -\rho_f g \eta n + \gamma \quad \text{on } S_F \tag{2'}$$

$$\sigma_b \nu = \delta \quad \text{on } \Gamma_1 \tag{10'}$$

If we assume that (γ, δ) belongs to $W^{2,\infty}(0,T; L^2(S_F)^3 \times L^2(\Gamma_1)^3)$, then there exists a unique (ω_f, ω_b) in $W^{2,\infty}(0,T; V_2)$ such that:

$$\int_\Omega 2\mu e(\omega_f)\cdot e(\varphi)\,\mathrm{d}x + \int_B a e(\omega_b)\cdot e(\psi)\,\mathrm{d}x = \int_{S_F} \gamma\cdot\varphi\,\mathrm{d}s + \int_{\Gamma_1} \delta\cdot\psi\,\mathrm{d}s \quad \forall(\varphi,\psi) \in V_2 \tag{28}$$

In this way, $\bar{u}_g := u_g - (\omega_f, 0, \omega_b, \omega_b)$ satisfies an evolution equation like (20), but with $(f_f^\star/\rho_f - \mathrm{d}\omega_f/\mathrm{d}t, \omega_f\cdot n, \omega_b - \mathrm{d}\omega_b/\mathrm{d}t, f_b/\rho_b - \mathrm{d}\omega_b/\mathrm{d}t)$ as second member and $\bar{u}_g^0 := u_g^0 - (\omega_f(0), 0, \omega_b(0), \omega_b(0))$ as initial data, so that the additional assumptions $\bar{u}_g^0 \in \mathrm{dom}(A_g)$ and $(f_f, f_b) \in W^{1,\infty}(0,T; L^2(\Omega)^3 \times L^2(B)^3)$ imply existence and uniqueness of u_g in $C^1([0,T]; H_g) \cap C^0([0,T]; \mathrm{dom}(A_g))$.

3.4. Superposed given rigid motion

We assume that the given motion of the frame attached to the body is such that the concept of mean reference configuration is meaningful. In that case, the small motions around this configuration may be described by:

$$\frac{\mathrm{d}u_g}{\mathrm{d}t} + A_g ug \ni (f_f^\star/\rho_f, 0, 0, f_b/\rho_b) + L(t)u_g + c(t), \quad u_g(0) = u_g^0 \tag{29}$$

where the element c of H_g and the element L of the space $\mathcal{L}(H_g)$ of bounded linear operators on H_g account for the additional acceleration terms in (1), (7). If we assume that the given motion of the frame is of class $W^{2,\infty}(0,T)$ then $L \in W^{2,\infty}(0,T;\mathcal{L}(H_g))$ and $c \in W^{1,\infty}(0,T;H_g)$, thus existence and uniqueness of a solution of (28) is clear [3].

References

[1] M.B. Orazov, Localization of the spectrum in the problem of normal oscillations of an elastic shell filled with a viscous incompressible fluid, Comput. Math. Math. Phys. (USSR) 25 (1985) 52–58.
[2] C. Conca, A. Osses, J. Planchard, Added mass and damping in fluid–structure interaction, Comput. Methods Appl. Mech. Engrg. 146 (1997) 387–405.
[3] H. Brezis, Opérateurs maximaux-monotones et semi-groupes non-linéaires de contraction, North–Holland, 1973.
[4] R. Temam, Problèmes mathématiques en plasticité, Gauthier-Villars, 1983.
[5] H.F. Trotter, Approximation of semi-groups of operators, Pacific J. Math. 28 (1958) 887–919.