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# An optimality condition on the minimum energy threshold in subcritical instabilities

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**Abstract**

For flows subject to subcritical instabilities the stability of the basic flow can be guaranteed only for perturbations of energy lower than a critical threshold $\delta$. The computation of this threshold for the Navier–Stokes equations is still out of reach. More surprisingly, this computation has not been attempted for low dimensional models of subcritical transition. In this Note guidelines are provided for the computation of the minimum energy threshold $\delta$ and of the corresponding nonlinear optimal perturbations. In particular it is demonstrated that nonlinear optimal perturbations are constrained by the requirement that they must satisfy a local minimum condition. These results are applied to the analysis of four-dimensional models proposed in F. Waleffe, Phys. Fluids 7 (1995) and Phys. Fluids 9 (1997). ***To cite this article: C. Cossu, C. R. Mecanique 333 (2005).***


**Résumé**

**Une condition d'optimalité pour le seuil minimal d'énergie dans les transitions sous-critiques.** Dans le cas d'instabilité sous-critique, la stabilité de l'écoulement de base peut être garantie seulement pour des perturbations d'énergie inférieure au seuil critique $\delta$. Le calcul direct de ce seuil est inaccessible pour les équations de Navier–Stokes. Plus surprenant est le fait que ce calcul n'a pas été tenté pour des modèles de basse dimension de transition sous-critique. Dans cette Note des indications générales sont fournies pour le calcul de $\delta$ et des perturbations non linéaires optimales associées. Notamment, nous démontrons que les perturbations non linéaires optimales doivent satisfaire une condition de minimum local. Ces résultats sont appliqués à l'analyse de systèmes à quatre dimensions proposés in F. Waleffe, Phys. Fluids 7 (1995) et Phys. Fluids 9 (1997). ***Pour citer cet article : C. Cossu, C. R. Mecanique 333 (2005).***


*Keywords:* Fluid mechanics; Instability; Dynamical systems; Optimal perturbations

*Mots-clés :* Mécanique des fluides ; Instabilité ; Systèmes dynamiques ; Perturbations optimales

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## 1. Framework

In many shear flows, fully developed turbulence is observed at sufficiently high Reynolds number $R$ even though the laminar basic state is linearly stable to infinitesimal perturbations. Examples of such so-called subcritical instability include plane Couette flow, (which is linearly stable for all $R$) and pipe Poiseuille flow. In such circumstances, asymptotic stability (i.e. the decay of all the disturbances as $t \to \infty$) of the basic state is guaranteed only if the energy of the allowable perturbations $\mathcal{E}$ is lower than some threshold $\delta$. In general $\delta$ is a function of the Reynolds number $R$. For sufficiently small $R < R_G$, $\delta$ is infinite and the flow is said to be globally stable in that all disturbances, whatever their initial amplitude, decay asymptotically in time. Various analyses have provided [1] bounds on $R_G$, as well as [2] bounds on $\delta(R)$. Indeed, various studies [3–5] have attempted to identify a power law relationship between $\delta$ and $R$ as $R \to \infty$ of the form $\delta^{1/2} \sim R^{\gamma}$ for some scaling coefficient $\gamma$. Nevertheless, despite great progress, an important gap often exists between upper and lower bounds provided by different studies and a formal theory predicting the dependence of $\delta$ on $R$ still seems out of reach for real flows determined by solutions of the Navier–Stokes equations. However, the situation is very different for low order reduced models which have been developed to mimic different aspects of subcritical transition in real flows. The principal aim of this Note is to compute, for specific low dimensional models, threshold amplitude as a function of $R$ as well as the associated optimal nonlinear perturbations. Therefore, we examine the nonlinear asymptotic stability of a linearly stable basic state $\mathbf{U}$ to a general perturbation $\mathbf{u}$. Defining the total generic flow field as $\mathbf{u}_G = \mathbf{U} + \mathbf{u}$, evolution equations can be derived for $\mathbf{u}$, which take the general form [1]:

$$\mathrm{d}\mathbf{u}/\mathrm{d}t = \mathcal{L}_R\,\mathbf{u} + \mathcal{N}(\mathbf{u}) \tag{1}$$

The laminar basic state $\mathbf{u}_G = \mathbf{U}$ of course corresponds to $\mathbf{u} = \mathbf{0}$. In (1), $\mathcal{L}_R$ is a linear operator which depends on $\mathbf{U}$ and $R$ while $\mathcal{N}$ is a nonlinear operator which we assume to be homogeneous, in the sense that $\mathcal{N}(\mathbf{0}) = \mathbf{0}$. The initial value problem corresponds to considering the evolution of $\mathbf{u}$ with time, which evolution is in general a function of $R$ and the specified initial conditions $\mathbf{u}_0$ given at $t_0$. Naturally it is also possible to consider the energy $\mathcal{E}$ of this perturbation, which may be defined in the conventional way using an appropriate inner product:

$$\mathcal{E}\big[\mathbf{u}(t, \mathbf{u}_0, R)\big] = \langle \mathbf{u}, \mathbf{u}\rangle/2, \qquad \mathcal{E}_0(\mathbf{u}_0) = \langle \mathbf{u}_0, \mathbf{u}_0\rangle/2 \tag{2}$$

Taking the inner product of (1) with $\mathbf{u}$ yields the evolution equation for $\mathcal{E}$ [1]:

$$\mathrm{d}\mathcal{E}/\mathrm{d}t = \langle \mathbf{u}, \mathcal{L}_R\,\mathbf{u}\rangle + \big\langle \mathbf{u}, \mathcal{N}(\mathbf{u})\big\rangle \tag{3}$$

It is important to note the Navier–Stokes equations [1,2], and most low dimensional models of subcritical transition [4] have the valuable property that the relevant nonlinear terms $\mathcal{N}$ are 'energy preserving', in the sense that $\langle \mathbf{u}, \mathcal{N}(\mathbf{u})\rangle = 0\ \forall \mathbf{u}$. The basin of attraction $\mathcal{S}_R$ of the laminar basic flow $\mathbf{u} = 0$ at fixed $R$ is given by the set of initial perturbations $\mathbf{u}_0$ such that $\lim_{t\to\infty} \mathcal{E}[\mathbf{u}(t, \mathbf{u}_0, R)] = 0$. Since we assume $\mathbf{U}$ linearly (strictly) stable, its basin of attraction has non-zero measure. The complementary set $\mathcal{U}_R$ is made of initial perturbations for which $\lim_{t\to\infty} \mathcal{E}[\mathbf{u}(t, \mathbf{u}_0, R)] \neq 0$. This set has zero measure if the laminar basic flow is globally stable. When $\mathcal{U}_R$ has non-zero measure, the *minimum threshold energy* can be defined as $\delta(R) = \min_{\mathbf{u}_0 \in \mathcal{U}_R} \mathcal{E}_0(\mathbf{u}_0)$. The nonlinear optimal perturbation (abbreviated henceforth as NLOP) is defined as the initial perturbation for which the minimum $\delta$ is attained. In the following we will refer also to *linear optimal perturbations*, denoted by LOP, maximizing the linear energy growth $\mathcal{E}^{(L)}/\mathcal{E}_0^{(L)}$ computed on the linear system $\mathrm{d}\mathbf{u}^{(L)}/\mathrm{d}t = \mathcal{L}_R\,\mathbf{u}^{(L)}$ over all the possible perturbations and over all possible times [6]. The maximum linear energy growth can be large if the linear operator $\mathcal{L}_R$ is non-normal (i.e. if it does not commute with its adjoint). It has been suggested that the potential for large transient growths dur to non-normality is a key mechanism in subcritical transition (see among others [6,3,2]). Linear optimals have been therefore computed for a variety of flows and are reviewed in [2].

## 2. A necessary condition for nonlinear optimality

Not all possible initial perturbations are suitable candidates to be nonlinear optimals (NLOP). Consider a neighborhood $\Delta t \ll 1$ of $t_0$. We expand the perturbation energy as $\mathcal{E}[\mathbf{u}(t_0 + \Delta t, \mathbf{u}_0, R)] = \mathcal{E}_0 + \Delta t\,(\mathrm{d}\mathcal{E}/\mathrm{d}t)_0 + (\Delta t^2/2)(\mathrm{d}^2\mathcal{E}/\mathrm{d}t^2)_0 + \mathrm{O}(\Delta t^3)$. If $\mathbf{u}_0 \in \mathcal{U}_R$ (the complementary set to the basin of attraction $\mathcal{S}_R$ of the laminar basic state), and $(\mathrm{d}\mathcal{E}/\mathrm{d}t)_0 < 0$ then $\mathbf{u}(t_0 + \Delta t, \mathbf{u}_0, R)$, with $\Delta t > 0$ also belongs to $\mathcal{U}_R$ because it is on the same trajectory in phase space, and has a perturbation energy $\mathcal{E} < \mathcal{E}_0$. Therefore $\mathbf{u}_0$ is not an NLOP because it has not the minimum $\mathcal{E}$ in $\mathcal{U}_R$. Analogously, in the case $(\mathrm{d}\mathcal{E}/\mathrm{d}t)_0 > 0$, a similar argument with $\Delta t < 0$ demonstrates that $\mathbf{u}_0$ is also not an NLOP. Indeed all NLOPs must be members of the set $\mathcal{Z}_R$ of perturbations with $\mathrm{d}\mathcal{E}/\mathrm{d}t = 0$ and furthermore satisfy $\mathrm{d}^2\mathcal{E}/\mathrm{d}t^2(t_0, \mathbf{u}_0, R) \geqslant 0$. By substitution in the evolution equation (3), it is easily verified that perturbations which are in $\mathcal{Z}_R$ must satisfy the condition $\langle \mathbf{u}, \mathcal{L}_R\,\mathbf{u}\rangle + \langle \mathbf{u}, \mathcal{N}(\mathbf{u})\rangle = 0$. For systems with energy-preserving nonlinear terms, the perturbations belonging to $\mathcal{Z}_R$ satisfy the simpler condition $\langle \mathbf{u}, \mathcal{L}_R\,\mathbf{u}\rangle = 0$, a condition which is independent of the amplitude of $\mathbf{u}_0$. This proves that:

**Theorem 2.1.** *In a nonlinear system with energy preserving nonlinear terms, a necessary condition for* $\mathbf{u}_0$ *to be a nonlinear optimal perturbation is that* $\langle \mathbf{u}_0, \mathcal{L}_R\,\mathbf{u}_0\rangle = 0$ *and* $\mathrm{d}^2\mathcal{E}/\mathrm{d}t^2(t_0, \mathbf{u}_0, R) \geqslant 0$.

## 3. Application to low dimensional models of subcritical transition

Several low dimensional models, reviewed in [4], have been considered to investigate qualitative features of subcritical transition. These models usually share properties of the Navier–Stokes equations that are thought to be relevant in subcritical transition. These include: (a) They admit the linearly stable laminar fixed point $\mathbf{u} = 0$, $\forall R$; (b) the linear operator $\mathcal{L}_R$ is non-normal in the sense defined above; (c) nonlinear terms preserve energy. We compute the minimum threshold energy $\delta$ and the associated nonlinear optimal perturbations for two of these models making use of the optimality condition stated in the previous section. The model systems we consider are four-dimensional. One, referred to henceforth as W95, has been inspired by the modeling of self-sustained processes in wall turbulence [7] while the other, referred to henceforth as W97, is inspired from a low dimensional Galerkin projection of the Navier–Stokes equations for a Couette-like shear flow [8]. Both models describe a nonlinear self-sustained process using the amplitude of streamwise vortices $v$, the amplitude of streamwise streaks $u$, the amplitude of sinuous perturbations of the streaks $w$ and the amplitude of the mean shear $m$ induced by these perturbations. The 4D state vector is defined as $\mathbf{u} = \{m, u, v, w\}^{\mathrm{T}}$ and the linear and nonlinear operators are defined as:

$$\mathcal{L}_R = \begin{bmatrix} -k_m^2/R & 0 & 0 & 0 \\ 0 & -k_u^2/R & \sigma_u & 0 \\ 0 & 0 & -k_v^2/R & 0 \\ 0 & 0 & 0 & -k_w^2/R - \sigma_m \end{bmatrix}; \qquad \mathcal{N}(\mathbf{u}) = \begin{Bmatrix} \sigma_m\, w^2 - \sigma_u\, u\, v \\ -\sigma_w\, w^2 + \sigma_u\, m\, v \\ \sigma_v\, w^2 \\ (\sigma_w\, u - \sigma_m\, m - \sigma_v\, v)\, w \end{Bmatrix} \tag{4}$$

The same coefficients as those considered in [7] and [8] have been selected.[1] The phase space dynamics of these systems have already been investigated in [7,8]. For model W95, a saddle-node bifurcation takes place at $R = 98.63$. The global stability of the basic flow is lost between $R = 98.63$ and $R = 101.03$ where another attractor exists with a minute basin of attraction. The basic flow is again globally stable between $R = 101.03$ and $R = 356$. Subsequently it loses again its global stability because a stable limit cycle appears and persists for larger $R$. For model W97 the first saddle-node bifurcation is at $R = 104.84$. The 'lower branch' solution is a saddle while the

[1] For the W95 model: $[k_m, k_u, k_v, k_w] = [3.16, 3.16, 3.16, 3.87]$, and $[\sigma_m, \sigma_u, \sigma_v, \sigma_w] = [0, 1, 1, 0.5]$. For the W97 model: $[k_m, k_u, k_v, k_w] = [1.57, 2.28, 2.77, 2.67]$, and $[\sigma_m, \sigma_u, \sigma_v, \sigma_w] = [0.31, 1.29, 0.22, 0.68]$.

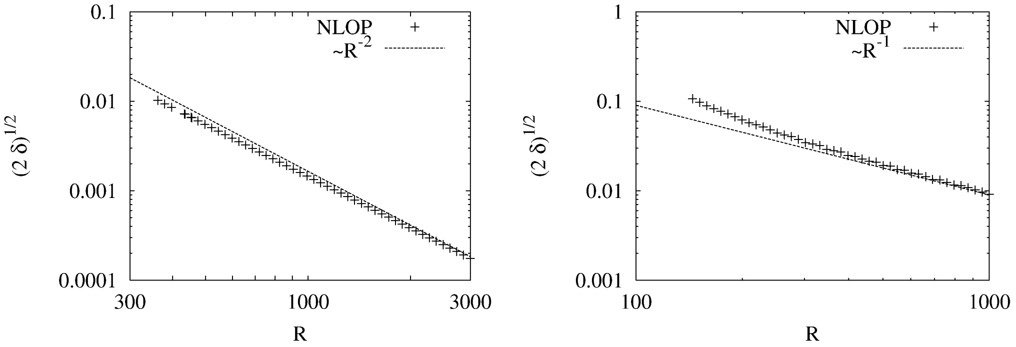


Fig. 1. Minimum threshold amplitude curve $(2\,\delta)^{1/2}(R)$ for respectively the W95 (left) and the W97 (right) model.

upper branch is an unstable fixed point, which becomes stable at $R_G = 138.06$, thus rendering the stability of the laminar basic flow only conditional for $R > R_G$. The perturbation energy is defined using the standard inner product as $\mathcal{E} = (m^2 + u^2 + v^2 + w^2)/2$ and it is easily found that the vectors belonging to $\mathcal{Z}_R$ must satisfy:

$$\sigma_u u_0 v_0 - \left[k_m^2 m_0^2 + k_u^2 u_0^2 + k_v^2 v_0^2 + (R\sigma_m + k_w^2) w_0^2\right]/R = 0 \tag{5}$$

Non-trivial solutions of Eq. (5) exist for $uv > 0$ and $R \geqslant R_E = 2k_u k_v/\sigma_u$. For the W95 model $R_E = 20$ while $R_E = 4.89$ for the W97 model. We computed the threshold $\delta$ and the corresponding NLOPs by selecting initial conditions on a sphere of sufficiently low $\mathcal{E}_0$ and then integrating the equations up to $t_{\max} = 100R$. Only 'potentially optimal' initial conditions in the $\mathcal{Z}_R$ set were considered. We randomly selected $m_0, u_0$ and $w_0$ and then retrieved $v_0$ solving Eq. (5) which is a second order algebraic equation in $v_0$. Real solutions for $v_0$ are found only when the discriminant of that equation is non-negative, which strongly reduces the number of initial conditions to be investigated. We then rescaled the solutions to $\mathcal{E}_0$ and excluded initial conditions for which $\mathrm{d}^2\mathcal{E}/\mathrm{d}t^2 < 0$. For sufficiently low $\mathcal{E}_0$ all the solutions end in a neighborhood of the laminar basic state $\mathcal{E}(t_{\max}) < \varepsilon$ demonstrating that the sphere is completely contained in the basin of attraction of the laminar basic state. The initial energy $\mathcal{E}_0$ is then increased by small steps $\Delta\mathcal{E}_0$ and the computations repeated at each step. The minimum threshold $\delta$ is found as the minimum value of $\mathcal{E}_0$ for which at least one solution does not satisfy the condition $\mathcal{E}(t_{\max}) < \varepsilon$. The corresponding initial condition represents the NLOP. This kind of computation was repeated for values of $R$ ranging from 300 to 3000 for the W95 model[2] and from 100 to 1000 for the W97 model. The minimum threshold amplitude curves $\sqrt{2\,\delta(R)}$ are shown in Fig. 1. These curves, originating at $R_G$, attain, for sufficiently large $R$, the asymptotic scalings, predicted using non-normal/nonlinear dominant balance arguments [3,4] ($\sim R^{-2}$ for the W95 model, as shown in [4], and $\sim R^{-1}$ for the W97 model, as it can be easily seen using arguments similar to those used for the 'Stockholm models' in [4]). The amplitude of the vortices is seen to be always the component of largest amplitude in the NLOPs and therefore in Fig. 2 we show the ratios $u_0/v_0$ and $w_0/v_0$ defining the 'shape' of the optimal perturbations at $t = t_0$ as a function of $R$. The optimal perturbations maximizing the linear energy growth (LOP) have $m_0^{(L)} = w_0^{(L)} = 0$ and, for sufficiently large $R$, the ratio $u_0^{(L)}/v_0^{(L)}$ of the linear optimals scales like $R^{-1}$ for both models. From Fig. 2 we see that, for both models, the ratio $u_0/v_0$ of the nonlinear optimal is very similar to the linear optimal ratio. However, although the NLOPs for the W97 model asymptotically converge to the linear optimal shape because $w_0$ becomes exponentially small, for the W95 model the NLOP remains different from the linear one because the $w/v$ component tends to a finite constant. This qualitative difference, however, does not seem to affect the agreement of the computed scaling of $\delta(R)$ with the scaling predicted by the non-normal/nonlinear dominant balance arguments.

[2] We neglected the small window of conditional stability situated between $R = 98.63$ and $R = 101.03$.

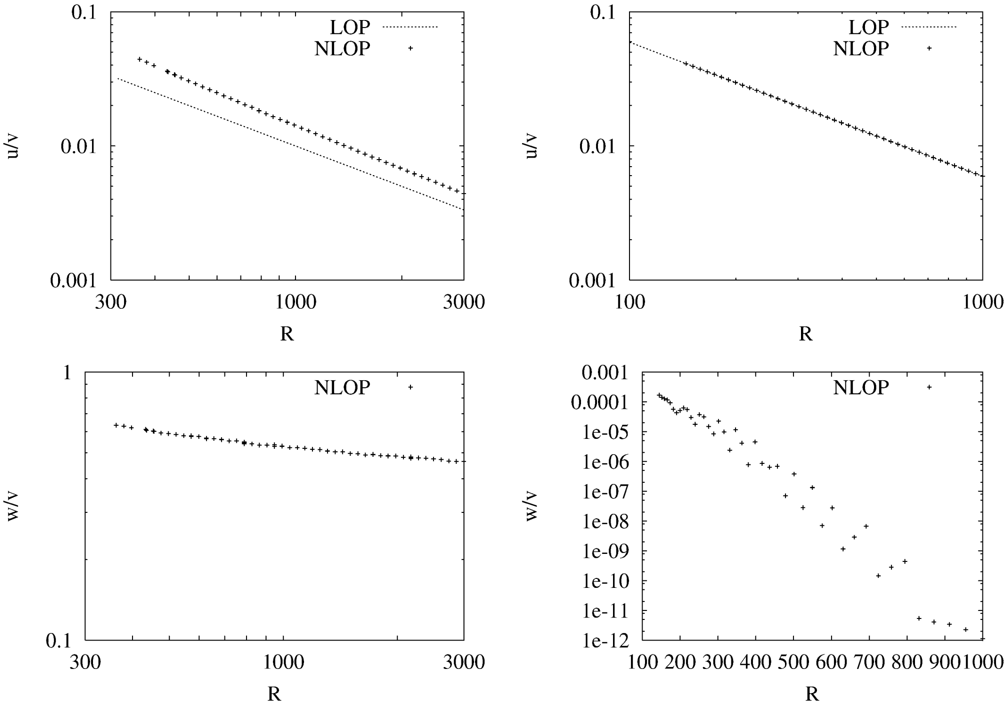


Fig. 2. Evolution with $R$ of the ratios $u_0/v_0$ (upper row) and $w_0/v_0$ (lower row) of nonlinear (NLOP) and linear (LOP) optimal perturbations for respectively the W95 (left) and the W97 (right) models. $w_0/v_0 = 0$ for the linear optimals and the curve is therefore not reported.

## 4. Summary

The scope of this study was to compute the minimum energy threshold $\delta(R)$ and the associated nonlinear optimal perturbations NLOP for low-dimensional systems of subcritical transition. We have demonstrated that a necessary condition for a perturbation to be nonlinearly optimal, in the sense that it is the one of minimum energy outside the basin of attraction of the laminar basic state, is that it realizes a local minimum for the perturbation energy i.e. $\mathrm{d}\mathcal{E}/\mathrm{d}t = 0$ and $\mathrm{d}^2\mathcal{E}/\mathrm{d}t^2 \geqslant 0$. For systems with energy preserving nonlinear terms the first condition of optimality is equivalent to the condition $\langle \mathbf{u}, \mathcal{L}_R \mathbf{u} \rangle = 0$ which can be determined by the analysis of the linear operator $\mathcal{L}_R$ and is independent on the amplitude of the perturbations. The enforcement of the optimality condition, reducing the number of perturbations to be investigated, has allowed the computation of $\delta(R)$ and the associated NLOPs for two four-dimensional models of subcritical transition [7,8]. The analysis of the results reveals that even if, in general, NLOPs do not have the same symmetries of the linear optimal perturbations, it is still to be noticed that: (a) the projection of NLOPs in the subspace optimizing the linear growth has almost the same shape of the linear optimal and (b) the minimum energy threshold asymptotically satisfy the $R^{\gamma}$-scalings predicted by using arguments based on dominant balance of non-normal growth and nonlinear-feedback [3,4]. However, this scaling, sets in only for relatively large values of $R$. It was conjectured [4] that for real flows the scaling coefficient is less than $-1$ for large $R$ but recent experimental results [5] suggest that, (e.g. for pipe Poiseuille flow) it assumes its upper limit $\gamma = -1$. However, in these experiments there is no attempt to optimize the *shape* of the initial condition in order to attain the *minimum* threshold energy. It would therefore be interesting to know if it is possible to get a lower scaling for the amplitude using initial conditions of optimized shape. This is the subject of current investigation.

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