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Second-order estimates for nonlinear isotropic composites with spherical pores and rigid particles

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Abstract

The 'second-order' nonlinear homogenization method (Ponte Castañeda, J. Mech. Phys. Solids 50 (2002) 737–757) is used to generate estimates of the Hashin–Shtrikman-type for the effective behavior of viscoplastic materials with isotropically distributed spherical pores or rigid particles. In the limiting case of an ideally plastic matrix with a dilute concentration of pores, the resulting estimates were found to exhibit a linear dependence on the porosity when the material is subjected to axisymmetric shear, but this dependence becomes singular for simple shear. In the process of this work, an alternative prescription for certain reference tensors used in the method is proposed, and shown to lead to more consistent estimates for the effective behavior than the earlier prescription. ***To cite this article: M. Idiart, P. Ponte Castañeda, C. R. Mecanique 333 (2005).***
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Résumé

Estimations du comportement effectif des composites isotropes non linéaires contenant des pores et des particules rigides sphériques. On utilise la méthode d'homogénéisation non linéaire proposée par Ponte Castañeda (J. Mech. Phys. Solids 50 (2002) 737–757), dite du second ordre, pour générer des estimations du type Hashin–Shtrikman pour le comportement effectif des matériaux viscoplastiques contenant des pores et des particules rigides sphériques. Dans le cas limite d'une matrice parfaitement plastique à faible concentration de pores, les estimations trouvées présentent une dépendance linéaire de la porosité sous un chargement de cisaillement axisymmétrique ; cependant cette dépendance devient singulière sous cisaillement simple. Lors de ce travail, certaines limites de la formulation de la méthode initialement proposée dans la référence ci-dessus ont été identifiées. En conséquence, des alternatives ont été testées. ***Pour citer cet article : M. Idiart, P. Ponte Castañeda, C. R. Mecanique 333 (2005).***
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Keywords: Porous media; Homogenization; Nonlinear behavior; Ideal plasticity

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Mots-clés : Milieux poreux ; Homogénéisation ; Comportement non linéaire ; Plasticité parfaite

1. Introduction

Much effort is still being devoted to developing methods capable of accurately estimating the effective behavior of nonlinear heterogeneous media [1]. A fairly general homogenization method has been introduced by Ponte Castañeda [2], which delivers estimates that are exact to second-order in the heterogeneity contrast and that do not violate rigorous bounds. This 'second-order' method, based on a variational principle, reduces to finding a set of constants that renders a certain functional stationary. To simplify the calculations, it was proposed, as an approximation in [2], to replace some of these (full) stationarity conditions by a set of partial stationarity conditions. In this Note, the method is used to generate estimates for the effective behavior of nonlinear composites with spherical pores or rigid particles. In the process of this work, some limitations of the approximation mentioned above were identified, and some alternatives were evaluated.

We consider composite materials made of N different homogeneous constituents, or *phases*, which are assumed to be *randomly* distributed in a specimen occupying a volume Ω, at a length scale that is much smaller than the size of Ω and the scale of variation of the loading conditions. The constitutive behavior of each phase is characterized by a *convex* potential function $u^{(r)}$ $(r = 1, \ldots, N)$, such that the stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ tensors are related by

$$\boldsymbol{\varepsilon} = \frac{\partial u^{(r)}}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}) \tag{1}$$

This constitutive relation can be used within the context of the deformation theory of plasticity, where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ represent the infinitesimal stress and strain, respectively. Relation (1) applies equally well to viscoplastic materials, in which case $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ represent the Cauchy stress and Eulerian strain rate, respectively.

We are concerned with the problem of finding the effective behavior of the composite, which is defined as the relation between the average stress $\bar{\boldsymbol{\sigma}} = \langle \boldsymbol{\sigma} \rangle$ and the average strain $\bar{\boldsymbol{\varepsilon}} = \langle \boldsymbol{\varepsilon} \rangle$, and can also be characterized [1] by an effective potential $\widetilde{U}$, such that

$$\bar{\boldsymbol{\varepsilon}} = \frac{\partial \widetilde{U}}{\partial \bar{\boldsymbol{\varepsilon}}}(\bar{\boldsymbol{\sigma}}), \qquad \widetilde{U}(\bar{\boldsymbol{\sigma}}) = \min_{\boldsymbol{\sigma} \in \mathcal{K}(\bar{\boldsymbol{\sigma}})} \sum_{r=1}^{N} c^{(r)} \big\langle u^{(r)}(\boldsymbol{\sigma}) \big\rangle^{(r)} \tag{2}$$

Here, $\langle \cdot \rangle$ and $\langle \cdot \rangle^{(r)}$ denote the volume averages over the composite (Ω) and over phase r ($\Omega^{(r)}$), respectively, $c^{(r)}$ is the volume fraction of phase r, and $\mathcal{K}(\bar{\boldsymbol{\sigma}}) = \{\boldsymbol{\sigma},\ \operatorname{div} \boldsymbol{\sigma} = \mathbf{0}$ in Ω, $\langle \boldsymbol{\sigma} \rangle = \bar{\boldsymbol{\sigma}}\}$ is the set of statically admissible stresses. Thus, the problem of estimating the effective behavior of the composite is equivalent to that of estimating the function $\widetilde{U}$.

2. Second-order homogenization estimates

The second-order method [2] delivers the following estimate for the effective potential of a general N-phase composite:

$$\widetilde{U}(\bar{\boldsymbol{\sigma}}) = \operatorname*{stat}_{\mathbf{M}_0^{(s)}} \left\{ \widetilde{U}_T(\bar{\boldsymbol{\sigma}}; \check{\boldsymbol{\sigma}}^{(s)}, \mathbf{M}_0^{(s)}) - \sum_{r=1}^{N} c^{(r)} V^{(r)}(\check{\boldsymbol{\sigma}}^{(r)}, \mathbf{M}_0^{(r)}) \right\} \tag{3}$$

where the stat*ionary* operation consists in setting the partial derivative of the argument with respect to the variable equal to zero. In this expression, $\widetilde{U}_T$ is the effective potential of a *linear* comparison composite (LCC) with the same microstructure as the nonlinear composite, and phase potentials $u_T^{(r)}$ given by

$$u_T^{(r)}(\boldsymbol{\sigma};\check{\boldsymbol{\sigma}}^{(r)},\mathbf{M}_0^{(r)}) = u^{(r)}(\check{\boldsymbol{\sigma}}^{(r)}) + \frac{\partial u^{(r)}}{\partial\boldsymbol{\sigma}}(\check{\boldsymbol{\sigma}}^{(r)})\cdot(\boldsymbol{\sigma}-\check{\boldsymbol{\sigma}}^{(r)}) + \frac{1}{2}(\boldsymbol{\sigma}-\check{\boldsymbol{\sigma}}^{(r)})\cdot\mathbf{M}_0^{(r)}(\boldsymbol{\sigma}-\check{\boldsymbol{\sigma}}^{(r)}) \tag{4}$$

where the $\check{\boldsymbol{\sigma}}^{(r)}$ are uniform reference stresses, and $\mathbf{M}_0^{(r)}$, symmetric, constant, fourth-order tensors (of compliances). The 'error functions' $V^{(r)}$ are defined as

$$V^{(r)}(\check{\boldsymbol{\sigma}}^{(r)},\mathbf{M}_0^{(r)}) = \operatorname*{stat}_{\hat{\boldsymbol{\sigma}}^{(r)}}\bigl\{u_T^{(r)}(\hat{\boldsymbol{\sigma}}^{(r)};\check{\boldsymbol{\sigma}}^{(r)},\mathbf{M}_0^{(r)}) - u^{(r)}(\hat{\boldsymbol{\sigma}}^{(r)})\bigr\} \tag{5}$$

where the $\hat{\boldsymbol{\sigma}}^{(r)}$ are uniform (stress) tensors in each phase, which are determined by the stationary condition in (5):

$$\frac{\partial u^{(r)}}{\partial\boldsymbol{\sigma}}(\hat{\boldsymbol{\sigma}}^{(r)}) - \frac{\partial u^{(r)}}{\partial\boldsymbol{\sigma}}(\check{\boldsymbol{\sigma}}^{(r)}) = \mathbf{M}_0^{(r)}\,(\hat{\boldsymbol{\sigma}}^{(r)}-\check{\boldsymbol{\sigma}}^{(r)}) \tag{6}$$

Note that the compliance tensors $\mathbf{M}_0^{(r)}$ correspond to 'generalized secant' approximations to the nonlinear stress-strain relations.

In turn, the stationary operation in (3) leads to additional conditions in each phase r, given by

$$(\hat{\boldsymbol{\sigma}}^{(r)}-\check{\boldsymbol{\sigma}}^{(r)})\otimes(\hat{\boldsymbol{\sigma}}^{(r)}-\check{\boldsymbol{\sigma}}^{(r)}) = \frac{2}{c^{(r)}}\frac{\partial\widetilde{U}_T}{\partial\mathbf{M}_0^{(r)}} = \bigl\langle(\boldsymbol{\sigma}-\check{\boldsymbol{\sigma}}^{(r)})\otimes(\boldsymbol{\sigma}-\check{\boldsymbol{\sigma}}^{(r)})\bigr\rangle^{(r)} \tag{7}$$

which relate the variables $\hat{\boldsymbol{\sigma}}^{(r)}$ to the variables $\check{\boldsymbol{\sigma}}^{(r)}$ and $\mathbf{M}_0^{(r)}$ through the (intraphase) field fluctuations (about the references $\check{\boldsymbol{\sigma}}^{(r)}$) in the LCC.

Then, using the fact that (3) and (5) are stationary with respect to the tensors $\mathbf{M}_0^{(r)}$ and $\hat{\boldsymbol{\sigma}}^{(r)}$, respectively, we can rewrite the estimate (3) as:

$$\widetilde{U}(\bar{\boldsymbol{\sigma}}) = \sum_{r=1}^{N} c^{(r)}\left[u^{(r)}(\hat{\boldsymbol{\sigma}}^{(r)}) - \frac{\partial u^{(r)}}{\partial\boldsymbol{\sigma}}(\check{\boldsymbol{\sigma}}^{(r)})\cdot(\hat{\boldsymbol{\sigma}}^{(r)}-\bar{\boldsymbol{\sigma}}^{(r)})\right] \tag{8}$$

where $\bar{\boldsymbol{\sigma}}^{(r)} = \langle\boldsymbol{\sigma}\rangle^{(r)}$ is the average of the stress over phase r in the LCC. Eqs. (6) and (7) determine the variables $\hat{\boldsymbol{\sigma}}^{(r)}$ and $\mathbf{M}_0^{(r)}$ for any choice of the reference tensors $\check{\boldsymbol{\sigma}}^{(r)}$, which remain to be specified.

Completely analogous expressions may be developed [2] starting from the dual formulation for the strain potentials $w^{(r)}$, which are the Legendre transforms of $u^{(r)}$ (so that $\boldsymbol{\sigma} = \partial w^{(r)}/\partial\boldsymbol{\varepsilon}(\boldsymbol{\varepsilon})$). This formulation involves a LCC with phase potentials $w_T^{(r)}$, given by second-order Taylor approximations to $w^{(r)}$ of the same form as (4), in terms of reference strains $\check{\boldsymbol{\varepsilon}}^{(r)}$ and tensors of moduli $\mathbf{L}_0^{(r)}$, and generates the following estimate for the effective strain potential

$$\widetilde{W}(\bar{\boldsymbol{\varepsilon}}) = \sum_{r=1}^{N} c^{(r)}\left[w^{(r)}(\hat{\boldsymbol{\varepsilon}}^{(r)}) - \frac{\partial w^{(r)}}{\partial\boldsymbol{\varepsilon}}(\check{\boldsymbol{\varepsilon}}^{(r)})\cdot(\hat{\boldsymbol{\varepsilon}}^{(r)}-\bar{\boldsymbol{\varepsilon}}^{(r)})\right] \tag{9}$$

where $\bar{\boldsymbol{\varepsilon}}^{(r)} = \langle\boldsymbol{\varepsilon}\rangle^{(r)}$ in the LCC, and the tensors $\hat{\boldsymbol{\varepsilon}}^{(r)}$ and $\mathbf{L}_0^{(r)}$ depend on the reference tensors $\check{\boldsymbol{\varepsilon}}^{(r)}$ and the second moments of the strain fluctuations (in the LCC) through equations analogous to (6) and (7).

Choice of reference tensors. Ideally, the estimates (8) and (9) for $\widetilde{U}$ and $\widetilde{W}$ should be Legendre duals of each other (i.e., no *duality gap*). These estimates would indeed satisfy this requirement if they were stationary with respect to the reference tensors $\check{\boldsymbol{\sigma}}^{(r)}$ and $\check{\boldsymbol{\varepsilon}}^{(r)}$, respectively (see Section 6 in [2] for details). In addition, this prescription for the references would lead to potentials $u_T^{(r)}$ and $w_T^{(r)}$ that would also be Legendre duals of each other, and the effective stress-strain relation of this LCC would coincide with that obtained by differentiation of (8) and (9). Unfortunately, it has not yet been possible to find a satisfactory solution to the resulting system of equations.

For this reason, it was suggested, as an approximation in [2], the use of the phase averages in the LCC as references, that is

$$\check{\boldsymbol{\sigma}}^{(r)} = \bar{\boldsymbol{\sigma}}^{(r)} \quad \text{and} \quad \check{\boldsymbol{\varepsilon}}^{(r)} = \bar{\boldsymbol{\varepsilon}}^{(r)} \tag{10}$$

This choice is physically appealing, for the right-hand side in (7) becomes the covariance tensor of the field fluctuations in phase r. Besides, this choice can be shown to render $\widetilde{U}_T$ and $\widetilde{W}_T$ stationary, thus *partially* satisfying the stationarity condition with respect to the references (see expression (3)). However, this approximation leads to estimates for $\widetilde{U}$ and $\widetilde{W}$ that are *not* Legendre duals of each other, i.e., there is a *duality gap*. But it should be noted that the phase potentials $u_T^{(r)}$ and $w_T^{(r)}$ of the LCC's are still Legendre duals of each other [2], provided $\bar{\boldsymbol{\sigma}}$ in (8) and $\bar{\boldsymbol{\varepsilon}}$ in (9) are taken to be related by the effective stress-strain relation of the LCC. As will be seen in the next section, the choice (10) can lead to inconsistencies in certain cases, and therefore, other prescriptions need to be considered.

A simple alternative consists in the choices

$$\check{\boldsymbol{\sigma}}^{(r)} = \bar{\boldsymbol{\sigma}} \quad \text{and} \quad \check{\boldsymbol{\varepsilon}}^{(r)} = \frac{\partial u^{(r)}}{\partial \boldsymbol{\sigma}}(\bar{\boldsymbol{\sigma}}) \tag{11}$$

where $\bar{\boldsymbol{\sigma}}$ is the overall stress in the LCC. Note that the requirement $(11)_2$ implies that the $\check{\boldsymbol{\varepsilon}}^{(r)}$ are not equal to $\bar{\boldsymbol{\varepsilon}}$, but it does imply that $u_T^{(r)}$ and $w_T^{(r)}$ remain Legendre duals of each other (in the sense mentioned above).

For a given choice of reference tensors, the estimates (8) and (9) require the computation of the effective potentials $\widetilde{U}_T$ and $\widetilde{W}_T$, which can be obtained using any *linear* homogenization method appropriate for composites with local potentials $u_T^{(r)}$ and $w_T^{(r)}$, and the same microstructure as the nonlinear composite. It can be verified that expressions (8) and (9), together with (10), as well as with (11), are exact to second order in the heterogeneity contrast, and therefore in agreement with the small-contrast expansion of Suquet and Ponte Castañeda [3]. It should be mentioned that Lahellec and Suquet [4] have provided an alternative formulation of the second-order method, which has some advantages relative to the original formulation [6], but still does not resolve the duality problem.

Choice of compliance tensors. The left-hand side of relation (7) is a rank-one tensor, whereas the right-hand side is, in general, of full rank. Therefore, equality cannot be enforced for all components of the tensorial relation, and only certain traces of it can be used. Consequently, the number of *independent* components of the tensors $\mathbf{M}_0^{(r)}$ can be at most equal to the number of components of $\hat{\boldsymbol{\sigma}}^{(r)}$. Thus, the estimates (8) cannot be fully stationary with respect to the variables $\mathbf{M}_0^{(r)}$.

For *isotropic*, incompressible phases with potentials depending only on the von Mises equivalent stress σ_e, it was proposed in Ref. [2] the use of *anisotropic*, incompressible tensors of the form

$$\mathbf{M}_0^{(r)} = (2\lambda_0^{(r)})^{-1}\mathbf{E}^{(r)} + (2\mu_0^{(r)})^{-1}\mathbf{F}^{(r)} \tag{12}$$

where $\mathbf{E}^{(r)}$ and $\mathbf{F}^{(r)}$ are projection tensors with principal axes aligned with the reference stresses $\check{\boldsymbol{\sigma}}^{(r)}$. Then, expression (7) reduces to

$$\hat{\sigma}_{\parallel}^{(r)} = \check{\sigma}_e^{(r)} \pm \sqrt{\frac{3}{2}\big\langle(\boldsymbol{\sigma} - \check{\boldsymbol{\sigma}}^{(r)}) \cdot \mathbf{E}^{(r)}(\boldsymbol{\sigma} - \check{\boldsymbol{\sigma}}^{(r)})\big\rangle^{(r)}}, \qquad \hat{\sigma}_{\perp}^{(r)} = \pm\sqrt{\frac{3}{2}\langle\boldsymbol{\sigma} \cdot \mathbf{F}^{(r)}\boldsymbol{\sigma}\rangle^{(r)}} \tag{13}$$

where $\hat{\sigma}_{\parallel}^{(r)} = (\frac{3}{2}\hat{\boldsymbol{\sigma}}^{(r)} \cdot \mathbf{E}^{(r)}\hat{\boldsymbol{\sigma}}^{(r)})^{1/2}$ and $\hat{\sigma}_{\perp}^{(r)} = (\frac{3}{2}\hat{\boldsymbol{\sigma}}^{(r)} \cdot \mathbf{F}^{(r)}\hat{\boldsymbol{\sigma}}^{(r)})^{1/2}$ are the 'parallel' and 'perpendicular' components of the traceless tensors $\hat{\boldsymbol{\sigma}}^{(r)}$, respectively. The sign of the square roots in (13) should be positive if $\check{\sigma}_e^{(r)} \leqslant \bar{\sigma}_e^{(r)}$, and negative otherwise, for consistency of (8) with the case of uniform fields (e.g., laminate, homogeneous limit). These same observations apply to the tensors $\mathbf{L}_0^{(r)}$ and $\hat{\boldsymbol{\varepsilon}}^{(r)}$ in the dual version.

3. Power-law composites

In this section we consider composite materials with phases characterized by isotropic, incompressible power-law potentials

$$u^{(r)}(\boldsymbol{\sigma}) = \frac{\varepsilon_0 \sigma_0^{(r)}}{1+n}\left(\frac{\sigma_e}{\sigma_0^{(r)}}\right)^{1+n}, \quad n = \frac{1}{m} \tag{14}$$

where $\sigma_0^{(r)}$ is the flow stress of phase r, m is such that $0 \leqslant m \leqslant 1$, ε_0 is a reference strain, and σ_e is the von Mises equivalent stress. Note that $m = 1$ and $m = 0$ correspond to linear and rigid-ideally plastic behaviors, respectively. For simplicity, we consider statistically isotropic microstructures, and phase potentials (14) with the same exponent m. It then follows that the effective potential can be written as

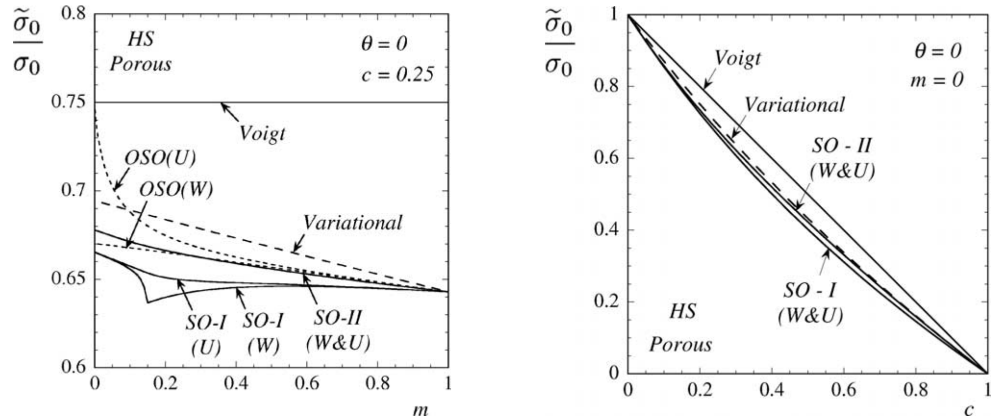
$$\widetilde{U}(\bar{\boldsymbol{\sigma}}) = \frac{\varepsilon_0 \tilde{\sigma}_0}{1+n}\left(\frac{\bar{\sigma}_e}{\tilde{\sigma}_0}\right)^{1+n} \tag{15}$$

where $\tilde{\sigma}_0$ is the effective flow stress, completely characterizing the effective behavior. In two-dimensional problems, such as transverse shear of a matrix with aligned fibers, $\tilde{\sigma}_0$ is a function of m, $\sigma_0^{(r)}$, and the volume fractions of the phases. In three dimensions, $\tilde{\sigma}_0$ also depends on the plastic phase angle θ, which in turn is related to the two invariants of the deviatoric stress $\bar{\boldsymbol{\sigma}}_d$ through $\cos(3\theta) = 4\det(\bar{\boldsymbol{\sigma}}_d)/\bar{\sigma}_e^3$.

The extreme cases of infinite contrast are of particular interest. The results given in the following subsections correspond to a matrix (phase 1) with flow stress $\sigma_0^{(1)} = \sigma_0$, with randomly distributed spherical pores or rigid particles (phase 2) at volume fraction $c^{(2)} = c$. Only the case of axisymmetric shear ($\theta = 0$) is considered in some detail. The Hashin–Shtrikman (HS) estimates of Willis [5] are used to estimate the effective behavior of the associated LCC. These estimates are known to be appropriate for (linear) particulate media at low to moderate concentrations, and are exact to second-order in the heterogeneity contrast. Both, the stress (U) and the strain (W) versions of the second-order (SO) estimates of the previous section are provided for two different choices of reference tensors. We denote by I the estimates associated with (10), whereas those associated with (11) are denoted by II. These estimates are compared with the 'original' HS second-order (OSO) estimates of Ref. [6], which do not make use of the field fluctuations in the linearization, as well as the corresponding 'variational' HS estimates of Ref. [7]. The latter are actually rigorous upper bounds for all other nonlinear HS estimates, and, in particular, for the second-order estimates. They all coincide, of course, for $m = 1$, where they reduce to the linear HS estimates. Also included for comparison purposes are the classical upper and lower bounds of Voigt and Reuss.

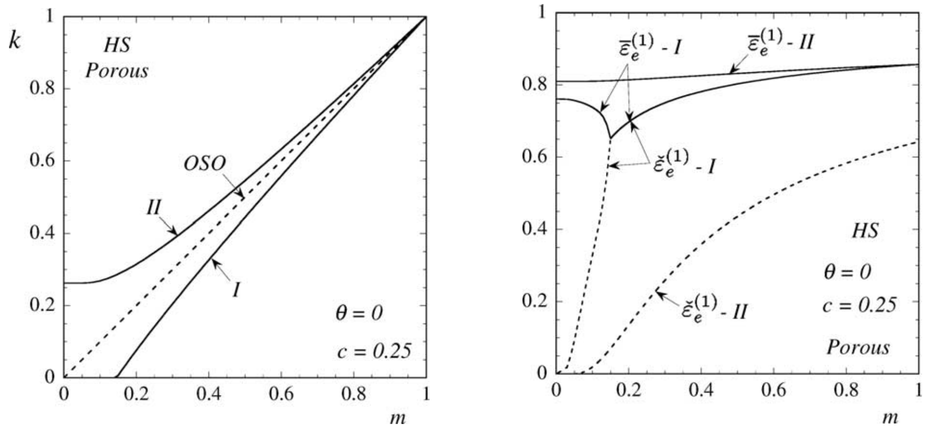
3.1. Porous materials

Fig. 1 provides upper bounds and estimates for the effective flow stress $\tilde{\sigma}_0$ for the porous case. We begin by noting that, unlike the OSO estimates (dashed lines), the SO-I estimates satisfy the HS variational bound (long-dashed lines) for all values of the nonlinearity exponent m (see Fig. 1(a)). Furthermore, the duality gap is found to vanish for $m = 0$, and it is negligible for most values of m, except in a small interval around $m^* \approx 0.15$. At this value of the nonlinearity exponent, the W-version presents a kink. This is related to the fact that, as will be explained shortly, the choice $(10)_2$ cannot be enforced for $m < m^*$ in this particular case. In contrast, both versions of the SO-II estimates are found to be smooth functions of the nonlinearity exponent, since choice (11) is consistent for all values of m. These estimates lie closer to the variational bound than the SO-I, still satisfying it for all m, and present a duality gap which is negligible for all m and even vanishes in the ideally-plastic limit ($m = 0$). It should be noted that the differences between the SO-I and SO-II estimates are not as significant as the enlarged scale in this figure might suggest. A fairer comparison is provided in Fig. 1(b), where estimates for the limiting case $m = 0$ are shown as a function of the concentration of pores c. The SO-II estimates are found to lie between the SO-I and the variational bound for all c, the differences being small. In fact, the SO-I and II estimates can be shown to agree



(a) (b)

Fig. 1. Upper bounds and estimates of the Hashin–Shtrikman (HS) type for the effective flow stress $\tilde{\sigma}_0$ of a power-law porous material subject to axisymmetric shear ($\theta = 0$): (a) as a function of the nonlinearity exponent m with a given concentration of pores ($c = 0.25$); (b) as a function of the pore concentration c with a rigid-ideally plastic matrix ($m = 0$).

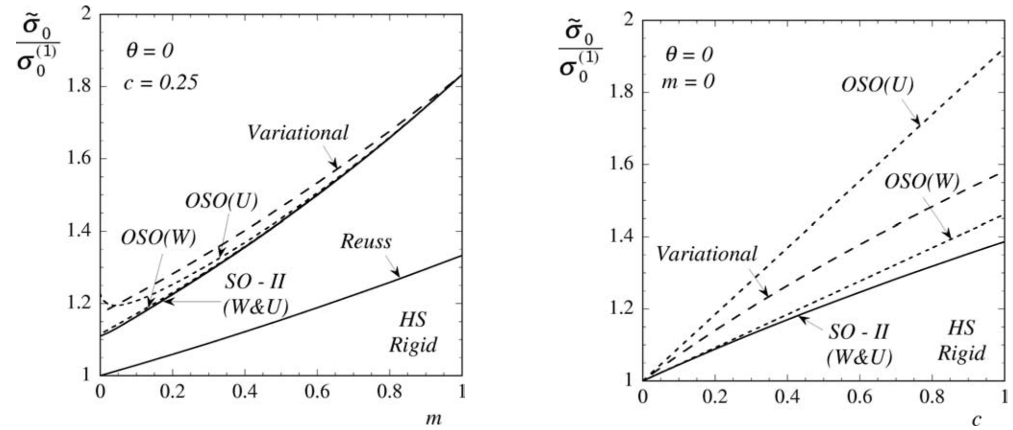


(a) (b)

Fig. 2. (a) Anisotropy ratio $k = \lambda_0/\mu_0$ of the matrix in the LCC; and (b) equivalent average strain $\bar{\varepsilon}_e^{(1)}$ (continuous lines) and reference $\check{\varepsilon}_e^{(1)}$ (dashed lines) in the matrix of the LCC, normalized by the equivalent applied strain $\bar{\varepsilon}_e$, as a function of the nonlinearity exponent m, for a power-law porous material with a given concentration of pores ($c = 0.25$), subject to axisymmetric shear ($\theta = 0$).

in the dilute limit, for any m, with the OSO(W) estimates, as given by the first-order expansion of expression (5.4) of [6] with $\theta = 0$, for small concentrations of pores.

Fig. 2(a) shows the 'anisotropy' ratio $k = \lambda_0/\mu_0$ of elastic moduli (see expression (12)) in the matrix of the LCCs associated with the second-order estimates of Fig. 1(a). The OSO estimates make use of a tangent compliance tensor, which for potentials (14) takes the form (12) with $k = m$, whereas the anisotropy of the more general compliance tensors used by the SO estimates depends not only on m but also on c. In the linear case ($m = 1$), these tensors are isotropic, so that $k = 1$, and as the nonlinearity increases they become progressively more anisotropic. The main observation in the context of this figure is that when prescription (10) is used, the associated k-I vanishes at a finite value m^* (already introduced in the context of Fig. 1(a)). In fact, for $m < m^*$, insisting on the prescription (10) for the references would lead to negative values of k, which is unacceptable since this implies a matrix with a negative definite compliance tensor in the LCC. The SO-I estimates provided in this Note were obtained by



(a) (b)

Fig. 3. Bounds and estimates of the Hashin–Shtrikman (HS) type for the effective flow stress $\tilde{\sigma}_0$ of a rigidly-reinforced power-law material subject to axisymmetric shear ($\theta = 0$): (a) as a function of the nonlinearity exponent m with a given concentration of particles ($c = 0.25$); (b) as a function of the particle concentration c for the case of a rigid-ideally plastic matrix ($m = 0$).

initially assuming an arbitrary $\check{\boldsymbol{\sigma}}^{(1)}$, with the corresponding reference strain given by $\check{\boldsymbol{\varepsilon}}^{(1)} = \partial u^{(1)}/\partial \boldsymbol{\sigma}(\check{\boldsymbol{\sigma}}^{(1)})$, and then taking the limit $\check{\boldsymbol{\sigma}}^{(1)} \to \bar{\boldsymbol{\sigma}}^{(1)}$. As can be seen in Fig. 2(b), the resulting $\check{\varepsilon}_e^{(1)}$-I and $\bar{\varepsilon}_e^{(1)}$-I coincide for $m \geqslant m^*$, in accordance with $(10)_2$, but for $m < m^*$ we have that $k = 0$ and the relation between $\check{\boldsymbol{\varepsilon}}^{(1)}$ and $\check{\boldsymbol{\sigma}}^{(1)}$ mentioned above no longer implies $(10)_2$. On the other hand, the alternative choice (11) leads to a well-behaved k-II that tends to some finite value, dependent on c, in the ideally-plastic limit. Moreover, $\bar{\boldsymbol{\varepsilon}}^{(1)}$-II is different from $\check{\boldsymbol{\varepsilon}}^{(1)}$-II for all values of m, and exhibits a smooth behavior even at high nonlinearities (see Fig. 2(b)).

In view of the smaller duality gap and the smoother behavior of the corresponding LCC, the prescription (11) is to be preferred to the earlier prescription (10). However, only comparisons with exact results will allow corroboration of this choice.

3.2. Rigidly-reinforced materials

Fig. 3 provides bounds and estimates for the effective flow stress $\tilde{\sigma}_0$ for the case of rigid reinforcement. The SO-I estimates are not shown for brevity, but it is worth mentioning that the associated k behaves similarly to the k-I shown in Fig. 2(a), for the reasons described above. Here, the SO-II estimates, unlike the OSO ones, are found to satisfy the bounds for all values of m, and exhibit essentially no duality gap (see Fig. 3(a)). Fig. 3(b) shows that the SO-II estimates lie below the corresponding OSO(W) estimates for all c, although the differences are small. In fact, they can be shown to agree in the dilute limit, as given by expression (5.3) of [6] with $\theta = 0$, for any m.

4. Final comments

Estimates of the HS type have also been obtained for the case of simple shear ($\theta = \pi/6$). The trends for $\tilde{\sigma}_0$ were found to be similar to those given in [8] for the in-plane shearing of 2D random fiber composites. Interestingly, for a dilute concentration of (cylindrical) pores in a rigid-ideally plastic matrix subject to simple shear, it was found in [8] that

$$\frac{\tilde{\sigma}_0}{\sigma_0} \sim 1 - \frac{3}{2}\left(\frac{c}{2}\right)^{2/3}$$

which is non-analytic at $c = 0$. For simple shear of (3D) spherical pores in a rigid-ideally plastic matrix, the corresponding dilute limit is found to be

$$\frac{\tilde{\sigma}_0}{\sigma_0} \sim 1 - \frac{1}{4} c |\ln c|$$

which is also non-analytic at $c = 0$, but with a *weaker* singularity. Weaker singularities in 3D than in 2D have already been found by Drucker [9] for the case of *periodic* arrays of pores. The question remains as to whether the singularities predicted by the second-order method for the random case may be indeed correct. That this might be the case is suggested by the comparisons with numerical results provided by Pastor and Ponte Castañeda [10]. In any case, the mere fact that the second-order method can capture some signature of the strongly nonlinear fields associated with ideally-plastic composites is already a positive result, as no other method to date seems to able to do so.

We conclude by emphasizing that the issue of the best choice for the reference tensors $\check{\boldsymbol{\sigma}}^{(r)}$ and $\check{\boldsymbol{\varepsilon}}^{(r)}$ in the context of the second-order method remains largely open. Nonetheless, the results provided in this Note suggest that the identification of $\check{\boldsymbol{\sigma}}^{(r)}$ with the macroscopic average $\bar{\boldsymbol{\sigma}}$ appears to give reasonable estimates. Although giving sensible estimates for most situations, the earlier choice for these variables (i.e., the phase averages $\bar{\boldsymbol{\sigma}}^{(r)}$) suggested in [2] can lead to inconsistent results for strong nonlinearities, if care is not taken to ensure that the LCC remains strongly elliptic. To avoid this complication, the use of the prescription (11) is recommended.

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References

[1] P. Ponte Castañeda, P. Suquet, Nonlinear composites, Adv. Appl. Mech. 34 (1998) 171–302.
[2] P. Ponte Castañeda, Second-order homogenization estimates for nonlinear composites incorporating field fluctuations: I – Theory, J. Mech. Phys. Solids 50 (2002) 737–757.
[3] P. Suquet, P. Ponte Castañeda, Small-contrast perturbation expansions for the effective properties of nonlinear composites, C. R. Acad. Sci. Paris, Ser. II 317 (1993) 1515–1522.
[4] N. Lahellec, P. Suquet, Nonlinear composites: a linearization procedure, exact to second-order in the contrast and for which the strain-energy and affine formulations coincide, C. R. Mecanique 332 (2004) 693–700.
[5] J.R. Willis, Bounds and self-consistent estimates for the overall moduli of anisotropic composites, J. Mech. Phys. Solids 25 (1977) 185–202.
[6] P. Ponte Castañeda, Exact second-order estimates for the effective mechanical properties of nonlinear composites, J. Mech. Phys. Solids 44 (1996) 827–862.
[7] P. Ponte Castañeda, The effective mechanical properties of nonlinear isotropic composites, J. Mech. Phys. Solids 39 (1991) 45–71.
[8] P. Ponte Castañeda, Second-order homogenization estimates for nonlinear composites incorporating field fluctuations: II – Applications, J. Mech. Phys. Solids 50 (2002) 759–782.
[9] D.C. Drucker, The continuum theory of plasticity on the macroscale and the microscale, J. Mater. 1 (1966) 873–910.
[10] J. Pastor, P. Ponte Castañeda, Yield criteria for porous media in plane strain: second-order estimates versus numerical results, C. R. Mecanique 330 (2002) 741–747.