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# On interfacial properties in gradient damaging continua

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**Abstract**

The bifurcation problem near an interface is considered for a heterogeneous body made of two different materials that damage following gradient constitutive relations. The roles of internal length scales on bifurcation are studied especially in the shortwavelength regime. It is shown that the interfacial complementing condition is always satisfied meaning that a minimum wavelength exists for the bifurcation mode. The regularization properties of gradient damage models are underlined. A simple plane strain problem is used to illustrate the results. The interface bifurcated modes are explicitly computed: their wavelengths turn out to be fixed by the gradient coefficient; the influence of the interface behaviour is also highlighted. ***To cite this article: A. Benallal, C. Comi, C. R. Mecanique 333 (2005).***


**Résumé**

**Sur quelques propriétés des interfaces dans les milieux à gradient.** Les phénomènes de bifurcation en présence d'interfaces sont analysés lorsque le comportement des matériaux constitutifs d'un solide hétérogène est gouverné par des modèles à gradient. Les roles des longueurs internes sont étudiés particulièrement aux faibles longueurs d'onde et on montre que la condition complémentaire aux interfaces est toujours satisfaite impliquant l'existence d'une longueur d'onde minimale pour le mode de bifurcation. Les effets régularisants sont soulignés et on illustre les résultats à l'aide d'un exemple en déformations planes où les modes de bifurcation sont explicitement calculés. ***Pour citer cet article : A. Benallal, C. Comi, C. R. Mecanique 333 (2005).***


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## 1. Introduction

The inelastic behaviour up to failure of quasi-brittle materials is often effectively described by damage models. Nevertheless, the necessity for the use of enriched models which account for the microstructure of the materials through a characteristic internal length has become clear in recent years. While from a physical point of view the introduction of nonlocality is justified by the discrete nature of the real damage process, from a mathematical point of view the introduction of an internal length is required in order to preserve the well-posedness of the boundary value problem. In macroscopically homogeneous solids the use of nonlocal gradient dependent models has been proved to be effective to restore ellipticity [1] and various gradient damage models have been proposed in the literature and used in numerical analyses [2,3]. However, the problem may also become ill-posed due to the loss of the boundary complementing condition and of the interfacial complementing condition. In particular the interfacial complementing condition is of main interest in the case of zone wise heterogeneous solids, i.e. solids constituted by two or more different materials. Problems in which two different quasi-brittle materials in contact (e.g. concrete and rock in dam engineering) should be considered are quite common in practice. A discussion of the regularization properties of nonlocal formulations with respect to these complementing conditions is still lacking, to the authors' knowledge. In this Note we consider an heterogeneous body, constituted by two parts, endowed with different materials properties and in particular with two different internal lengths. The issue of the well-posedness is addressed with particular attention to the interfacial complementing condition. The results obtained are illustrated by a simple plane strain example.

## 2. Gradient damage model

The isotropic elastic-damage model considered is based on the definition of a free energy density $\Psi$ depending on a scalar damage variable $D$ and a scalar kinematic internal variable $\kappa$,

$$\Psi(\boldsymbol{\varepsilon}, D, \kappa) = \frac{1}{2}(1-D)\boldsymbol{\varepsilon} : \boldsymbol{E} : \boldsymbol{\varepsilon} + k(1-\kappa)\sum_{i=0}^{n}\frac{n!}{i!}\ln^i\left(\frac{c}{1-\kappa}\right) \tag{1}$$

Here $\boldsymbol{\varepsilon}$ is the small strain tensor, $\boldsymbol{E}$ is the undamaged elastic tensor, $k$, $c$ and $n$ are material parameters and, by definition, $0! = 1$. The stresses $\boldsymbol{\sigma}$, the elastic energy release rate $Y$ and the internal variable $\chi$ are defined through the state equations

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = (1-D)\boldsymbol{E} : \boldsymbol{\varepsilon}, \quad Y = -\frac{\partial \Psi}{\partial D} = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{E} : \boldsymbol{\varepsilon}, \quad \chi = -\frac{\partial \Psi}{\partial \kappa} = k\ln^n\left(\frac{c}{1-\kappa}\right) \tag{2}$$

For the model, in its local format, the activation and evolution of the damage process is governed by an activation function $f_{\text{local}}$, depending on the static variables, by loading–unloading complementarity conditions and by associative evolution equations

$$f_{\text{local}} = Y - \chi \leqslant 0; \quad \dot{\gamma} \geqslant 0; \quad f\dot{\gamma} = 0; \quad \dot{D} = \frac{\partial f}{\partial Y}\dot{\gamma} = \dot{\gamma}; \quad \dot{\kappa} = -\frac{\partial f}{\partial \chi}\dot{\gamma} = \dot{\gamma} \tag{3}$$

where $\dot{\gamma}$ is a non-negative damage multiplier.

The nonlocal gradient-dependent model is obtained by adding to the loading function a term depending on the spatial Laplacian of the damage variable

$$f = f_{\text{local}} + \omega\nabla^2 D \tag{4}$$

As shown e.g. in [1], the material parameter $\omega$ is proportional to the square of the material characteristic length $l_c$, measuring the spatial range of the microscopic interaction between different material points in the damage

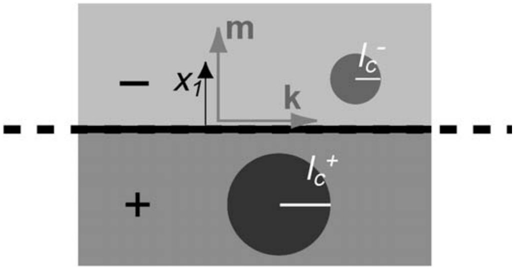



Fig. 1. Sketch of the problem analyzed.

mechanisms. Due to the presence of the gradient term, for finite bodies, proper boundary conditions should be added. A common choice, thermodynamically motivated in [4] can be written

$$\nabla \dot{D} \cdot \boldsymbol{n} = 0$$

$\boldsymbol{n}$ being the normal to the boundary $\Gamma$.

## 3. Bifurcation at the interface

Let us consider two semi-infinite bodies, labeled by $-$ and $+$, described by the above gradient damage model, endowed with different internal lengths and bonded together at the interface $x_1 = 0$ (Fig. 1). Assuming full loading conditions, zero body forces and small strains, the rate problem, for each part, can be expressed

$$\nabla \cdot \dot{\boldsymbol{\sigma}} = \mathbf{0}, \quad \dot{\boldsymbol{\varepsilon}} = \frac{1}{2}(\nabla \dot{\boldsymbol{u}} + \nabla^{\mathrm{T}} \dot{\boldsymbol{u}}), \quad \dot{f} = \dot{Y} - \dot{\chi} + \omega \nabla^2 \dot{D} = 0 \tag{5}$$

Using the state law (2a), the field equations are better written in the form

$$\nabla \cdot \big[(1 - D)\boldsymbol{E} : \nabla \dot{\boldsymbol{u}}\big] - \nabla \dot{D} \boldsymbol{E} : \boldsymbol{\varepsilon} = \mathbf{0} \tag{6}$$

$$\omega \nabla^2 \dot{D} - h\dot{D} + \boldsymbol{\varepsilon} : \boldsymbol{E} : \nabla \dot{\boldsymbol{u}} = 0 \tag{7}$$

having set $h = \mathrm{d}\chi/\mathrm{d}\kappa$. To these relations, written for $+$ and $-$ parts, one should add interface conditions on traction and displacements rates and also, for the gradient model under consideration, on damage and gradient of damage rates. These conditions can be taken in the form, $\boldsymbol{m}$ being the normal to the interface

$$\dot{\boldsymbol{\sigma}}^+ \cdot \boldsymbol{m} = \dot{\boldsymbol{\sigma}}^- \cdot \boldsymbol{m}, \qquad \dot{\boldsymbol{\sigma}} \cdot \boldsymbol{m} = \boldsymbol{K} \cdot (\dot{\boldsymbol{u}}^+ - \dot{\boldsymbol{u}}^-) \tag{8}$$

$$\nabla \dot{D}^+ \cdot \boldsymbol{m} = \nabla \dot{D}^- \cdot \boldsymbol{m}, \qquad \nabla \dot{D} \cdot \boldsymbol{m} = S(\dot{D}^+ - \dot{D}^-) \tag{9}$$

While conditions (8a,b) have a clear mechanical meaning, namely equilibrium at the interface and interface constitutive law respectively, conditions (9a,b) are required by the presence of damage gradient in the loading function but do not have a clear mechanical interpretation. Their particular form has been selected in analogy with (8). The interface stiffness parameters in $\boldsymbol{K}$, assumed diagonal for simplicity, can vary from zero (free surface condition) to infinite (perfect bonding). Analogously $S$, which dimensionally is the inverse of a length, varies from zero, thus imposing to the normal gradient of damage to be zero at the interface ('free surface' like condition) to infinite, thus imposing the continuity of damage rates at the interface.

As outlined in [5], the interfacial complementing condition bears interest only in the elliptic regime. For the situation considered here, ellipticity always holds as was shown in [1]. The system (6), (7) is a system of mixed order in contrast to the local case. To check its ellipticity and also the complementing condition, it is sufficient to look at its principal part (in the sense of Agmon–Douglis–Nirenberg [6]). In this case, this amounts to considering only the higher derivatives in (6) and (7),

$$\nabla\cdot\left[(1-D)\boldsymbol{E}:\nabla\dot{\boldsymbol{u}}\right]=\boldsymbol{0},\qquad \omega\nabla^2\dot{D}=0 \tag{10}$$

associated to the boundary conditions. We obtain two decoupled problems for $\dot{\boldsymbol{u}}$ and $\dot{D}$. For $D\neq 1$ and $\omega\neq 0$, these two problems have no bounded solution other than the trivial one $\dot{\boldsymbol{u}}=\boldsymbol{0}$ and $\dot{D}=0$, therefore the interfacial condition is always satisfied. As a consequence an upper critical wavenumber exists under which system (6)–(9) has no bifurcated solution. To obtain this wavenumber, one should consider the complete bifurcation problem.

To investigate the existence of bifurcation starting from a homogeneous state in each part, solution of the system (6)–(9) are sought in the form (cp e.g. [5,7])

$$\begin{aligned}\dot{\boldsymbol{u}}&=\boldsymbol{w}(x_1)\exp(\mathrm{i}\xi\boldsymbol{k}\cdot\boldsymbol{x})=\boldsymbol{a}\exp(\xi\tau\boldsymbol{m}\cdot\boldsymbol{x}+\mathrm{i}\xi\boldsymbol{k}\cdot\boldsymbol{x}),\\ \dot{D}&=d(x_1)\exp(\mathrm{i}\xi\boldsymbol{k}\cdot\boldsymbol{x})=g\exp(\xi\tau\boldsymbol{m}\cdot\boldsymbol{x}+\mathrm{i}\xi\boldsymbol{k}\cdot\boldsymbol{x})\end{aligned} \tag{11}$$

where $\boldsymbol{k}$ is a unit vector of components $0, k_2, k_3$ lying in the interface plane, $\xi$ is the wave number. Functions $\boldsymbol{w}(x_1)$ and $d(x_1)$ will in general differ for $+$ and $-$ parts, while the wavenumber $\xi$ is taken to be the same for the two parts, as implied by the interfacial conditions. Direct substitution of the fields (11) into the governing equations (6)–(7) gives, for the two bodies, the following equation:

$$\left[\boldsymbol{N}\cdot\boldsymbol{H}(\xi,\tau)\cdot\boldsymbol{N}\right]\cdot\boldsymbol{a}=\boldsymbol{0},\qquad \boldsymbol{H}(\xi,\tau)=(1-D)\boldsymbol{E}-\frac{\boldsymbol{E}:\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}:\boldsymbol{E}}{h+\omega\xi^2(1-\tau^2)} \tag{12}$$

where we have set $\boldsymbol{N}=\tau\boldsymbol{m}+\mathrm{i}\boldsymbol{k}$. Eq. (12) has non trivial solutions if its determinant vanishes, i.e.

$$\frac{\mu(1-\tau^2)F(\xi,\tau)}{h+\omega\xi^2(1-\tau^2)}=0\quad\text{with } F(\xi,\tau)=(\lambda+2\mu)(1-D)\xi^2(\tau^2-1)^3\omega+G(\tau) \tag{13}$$

In (13) $\lambda$ and $\mu$ are the elastic Lamé constants and $G(\tau)$ is a biquadratic function of $\tau$, depending on the strain tensor.

It is recalled here that in the elliptic regime which holds everywhere and at every time (except for $D=1$), Eq. (13) has exactly four roots $\tau_j^+$ with positive real part and four other roots $\tau_j^-$ with negative real part. These are $\tau_1^\pm=\pm 1$ and the roots of $F(\xi,\tau)=0$. This equation is a cubic equation (in $\tau^2$) with real coefficients and it has therefore two real solutions $\tau_2^\pm$ and four mutually conjugate roots $\tau_3^\pm$ and $\tau_4^\pm$ (due to ellipticity). These solutions depend on the wavenumber $\xi$. To obtain the general bounded solution, one should consider only the roots with negative real parts in the $(-)$ part of the body and only those with positive real parts in the $(+)$ part of the body (see Fig. 1). The eigenvectors $\boldsymbol{a}_j^\pm$ associated to $\tau_j^\pm$ (see (12)) are obtained as

$$\begin{aligned}\boldsymbol{a}_1^\pm&=-\left[(\boldsymbol{m}\times\boldsymbol{k})\cdot(\boldsymbol{\varepsilon}:\boldsymbol{E}^\pm\cdot\boldsymbol{N}_1^\pm)\right]\cdot\boldsymbol{N}_1^\pm+(\boldsymbol{N}_1^\pm\boldsymbol{\varepsilon}:\boldsymbol{E}^\pm\cdot\boldsymbol{N}_1^\pm)\cdot(\boldsymbol{m}\times\boldsymbol{k})\\ \boldsymbol{a}_j^\pm&=\left[\boldsymbol{N}_j^\pm\cdot(1-D)\boldsymbol{E}^\pm\cdot\boldsymbol{N}_j^\pm\right]^{-1}\cdot(\boldsymbol{N}_j^\pm\cdot\boldsymbol{E}^\pm:\boldsymbol{\varepsilon})\quad\text{for } j=2,3,4\end{aligned} \tag{14}$$

where $\times$ is the vectorial product. Assuming $\tau_j$ distinct, the general solutions $\boldsymbol{w}^\pm(x_1)$ and $d^\pm(x_1)$ can be expressed as

$$\boldsymbol{w}^\pm(x_1)=\sum_{j=1}^4 b_j^\pm\boldsymbol{a}_j^\pm\,\mathrm{e}^{\xi\tau_j^\pm\boldsymbol{m}\cdot\boldsymbol{x}},\qquad d^\pm(x_1)=\sum_{j=1}^4 b_j^\pm\frac{(\boldsymbol{\varepsilon}:\boldsymbol{E}\cdot\boldsymbol{N}_j^\pm)\cdot\boldsymbol{a}_j^\pm}{h+\omega\xi^2(1-(\tau_j^\pm)^2)}\,\mathrm{e}^{\xi\tau_j^\pm\boldsymbol{m}\cdot\boldsymbol{x}}=\sum_{j=2}^4 b_j^\pm\,\mathrm{e}^{\xi\tau_j^\pm\boldsymbol{m}\cdot\boldsymbol{x}} \tag{15}$$

where $b_j^\pm$ are arbitrary coefficients. The last equality being a consequence of (14) and (13). The stress vectors and the gradients of damage are obtained as

$$\begin{aligned}\dot{\boldsymbol{\sigma}}^\pm\cdot\boldsymbol{m}&=\xi\sum_{j=1}^4 b_j^\pm\boldsymbol{r}_j^\pm\exp(\xi\tau_j^\pm\boldsymbol{m}\cdot\boldsymbol{x}+\mathrm{i}\xi\boldsymbol{k}\cdot\boldsymbol{x}),\\ \nabla\dot{D}^\pm\cdot\boldsymbol{m}&=\xi\sum_{j=2}^4 b_j^\pm\boldsymbol{m}\cdot\boldsymbol{N}_j^\pm\exp(\xi\tau_j^\pm\boldsymbol{m}\cdot\boldsymbol{x}+\mathrm{i}\xi\boldsymbol{k}\cdot\boldsymbol{x})\end{aligned} \tag{16}$$

with $\boldsymbol{r}_j = \boldsymbol{m} \cdot \boldsymbol{H}(\xi, \tau_j) \cdot \boldsymbol{N}_j \cdot \boldsymbol{a}_j$. Using these last relations together with (15), the interface conditions read

$$\sum_{j=1}^{4} b_j^+ \boldsymbol{r}_j^+ = \sum_{j=1}^{4} b_j^- \boldsymbol{r}_j^-, \quad \sum_{j=1}^{4} b_j^+ \boldsymbol{a}_j^+ = \sum_{j=1}^{4} b_j^- \boldsymbol{a}_j^-, \quad \sum_{j=2}^{4} b_j^+ \tau_j^+ = \sum_{j=2}^{4} b_j^- \tau_j^-, \quad \sum_{j=2}^{4} b_j^+ = \sum_{j=2}^{4} b_j^- \tag{17}$$

This leads to a linear system in the unknown coefficients $b_j^+, b_j^-$

$$\boldsymbol{M}\boldsymbol{b} = \boldsymbol{0} \tag{18}$$

and the bifurcation condition becomes $\det \boldsymbol{M} = 0$. This last equation gives the critical wavenumber which fixes the minimum wavelength of the bifurcated modes at the interface.

## 4. Plane strain example

Results are presented in this section for plane strain conditions with

$$\epsilon_{12} = 0, \qquad \epsilon_1 = \epsilon_2 = \epsilon \tag{19}$$

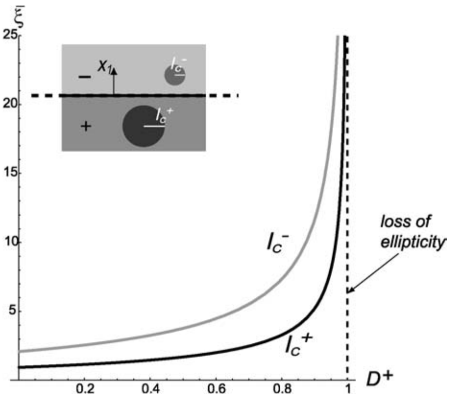
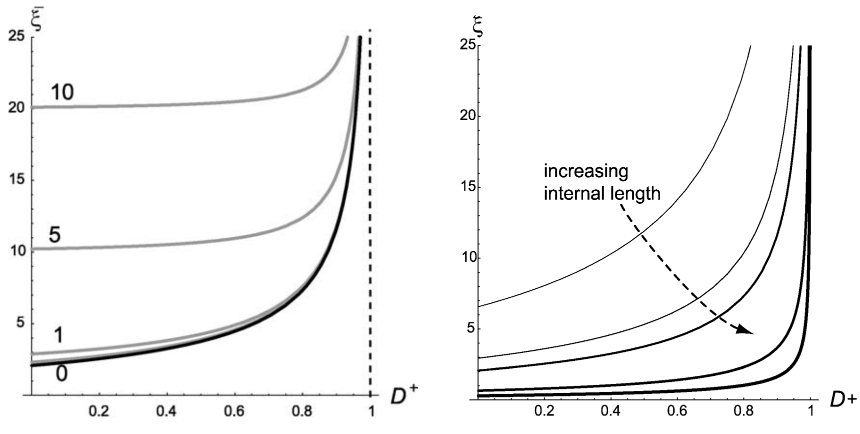



Fig. 2. Evolution of the wavenumber of the bifurcated mode at the interface: influence of the different internal lengths ($S = 0$, homogeneous initial state, $\lambda^+ = \lambda^- = 0$, $\mu^+/\mu^- = 1$).




(a) (b)

Fig. 3. Evolution of the wavenumber; (a) varying interface condition ($S$); (b) varying gradient coefficient ($\omega^+$).

For this 2-D problem Eq. (12) has the following 6 eigenvalues

$$\tau_{1,2} = -1, \qquad \tau_{3,4} = 1$$
$$\tau_{5,6} = \pm\frac{\sqrt{(1-D)(\lambda+2\mu)(h+\omega\xi^2) - 4(\lambda+\mu)^2\epsilon^2}}{\xi\sqrt{(1-D)(\lambda+2\mu)\omega}} \tag{20}$$

Assuming perfect bonding ($\boldsymbol{K} = \boldsymbol{0}$), the interfacial conditions (8) and (9) become

$$\dot\sigma_{11}^+ - \dot\sigma_{11}^- = 0, \quad \dot\sigma_{12}^+ - \dot\sigma_{12}^- = 0, \quad \dot u_1^+ - \dot u_1^- = 0, \quad \dot u_2^+ - \dot u_2^- = 0$$
$$\nabla_1 \dot D^+ - \nabla_1 \dot D^- = 0, \qquad \nabla_1 \dot D - S(\dot D^+ - \dot D^-) = 0 \tag{21}$$

Substituting (20) into (11) and the resulting expressions in (21) one obtains a system of 6 equations in the 12 unknown $b_j^+, b_j^-$. The remaining equations come from the condition of boundedness at infinite of the solution. The resulting homogeneous system, Eq. (18), has nontrivial solutions if $\det \boldsymbol{M} = 0$. This equation can be explicitly written and solved in terms of critical wavenumber $\xi$. Fig. 2 shows the evolution of the nondimensional wavenumber $\bar\xi = \xi\sqrt{\frac{\omega^+}{\mu^+}}$ as a function of the damage $D^+$. When two different material lengths $l_c^+$ and $l_c^-$ are present, two solutions are found and the bifurcated mode at the interface corresponds to the smallest internal length (gray curve in Fig. 2). The dashed line corresponds to the loss of ellipticity, which for the gradient model only occur in the limit $D^+ \to 1$. Fig. 3(a) illustrates the influence of the interface behaviour assumed in Eq. (21). The different curves refer to different values (marked in the figure) of the normalized interface parameter $\overline{S} = S\sqrt{\frac{\omega^+}{\mu^+}}$, with $\omega^- = \omega^+$ and $\mu^- = \mu^+$. As $S$ increases, bifurcation for a given wavenumber $\bar\xi$ is anticipated (i.e. it corresponds to a lower damage value). In Fig. 3(b) the results obtained with different gradient coefficients $\omega^+$ (or different internal lengths) and fixed $\omega^-$ are shown (the curves refer to the range 0.1–50 of the ratio $\omega^+/\omega^-$).

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