Available online at www.sciencedirect.com
SCIENCE DIRECT®
C. R. Mecanique 333 (2005) 493–498
COMPTES RENDUS MECANIQUE
http://france.elsevier.com/direct/CRAS2B/

Justification of the kinematic assumptions for thin-walled rods with shallow profile

Lionel Grillet [a], Aziz Hamdouni [b,*], Olivier Millet [c]

[a] *Laboratoire de modélisation mécanique et de mathématiques appliquées (L3MA), université de Poitiers, France*
[b] *Laboratoire d'étude des phénomènes de transfert appliqués au bâtiment, Université de la Rochelle, avenue Michel-Crepeau, 17042 la Rochelle cedex 1, France*
[c] *Laboratoire de mécanique de Lille, CNRS, université de Lille 1, France*

Received 27 October 2004; accepted 30 December 2004

Available online 3 May 2005

Presented by Évariste Sanchez-Palencia

Abstract

In this Note we present a justification of the kinematic assumptions for thin-walled rods with shallow profile. These assumptions are fundamental to writing the one-dimensional equilibrium equations for such structures. The obtained kinematics are different from the Vlassov case, which is only valid for strongly curved profiles. They are also different from the that classically used in shell theory. The justification given in this Note is based on an asymptotic approach. ***To cite this article: L. Grillet et al., C. R. Mecanique 333 (2005).***

Résumé

Justification des hypothèses cinématiques des poutres à paroi mince faiblement courbées. Dans cette Note on présente une justification des hypothèses cinématiques des poutres à paroi mince faiblement courbées. Ces hypothèses sont à la base de l'écriture des équations d'équilibre de ces structures. Le modèle cinématique obtenu est différent à la fois, de celui utilisé par Vlassov et qui n'est valable que pour les poutres à paroi mince fortements courbées, et de celui utilisé en théorie des coques. La justification donnée ici utilise une approche asymptotique. ***Pour citer cet article : L. Grillet et al., C. R. Mecanique 333 (2005).***

Keywords: Computational solid mechanics; Thin-walled rod; Elasticity; Asymptotic methods

Mots-clés : Mécanique des solides numérique ; Poutre à paroi mince ; Élasticité ; Méthodes asymptotiques

* Corresponding author.
E-mail address: ahamdoun@univ-lr.fr (A. Hamdouni).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2004.12.007

Version française abrégée

En théorie linéaire des poutres à paroi mince, le modèle de Vlassov [1] est devenu d'usage classique. Ce modèle s'obtient à partir d'hypotèses cinématiques sur les déplacements tridimensionnels et d'hypothèses sur les contraintes. Ces hypothèses, ainsi que les équations d'équilibre qui s'en déduisent, ont été justifiées pour les poutres voiles fortement courbées par une approche asymptotique dans [2–4]. Cependant les hypothèses cinématiques de Vlassov ne sont pas suffisantes dans le cas d'une poutre à paroi mince faiblement courbée comme l'a montré Gjelsvik [5]. Ce dernier introduit des termes correctifs par analogie avec les hypothèses cinématiques de Kirchhof–Love en théorie des plaques minces. Cependant, la cinématique ainsi obtenue ne correspond pas à celle de Vlassov quand on fait tendre la courbure de la section droite de la poutre vers zéro. Nous proposons dans cette Note de justifier ces hypothèses cinématiques pour les poutres à paroi mince, faiblement courbées, à partir d'une analyse asymptotique des équations d'équilibre tridimensionnelles de la structure mince.

Pour une poutre à paroi mince, les nombres sans dimension caractérisant la géométrie de la structure sont : l'inverse de l'élancement ε (rapport du diamètre de la poutre sur sa longueur), la finesse η (rapport de l'épaisseur de la poutre sur son diamètre) et la coubure adimensionnelle ν (produit de la courbure de référence de la section droite de la poutre par son épaisseur). L'hypothèse de poutre à paroi mince correspond à ε et η petit devant 1 mais du même ordre de grandeur : $\eta = \mathrm{O}(\varepsilon)$. La poutre est dite faiblement courbée si l'ordre de grandeur de ν est petit devant ε et η, soit $\nu = \mathrm{O}(\varepsilon^2)$. Les nombres sans dimension caractérisant les efforts subis par la structure sont similaires à ceux introduit dans [7]. On considère ici des nombres sans dimension $\mathcal{F}$ caractérisant les efforts extérieurs volumiques et des nombres sans dimension $\mathcal{G}$ caractérisant les efforts extérieurs surfaciques.

Pour des efforts d'un ordre de grandeur suffisamment faible, mais induisant un couple de torsion plus élevé (voir Section 2 et les hypothèses de la Proposition 4.1), la forme générale du déplacement ainsi obtenu au premier ordre du développement asymptotique est donnée par les Éqs. (2) à (4) de la Proposition 4.1. Ce champ de déplacement vérifie la condition de non distorsion introduite par Vlassov [1] (résultat similaire au cas fortement courbée [3]) ainsi que l'hypothèse de Kirchhof–Love (qui n'est pas valide dans le cas fortement courbée). Ces conditions s'écrivent :

$$\frac{\partial u_3^0}{\partial s} + \frac{\partial u_t^0}{\partial x_3} = 0 \quad \text{(non distorsion)}$$

et

$$\mathbf{grad}\, u_n^0 + \frac{\partial \mathbf{V}}{\partial r} = 0 \quad \text{(Kirchhof–Love)}$$

où l'on note e_3 le vecteur unitaire de l'axe de la poutre, (t, n) la base de Frenet du profile de la section plane de la poutre, (s, r, x_3) les coordonnées curvilignes associées à la base orthonormée directe (t, n, e_3). De ces hypothèses cinématiques se déduisent les équations unidimensionnelles de la structure.

1. Introduction

In linear theory of thin-walled rods, the Vlassov model [1] is the model classically used. It is generally obtained from kinematic assumptions on the three-dimensional displacements and on the stresses as well. These kinematic assumptions and the associated equilibrium equations have been justified by asymptotic methods in the case of strongly curved profiles in [2–4]. However, for shallow profiles, Gjelsvik proved that the Vlassov kinematic assumptions are not sufficient [5]. So that he introduced supplementary terms by analogy with the Kirchhoff–Love assumptions in plate theory. However, the kinematics so obtained are not the limit of the Vlassov case when the curvature of the cross section tends towards zero. In this Note, we propose to justify the kinematic assumptions for thin-walled rods with shallow profile, from an asymptotic analysis of three-dimensional equations. Then the equilibrium equations of the structure can be deduced from these kinematic assumptions.

2. The three-dimensional problem

We recall here the notations of [3] where the reader can refer for more details. Let us consider a thin-walled rod with open cross-section and $2h$ thickness, whose middle surface ω^* is an open cylindrical surface of $\mathbb{R}^3$. We denote L the length of the rod in the direction e_3, d its diameter and $2h$ its thickness. Thus the thin-walled rod occupies the set $\overline{\Omega}^* = \overline{\omega}^* \times [-h, h]$ of $\mathbb{R}^3$ in its reference configuration. We call Γ_1^* and Γ_2^* the extreme faces, Γ_g^* and Γ_d^* the lateral faces, $\Gamma_\pm^* = \omega^* \times \{\pm h\}$ the upper and lower faces. On the other hand, we call $\mathcal{C}^*$ the intersection curve between the middle surface ω^* and the cross-section. The orthogonal projection m^* of a generic point of the rod M^* on the middle surface is located by its Cartesian coordinates $x^* = (x_1^*, x_2^*)$ or by its curvilinear abscisse s^* along $\mathcal{C}^*$. The origin s_0^* of the curvilinear abscisse is an arbitrary chosen point of $\mathcal{C}^*$. We note n the unit normal and t the unit tangent vector of $\mathcal{C}^*$. Finally, x_3^* denotes the position of the cross-section containing M^* on the axis $(0x_3^*)$.

It can be proved (see [3,4,6]) that the dimensionless numbers characterizing the elastic behavior of thin-walled rods are the following:

$$\varepsilon = \frac{d}{L} \ll 1, \quad \eta = \frac{h}{d} = \mathrm{O}(\varepsilon), \quad \nu = hc_r = \mathrm{O}(\varepsilon^2), \quad \mathcal{F}_\alpha = \frac{hf_{\alpha r}}{\mu}, \quad \mathcal{G}_\alpha = \frac{g_{\alpha r}}{\mu}, \quad \alpha = \{t, n, e_3\} \tag{1}$$

The inverse of the shooting-pain ε (ratio of the diameter to the length), the relative thickness η (ratio of the thickness to the diameter) and the dimensionless curvature ν (product of the reference curvature with the thickness) characterize the geometry of the rod. The thin-walled rod assumption corresponds to $\varepsilon \ll 1$ and $\eta \ll 1$, with ε and η of the same order of magnitude. Moreover for the shallow rods considered here, the dimensionless curvature ν is small with respect to ε and η, so that we have $\nu = hc_r = \mathrm{O}(\varepsilon^2)$.

The dimensionless numbers characterizing the forces are similar to those introduced in [7]. $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ characterize respectively the level of applied body and surface forces in Frenet basis (t, n, e_3). We consider here levels of applied forces such as: $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^6$, $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^7$. The normal forces are split into two parts. The first $\mathcal{G}_n^0 = \mathcal{F}_n^0 = \varepsilon^6$ induces a torque but has no resultant:

$$\int_{s_g}^{s_d}\int_{-1}^{1} f_n^0 \,\mathrm{d}r\,\mathrm{d}s + \int_{s_g}^{s_d} \left[g_n^{0+} - g_n^{0-}\right] \mathrm{d}s = 0$$

The second $\mathcal{G}_n^1 = \mathcal{F}_n^1 = \varepsilon^8$ of higher order has a resultant different from zero.

3. Asymptotic expansion of equations

Using reference physical data of the problem, the three-dimensional equilibrium equations of linear elasticity are decomposed in Frenet basis and written in a dimensional form. We obtain in Ω:

$$\begin{cases} \dfrac{\partial \sigma_{tn}}{\partial r} + \left(1 + \varepsilon^2 rc + (\varepsilon^2 rc)^2 + \cdots\right)\left(\varepsilon \dfrac{\partial \sigma_{tt}}{\partial s} - 2\varepsilon^2 c\sigma_{tn}\right) + \varepsilon^2 \dfrac{\partial \sigma_{t3}}{\partial x_3} = -\varepsilon^7 f_t \\ \dfrac{\partial \sigma_{nn}}{\partial r} + \left(1 + \varepsilon^2 rc + (\varepsilon^2 rc)^2 + \cdots\right)\left(\varepsilon \dfrac{\partial \sigma_{tn}}{\partial s} + \varepsilon^2 c\sigma_{tt} - \varepsilon^2 c\sigma_{nn}\right) + \varepsilon^2 \dfrac{\partial \sigma_{n3}}{\partial x_3} = -\varepsilon^6 f_n^0 - \varepsilon^8 f_n^1 \\ \dfrac{\partial \sigma_{n3}}{\partial r} + \left(1 + \varepsilon^2 rc + (\varepsilon^2 rc)^2 + \cdots\right)\left(\varepsilon \dfrac{\partial \sigma_{t3}}{\partial s} - \varepsilon^2 c\sigma_{n3}\right) + \varepsilon^2 \dfrac{\partial \sigma_{33}}{\partial x_3} = -\varepsilon^6 f_3 \end{cases}$$

where the variables without a star are dimensionless. The boundary conditions on the upper and lower faces can be written for $r = \pm 1$:

$$\sigma_{tn} = \varepsilon^7 g_t^{\pm}, \qquad \sigma_{nn} = \varepsilon^6 g_n^{0\pm} + \varepsilon^8 g_n^{1\pm}, \qquad \sigma_{n3} = \varepsilon^6 g_3^{\pm}$$

Finally, the boundary conditions on the lateral faces for $s = s_+$ and $s = s_-$ become:

$$\sigma_{tt} = 0, \qquad \sigma_{tn} = 0, \qquad \int_{-1}^{1} \sigma_{t3}\,\mathrm{d}r = 0$$

where $\sigma_{\alpha\beta}$ denotes the dimensionless components of the stresses whose expression depends on ε and on the components (u_t, u_n, u_3) of U in Frenet basis (t, n, e_3). The expression is rather complex and not given here.

With the order of magnitude of the applied forces considered here, and for the given geometry, we can prove that the corresponding reference scales for the displacements are:

$$u_{nr} = h, \qquad u_{tr} = \varepsilon h, \qquad u_{3r} = \varepsilon^2 h$$

Now we search the solution U of the three-dimensional problem as a formal series with respect to ε:

$$U = U^0 + \varepsilon U^1 + \varepsilon^2 U^2 + \cdots$$

This implies as well an expansion of σ with respect to ε:

$$\sigma = \sigma^0 + \varepsilon\sigma^1 + \varepsilon^2\sigma^2 + \cdots$$

The asymptotic expansion of the dimensionless equations enables us to determine the expression of the leading term U^0 of the expansion of the displacement. This constitutes the search kinematics for thin-walled rods with shallow profile which is explained in the following result.

4. Three-dimensional kinematics for thin-walled rods with shallow profile

Proposition 4.1. *For levels of applied forces such as $\mathcal{F}_n^0 = \mathcal{G}_n^0 = \varepsilon^6$, $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^6$, $\mathcal{F}_t = \mathcal{G}_t = \varepsilon^7$ and $\mathcal{F}_n^1 = \mathcal{G}_n^1 = \varepsilon^8$, the leading term (u_t^0, u_n^0, u_3^0) of the expansion of the displacement verifies*:

$$u_n^0 = \bar{u}_n^0 + s\bar{\Theta}^0 \tag{2}$$

$$u_t^0 = \bar{u}_t^0 + (\alpha - \alpha_0)\bar{u}_n^0 + a\bar{\Theta}^0 - r\bar{\Theta}^0 \tag{3}$$

$$u_3^0 = \bar{u}_3^0 - s\frac{\mathrm{d}\bar{u}_t^0}{\mathrm{d}x_3} - \xi\frac{\mathrm{d}\bar{u}_n^0}{\mathrm{d}x_3} - \omega\frac{\mathrm{d}\bar{\Theta}^0}{\mathrm{d}x_3} - r\left(\frac{\mathrm{d}\bar{u}_n^0}{\mathrm{d}x_3} + s\frac{\mathrm{d}\bar{\Theta}^0}{\mathrm{d}x_3}\right) \tag{4}$$

with

$$\frac{\mathrm{d}a}{\mathrm{d}s} = sc, \qquad \frac{\mathrm{d}\omega}{\mathrm{d}s} = a, \qquad \frac{\mathrm{d}\xi}{\mathrm{d}s} = (\alpha - \alpha_0)$$

where $\bar{u}_t^0$, $\bar{u}_n^0$, $\bar{u}_3^0$ denote the components of the displacement of the origin m_o of the curvilinear abscissa s, α_0 the angle $(\widehat{e_1, t})$ around m_0, and $\overline{\Theta}^0$ the twist angle of a section around e_3. The functions $\bar{(\cdot)}$ depend on x_3 only.

Proof. We only give here the main steps of the proof.

– From $(\mathcal{P}_0)$, we deduce immediately that u_n^0 does not depend on r. So that we have $u_n^0 = \tilde{u}_n^0$, and the leading term of the expansion of the stresses is $\sigma^0 = 0$.

– From $(\mathcal{P}_1)$ we obtain:

$$u_t^0 = -r\frac{\partial\tilde{u}_n^0}{\partial s} + \tilde{u}_t^0, \qquad u_n^1 = \tilde{u}_n^1 \tag{5}$$

where $\tilde{u}_t^0$ and $\tilde{u}_n^1$ do not depend on r. Once again, the stresses at order one are equal to zero: $\sigma^1 = 0$.

– Now the problem $(\mathcal{P}_2)$ leads to:

$$\sigma^2_{tn} = \sigma^2_{nn} = \sigma^2_{n3} = 0$$

so that

$$u^1_t = -r\frac{\partial \tilde{u}^1_n}{\partial s} + \tilde{u}^1_t, \qquad u^2_n = \frac{r^2}{2}\frac{\beta}{\beta+2}\frac{\partial^2 \tilde{u}^0_n}{\partial s^2} - r\frac{\beta}{\beta+2}\tilde{\psi}^0_n + \tilde{u}^2_n, \qquad u^0_3 = -r\frac{\partial \tilde{u}^0_n}{\partial x_3} + \tilde{u}^0_3 \tag{6}$$

where we set:

$$\tilde{\psi}^0_n = \frac{\partial \tilde{u}^0_t}{\partial s} - c\tilde{u}^0_n$$

Then we express the stresses σ^2_{tt} and σ^2_{33} with respect to $\tilde{u}^0_n$ and $\tilde{\psi}^0_n$. Using problem $(\mathcal{P}_3)$, we obtain:

$$\sigma^2_{tt} = -4\frac{(\beta+1)}{(\beta+2)}\frac{\partial^2 \tilde{u}^0_n}{\partial s^2}r, \qquad \sigma^2_{33} = -2\frac{\beta}{(\beta+2)}\frac{\partial^2 \tilde{u}^0_n}{\partial s^2}r \tag{7}$$

We then deduce that:

$$\sigma^3_{tn} = 4\frac{\beta+1}{\beta+2}\frac{\partial^3 \tilde{u}^0_n}{\partial s^3}\frac{r^2-1}{2}, \qquad \sigma^3_{nn} = 0, \qquad \sigma^3_{n3} = 0 \tag{8}$$

– Problem $(\mathcal{P}_4)$ leads to:

$$\frac{\partial^2 \tilde{u}^0_n}{\partial s^2} = 0$$

so that we have

$$\tilde{u}^0_n = \bar{u}^0_n + s\overline{\Theta}^0$$

where $\overline{\Theta}^0$ and $\bar{u}^0_n$ depend on x_3 only. In the last expression, $\overline{\Theta}^0$ characterizes the twist angle of the sections, and $\bar{u}^0_n$ the normal displacement of the origin m_0 for $s = 0$. Thus expression (2) of Proposition 4.1 is justified.

On the other hand, as a consequence of problem $(\mathcal{P}_3)$, we obtain:

$$\frac{\partial \tilde{u}^0_t}{\partial s} - cu^0_n = 0$$

which leads to

$$u^0_t = \bar{u}^0_t + (\alpha - \alpha_0)\bar{u}^0_n + (a - r)\overline{\Theta}^0$$

where we set

$$\frac{\mathrm{d}a}{\mathrm{d}s} = cs, \qquad \frac{\mathrm{d}\alpha}{\mathrm{d}s} = c$$

This corresponds to Eq. (3) of Proposition 4.1.

– To finish, let us express σ^3_{t3} with respect to $\overline{\Theta}^0$, $\tilde{u}^0_3$ and $\tilde{u}^0_t$. Replacing the obtained expression in problem $\mathcal{P}_4$, and using the boundary condition

$$\int_{-1}^{1} \sigma^3_{t3}\,\mathrm{d}r = 0$$

for $s = s^+$ and $s = s^-$, we obtain:

$$\frac{\partial \tilde{u}^0_3}{\partial s} + \frac{\partial \tilde{u}^0_t}{\partial x_3} = 0 \tag{9}$$

This equation constitutes the non-distortion assumption introduced by Vlassov [1]. The integration of this equation then leads to:

$$u_3^0 = \bar{u}_3^0 - s\frac{\mathrm{d}\bar{u}_t^0}{\mathrm{d}x_3} - \xi\frac{\mathrm{d}\bar{u}_n^0}{\mathrm{d}x_3} - \omega\frac{\mathrm{d}\overline{\Theta}^0}{\mathrm{d}x_3} + r\left(\frac{\mathrm{d}\bar{u}_n^0}{\mathrm{d}x_3} + s\frac{\mathrm{d}\overline{\Theta}^0}{\mathrm{d}x_3}\right)$$

with

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \alpha - \alpha_0, \qquad \frac{\mathrm{d}\omega}{\mathrm{d}s} = a$$

which ends the proof of Proposition 4.1.

Therefore, for a strongly curved profile [3], the displacement field of Proposition 4.1 verifies the non-distortion assumption introduced by Vlassov [1]:

$$\frac{\partial u_3^0}{\partial s} + \frac{\partial u_t^0}{\partial x_3} = 0$$

However, it verifies as well the Kirchhoff–Love assumption (*which is not valid for strongly curved profiles*):

$$\mathbf{grad}\, u_n^0 + \frac{\partial \mathbf{V}}{\partial r} = 0$$

where **grad** denotes the gradient operator with respect to the coordinates (s, x_3) of the middle surface of the thin-walled rod, and **V** the tangential displacement whose components are (u_t^0, u_3^0). Thus we have justified by asymptotic expansion the three-dimensional kinematics of a thin-walled rod with shallow profile, which differs from that of Vlassov. Such a justification does not seem to have any equivalent in the literature. Let us notice that from these kinematics that we can deduce the one-dimensional equilibrium equations, by going on the asymptotic expansion of equations up to order five. □

References

[1] B.Z. Vlassov, Pièces longues en voiles minces, Eyrolles, France, 1962.

[2] J.M. Rodriguez, J.M. Viaño, Asymptotic derivation of a general linear model for thin-walled elastic rods, Comput. Methods Appl. Mech. Engrg. 147 (1997) 287–321.

[3] L. Grillet, A. Hamdouni, C. Allery, Modèle asymptotique linéaire de poutres voiles fortement courbées, C. R. Acad. Sci. Paris, Sér. IIb 328 (2000) 587–592.

[4] L. Grillet, Modèles asymptotiques linéaires et non linéaires de poutres voiles élastiques, thèse de doctorat, Université de Poitiers, 2003.

[5] A. Gjelsvik, The Theory of Thin-Walled Bars, Wiley, NY, 1981.

[6] L. Grillet, A. Hamdouni, O. Millet, An asymptotic nonlinear model for thin-walled rods, C. R. Mécanique 332 (2004) 123–128.

[7] J. Sanchez-Hubert, E. Sanchez-Palencia, Introduction aux méthodes asymptotiques et à l'homogénéisation, Masson, 1992.