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Data completion via an energy error functional

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Abstract

We propose, in this Note, a new procedure for data completion based on the minimization of an energy like error functional. The efficiency of the proposed method is illustrated by a thermostatic application. ***To cite this article: S. Andrieux et al., C. R. Mecanique 333 (2005).***

Résumé

Complétion de données via une fonctionnelle d'erreur énergétique. Nous proposons, dans ce travail, une nouvelle méthode de reconstruction de données basée sur la minimisation d'une fonctionnelle type écart en énergie. L'éfficacité de la méthode est illustrée par une application issue de la thermostatique. ***Pour citer cet article : S. Andrieux et al., C. R. Mecanique 333 (2005).***

Keywords: Heat transfer; Data completion; Inverse problem; Cauchy problem

Mots-clés : Transferts thermiques ; Complétion de données ; Problème inverse ; Problème de Cauchy

1. Introduction

We consider in this work the problem of recovering lacking data on some part of the boundary of a domain from overspecified boundary data on the remaining part of the boundary. This kind of problem may occur very often in engineering sciences. The reconstruction of physical variables from lacking data is highly useful in many industrial

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processes. The more common problem, borrowed from thermostatics, consists in recovering the temperature in a given domain when the temperature distribution and the heat flux along the accessible region of the boundary are known. For the reader's convenience we shall be presenting the issue in the framework of the thermostatic case, which is similar to the electrostatic case encountered in electric impedance tomography. Note that this procedure is obviously extendable to elastostatics.

Given a flux Φ and the corresponding temperature T on Γ_c, one wants to recover the corresponding flux and temperature on the remaining part of the boundary Γ_i, where Γ_c and Γ_i constitute a partition of the whole boundary $\partial\Omega$. The problem can therefore be set out as follows: find (φ, t) on Γ_i such that:

$$\begin{cases} \nabla\cdot k\nabla u = 0 & \text{in } \Omega \\ k\nabla u\cdot n = \Phi & \text{on } \Gamma_c \\ u = T & \text{on } \Gamma_c \end{cases} \tag{1}$$

This problem has been known, since Hadamard [1], to be illposed in the sense that the dependence of u on the data (Φ, T) is known to be not continuous. We propose, in this Note, to reconstruct the lacking data using the energy error function introduced in [2] in the framework of parameter identification (see also [6] and [8]).

2. Data completion

Observe that, when the complete data are available on Γ, we have an overspecified boundary value problem:

$$\begin{cases} \nabla\cdot k\nabla u = 0 & \text{in } \Omega \\ u = T, \quad k\nabla u\cdot n = \phi & \text{on } \Gamma_c \\ u = t, \quad k\nabla u\cdot n = \varphi & \text{on } \Gamma_i \end{cases} \tag{2}$$

The approach in the error functional method is to consider, for a given pair (η, τ), the following two mixed problems whose solutions are denoted by u_1 and u_2:

$$\begin{cases} \nabla\cdot k\nabla u_1 = 0 & \text{in } \Omega \\ u_1 = T & \text{on } \Gamma_c \\ k\nabla u_1\cdot n = \eta & \text{on } \Gamma_i \end{cases} \tag{3}$$

$$\begin{cases} \nabla\cdot k\nabla u_2 = 0 & \text{in } \Omega \\ u_2 = \tau & \text{on } \Gamma_i \\ k\nabla u_2\cdot n = \Phi & \text{on } \Gamma_c \end{cases} \tag{4}$$

and to build an error functional on the pair (η, τ) by using an energy norm for the comparison of the fields u_1 and u_2. These fields are obviously equal only when the pair (η, τ) meets the real data (φ, t) on the boundary Γ_i. We propose then to solve the data completion problem via the following minimization:

$$\begin{cases} (\varphi, t) = \arg\min KV(\eta,\tau) \equiv \arg\min \int_\Omega (k\nabla u_1 - k\nabla u_2)\cdot(\nabla u_1 - \nabla u_2) \\ \eta \in H^{-1/2}(\Gamma_i),\ \tau \in H^{1/2}(\Gamma_i),\ u_1 \text{ and } u_2 \text{ being the solution of (3) and (4)} \\ \text{(here and from now on } H^{-1/2}(\Gamma_i) \text{ denotes the topological dual of } H^{1/2}(\Gamma_i)) \end{cases} \tag{5}$$

Using the properties of the u_i fields, it is straightforward to derive an alternative expression of the *KV* functional:

$$KV(\eta,\tau) = \int_{\Gamma_i} (\eta - k\nabla u_2\cdot n)(u_1 - \tau) + \int_{\Gamma_c} (k\nabla u_1\cdot n - \Phi)(T - u_2) \tag{6}$$

Furthermore, noticing that by the superposition principle the u_i fields are affine with respect to the pair (η, τ), it is easy to evaluate the gradient of the error functional:

$$\begin{cases} \dfrac{\partial KV}{\partial \eta}(\psi) = \displaystyle\int_{\Gamma_i} \big(u_1(\eta) - \tau\big)\psi \, \mathrm{d}\Gamma_i + \int_{\Gamma_c} k \frac{\partial u_1^*(\psi)}{\partial n} \big(T - u_2(\tau)\big) \, \mathrm{d}\Gamma_c \\ \text{for all } \ \psi \text{ in } H^{-1/2}(\Gamma_i) \end{cases} \tag{7}$$

$$\begin{cases} \dfrac{\partial KV}{\partial \tau}(h) = \displaystyle\int_{\Gamma_i} \left[k \frac{\partial u_2}{\partial n}(\tau) - \eta\right] h + \int_{\Gamma_c} \left[\Phi - k \frac{\partial u_1}{\partial n}(\eta)\right] u_2^*(h) \\ \text{for all } \ h \text{ in } H^{1/2}(\Gamma_i) \end{cases} \tag{8}$$

where u_1^* and u_2^* are solutions to:

$$\begin{cases} \nabla \cdot k \nabla u_1^* = 0 & \text{in } \Omega \\ u_1^* = 0 & \text{on } \Gamma_c \\ k \nabla u_1^* \cdot n = \psi & \text{on } \Gamma_i \end{cases} \qquad \begin{cases} \nabla \cdot k \nabla u_2^* = 0 & \text{in } \Omega \\ u_2^* = h & \text{on } \Gamma_i \\ k \nabla u_2^* \cdot n = 0 & \text{on } \Gamma_c \end{cases} \tag{9}$$

However, the components of the gradient can be computed in a more efficient way using the adjoint method, which allows one to evaluate the gradient in any direction using only the determination of two adjoint fields v_1 and v_2:

$$\begin{cases} \dfrac{\partial KV(\eta, \tau)}{\partial \eta}(\psi) = -\displaystyle\int_{\Gamma_i} 2 v_1 \psi \\ \dfrac{\partial KV(\eta, \tau)}{\partial \tau}(h) = -\displaystyle\int_{\Gamma_i} 2(\eta - k \nabla u_2 \cdot n - k \nabla v_2 \cdot n) h \end{cases} \tag{10}$$

with:

$$\begin{cases} \nabla \cdot k \nabla v_1 = 0 & \text{in } \Omega \\ v_1 = 0 & \text{on } \Gamma_c \\ k \nabla v_1 \cdot n = k \nabla u_2 \cdot n - \eta & \text{on } \Gamma_i \end{cases} \tag{11}$$

$$\begin{cases} \nabla \cdot k \nabla v_2 = 0 & \text{in } \Omega \\ v_2 = 0 & \text{on } \Gamma_i \\ k \nabla v_2 \cdot n = k \nabla u_1 \cdot n - \Phi & \text{on } \Gamma_c \end{cases} \tag{12}$$

Remark 1.

(i) The energy-like error functional reaches its minimum at $u_1 = u_2 + Cte = u$, where u is the unique solution to our data recovering problem.

(ii) The energy-like error functional is convex, positive with a minimum equal to zero.

(iii) Observe that the components of the KV functional gradients involve integrals on the whole boundary of the domain. Furthermore the two fields u_1 and u_2 are truly coupled via the energy functional as illustrated by (7) and (8).

(iv) The alternative form of the error functional (6), makes possible the comparison of the proposed approach with more classical least square error methods: here both Neumann and Dirichlet errors are naturally mixed and no dimensional factor is needed for that purpose. The Dirichlet error is weighted by the Neumann error.

(v) The method proposed in this work is related to the one introduced by Koslov et al. in [3] and widely numerically tested (see [4] and references therein). In Koslov's work the authors proposed a method for solving the problem under consideration based on an alternating iterative procedure. This procedure consists in obtaining successive solutions of wellposed mixed boundary value problems for the original equation. The method has been proved to be convergent. Notice that our approach generalizes that of Koslov in so far as the alternating method can be viewed as the energy-like error functional minimization by a relaxation procedure in the φ and t directions.

3. Numerical experiments

To explore the efficiency of the proposed data matching procedure, we start with the reconstruction of a temperature field in a pipeline of infinite length. We assume that the temperature does not depend on the longitudinal coordinate. We deal, therefore with a bidimensional problem.

This application may arise in several industrial processes. We describe an example borrowed from fluid mechanics consisting in the evaluation of the heat data at the internal wall of a pipeline. Such data may be necessary for the simulation of the heat transfer taking place in a fluid flowing within the pipeline. The knowledge of this temperature is necessary for controlling the safety of the material: a stratified inner fluid may generate mechanical stresses, which may cause damage such as cracks. From the experimental viewpoint, thermocouples are located at the external boundary of the pipe and the heat exchange conditions with the environment are known. The mathematical setting corresponds to a Cauchy problem with given data at the external wall. We consider four numerical trials corresponding to three analytical cases, an isotropic and two anisotropic materials; the fourth experiment is devoted to a practical case corresponding to stratified inner fluid. The cross section Ω is an annular thick domain with radii $r_1 = 1$ and $r_2 = 0.5$. The internal boundary is denoted Γ_i on which the data are lacking and the external one Γ_c on which the data are overspecified. For the first analytical example the data are provided by the harmonic function $u = e^x \cos(y)$.

The minimization of the error functional is achieved by ensuring the optimality condition of first order. All the calculations are run under *Matlab Software* environnement using the finite element formulation [5].

Fig. 1 shows the distribution of the reconstructed temperature u_1 on Ω as well as the absolute error $(u_{\text{exact}} - u_1)$. Fig. 2 plots the reconstructed temperatures u_1, u_2 and the exact temperature u_{exact} on Γ_i, as well as the exact flux and that reconstructed. Note that in the two cases the reconstructed fields are in close agreement with the exact ones. The proposed approach also works for orthotropic materials, where the conductivity k is a tensor field no longer proportional to the identity tensor of order two:

$$k = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} \tag{13}$$

Figs. 3 and 4 show the reconstructed temperature and the heat flux for $\epsilon = 0.1$. Figs. 5 and 6 show the reconstructed temperature and the heat flux for $\epsilon = 0.01$. Notice, here again, that the reconstruction remains very satisfactory. The data used in the fourth example are generated by the finite element computation of the above problem with Dirichlet data: $T = 20\,^\circ\text{C}$ on Γ_c, $T = 50\,^\circ\text{C}$ on the lower half circle of Γ_i and $T = 250\,^\circ\text{C}$ on the upper half one. Fig. 7 shows the reconstructed temperature and flux, and even in this case where the data is singular, they are in good agreement with the actual ones. The data matching procedure consists in a minimization algorithm

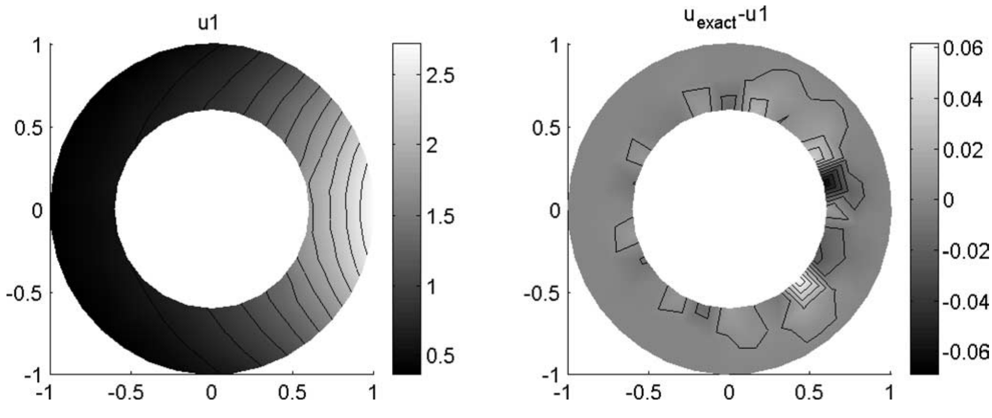

Fig. 1. Reconstructed temperature u_1 and $u_{\text{exact}} - u_1$ on Ω.

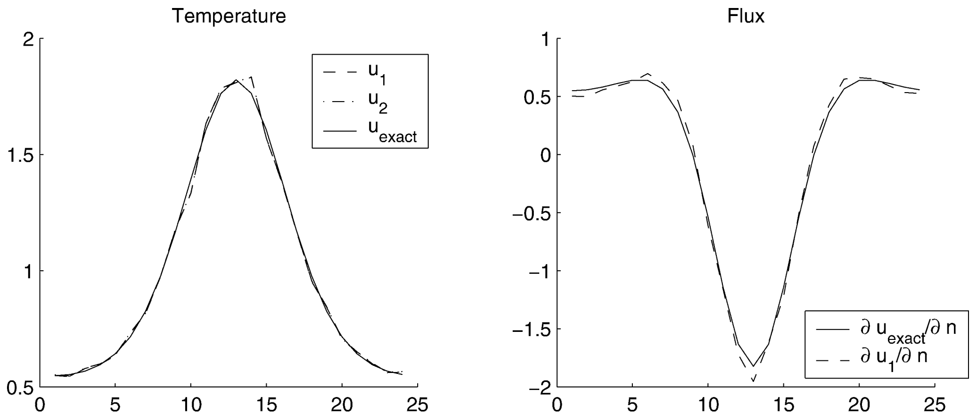

Fig. 2. Reconstructed temperature and flux on Γ_i.

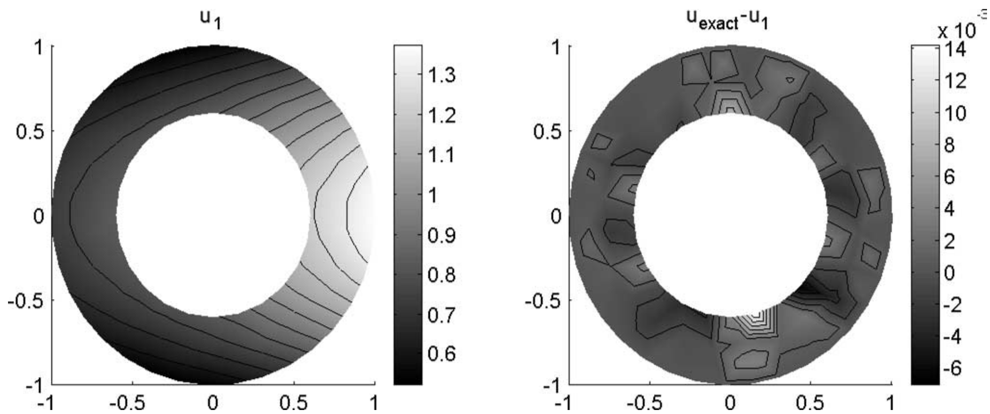

Fig. 3. Reconstructed temperature u_1 and $u_{\text{exact}} - u_1$ on Ω for $\epsilon = 0.1$.

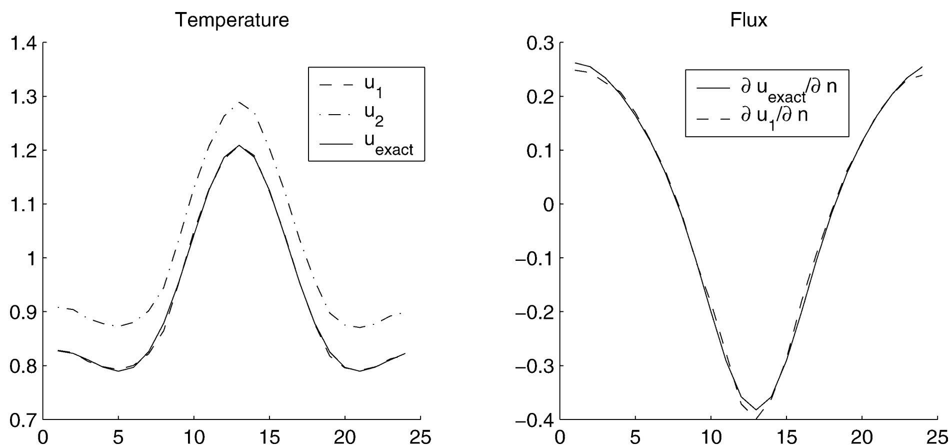

Fig. 4. Reconstructed temperature and flux on Γ_i for $\epsilon = 0.1$.

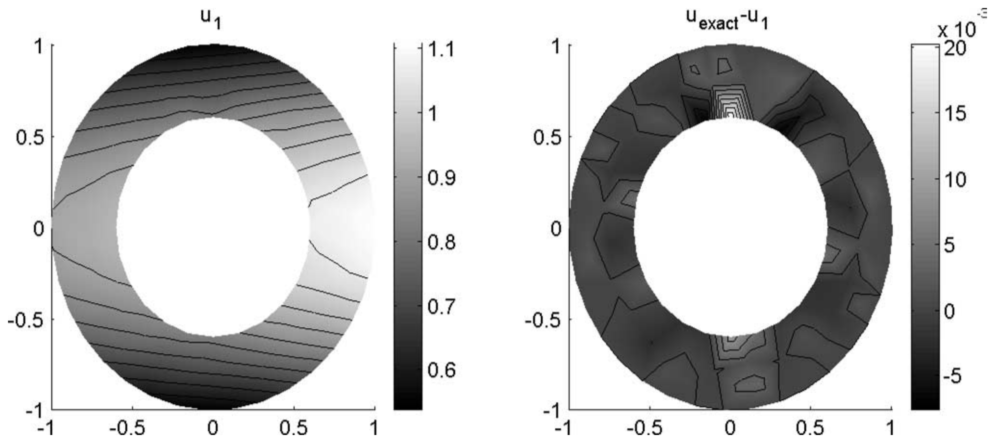

Fig. 5. Reconstructed temperature u_1 and $u_{\text{exact}} - u_1$ on Ω for $\epsilon = 0.01$.

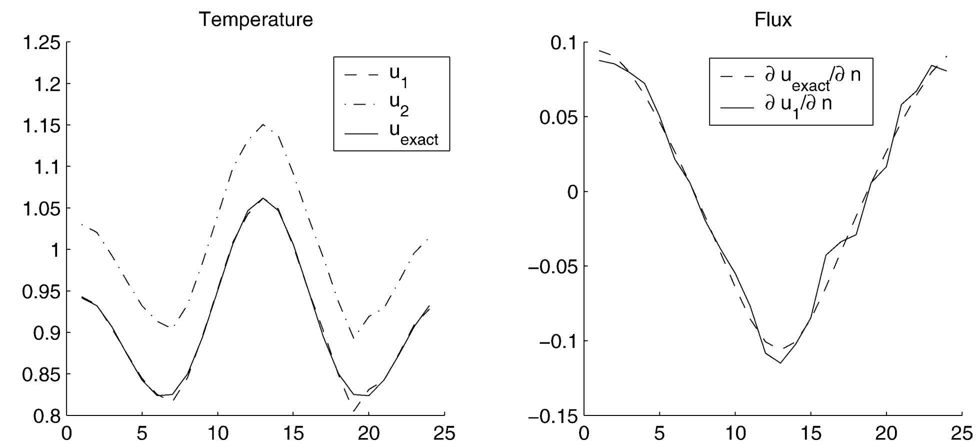

Fig. 6. Reconstructed temperature and flux on Γ_i for $\epsilon = 0.01$.

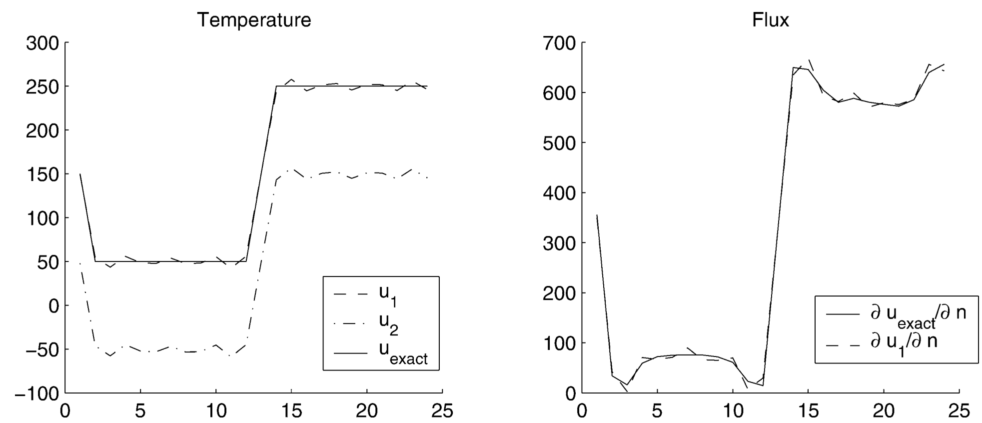

Fig. 7. Reconstructed temperature and flux on Γ_i.

(Conjugate Gradient Method) where the gradient is evaluated by the discrete adjoint state method. It requires four computations of a direct solution at each iteration. In the examples shown here, the number of iterations to achieve convergence ranges from 4 to 8, which correspond to 16 to 32 resolutions of the direct problem.

4. Comments

We have proposed in this Note a method for data matching based on the minimization of an energy error functional. This method is general, it has wide applications ranging from the bioelectrical field to mechanical engineering. We tested successfully the matching method in the case of temperature and heat flux recovery. This procedure compares here very favorably with iterative data matching existing method.

A numerical exploration of the method, together with applications to the non-destructive inspection of materials, seeking the quantitative determination of internal flaws (s.t. crack corrosion) is the aim of the forthcoming paper [7].

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