Available online at www.sciencedirect.com
SCIENCE DIRECT®

C. R. Mecanique 333 (2005) 305–310

COMPTES RENDUS
MECANIQUE
http://france.elsevier.com/direct/CRAS2B/

Singular perturbations in shape optimization for the Dirichlet Laplacian

Serguei A. Nazarov [a], Jan Sokolowski [b]

[a] *Institute of Mechanical Engineering Problems, Laboratory of Mathematical Methods, Russian Academy of Sciences, V.O. Bol'shoi 61, 199178 St. Petersburg, Russia*
[b] *Institut Elie Cartan, laboratoire de mathematiques, université Henri Poincare Nancy I, BP 239, 54506 Vandoeuvre-les-Nancy cedex, France*

Received 30 November 2004; accepted after revision 8 February 2005

Presented by Évariste Sanchez-Palencia

Abstract

A shape optimization problem is considered for the Dirichlet Laplacian. Asymptotic analysis is used in order to characterise the optimal shapes which are finally given by a singular perturbation of the smooth initial domain. ***To cite this article: S.A. Nazarov, J. Sokolowski, C. R. Mecanique 333 (2005).***

Résumé

Perturbations singulières en optimisation des formes pour le Laplacien avec conditions de Dirichlet. Un problème d'optimisation de forme est posé pour l'énergie du Laplacien avec conditions de Dirichlet. Des formes optimales obtenues par l'analyse asymptotique sont données par une perturbation singulière du domain initial régulier. ***Pour citer cet article : S.A. Nazarov, J. Sokolowski, C. R. Mecanique 333 (2005).***

Keywords: Computational solid mechanics; Dirichlet Laplacian; Shape optimization problem

Mots-clés : Mécanique des solides numérique ; Laplacien de Dirichlet ; Problème d'optimisation de forme

1. Introduction

The classical problem of minimization of the energy functional for the Dirichlet Laplacian is addressed. In the literature the existence, regularity and stability of optimal shapes for the energy functional is usually considered under the perimeter constraints. In our Note the perimeter constraint is relaxed, and we analyse the shape optimization problem without any regularization term added to the energy functional. On the other hand, we introduce the small

E-mail addresses: serna@snark.ipme.ru (S.A. Nazarov), sokolows@iecn.u-nancy.fr (J. Sokolowski).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2005.02.006

parameter $\varepsilon > 0$ which measures the quantity of the material we could add to the initial geometrical domain, see (6), (7) for the definition of the class of admissible domains. We observe that within our admissible class and without any bounds imposed on the perimeter of the boundary, there is no boundary variation of the domain which maximizes the first order shape variation of the energy functional. Actually, without such a regularization constraint, of common use in the shape optimization, the Dirac mass supported at a point s_* of the boundary maximizes the directional derivative of the shape functional. Such a generalized solution is useless both from practical and theoretical points of view, in addition it destroys the asymptotic procedure used for the construction of solutions.

In order to solve the shape optimization problem within the admissible class, the new design function $H(s)$, $s \in \Gamma^0$, is defined in (14). We are looking for an asymptotic solution of the problem, i.e., for sufficiently small positive boundary variations of the reference domain Ω^0 which increases the volume of the domain by the value $\varepsilon > 0$. We construct three terms asymptotics of the energy functional and find out that the energy minimization involves the classical Steklov–Poincaré operator. The operator makes well-posed the variational inequality which we derive for optimal variations of admissible domains. This way an asymptotic solution of shape optimization problem in form (28) is obtained by solving variational inequality (20), which admits a solution for sufficiently small $\varepsilon > 0$.

The shape optimization problem is formulated in Section 2, basic known properties of Steklov–Poincaré operator are given in Section 3, the leading term of asymptotics of the energy functional is determined in Section 4, the variational inequality which defines the optimal shapes is introduced in Section 5, and finally the asymptotic optimal shapes are given in Section 6.

2. Formulation shape optimization problem

We denote by u_0 a solution to the Dirichlet problem in the domain Ω^0 bounded by a simple, smooth contour Γ^0:

$$-\Delta u_0(x) = f(x), \quad x \in \Omega^0, \qquad u_0(x) = 0, \quad x \in \Gamma^0 = \partial\Omega^0 \tag{1}$$

Problem (1) admits the unique solution $u_0 \in H^{2,2}(\Omega^0) \cap H_0^{1,2}(\Omega^0)$ for any $f \in L_2(\Omega)$, it is assumed for the simplicity of presentation that the right-hand side $f \geqslant 0$ has the support $\operatorname{supp} f \subset \Omega^0$. The potential energy of the solution is denoted by $\mathcal{E}_0 = \mathcal{E}(u_0; \Omega^0)$, and takes the form

$$\mathcal{E}(u_0; \Omega^0) = \frac{1}{2}\int_{\Omega^0} \left|\nabla u_0(x)\right|^2 \mathrm{d}x - \int_{\Omega^0} f(x)u_0(x)\,\mathrm{d}x = -\frac{1}{2}\int_{\Omega^0} f(x)u_0(x)\,\mathrm{d}x \tag{2}$$

Let $\mathcal{U}$ be an open neighbourhood of the contour Γ^0, (n, s) is a curvilinear coordinate system on Γ^0, s is the length parameter, n is the oriented distance to Γ^0 such that $n > 0$ outside of Ω^0. Given a small amount of material measured by $\varepsilon > 0$, the domain Ω^0 is replaced by the new domain Ω^ε, such that its boundary is parametrized by a function h

$$\Gamma^\varepsilon = \left\{x \in \mathcal{U};\ s \in \Gamma^0,\ n = h(s)\right\} \tag{3}$$

and h should be chosen in such a way that the increment $\Delta\mathcal{E}$ of the potential energy is minimized

$$\Delta\mathcal{E} = \mathcal{E}_\varepsilon - \mathcal{E}_0 = \mathcal{E}(u_\varepsilon; \Omega^\varepsilon) - \mathcal{E}(u_0; \Omega^0) \tag{4}$$

By u^ε is denoted a solution to the Dirichlet problem in the perturbed domain

$$-\Delta u_\varepsilon(x) = f(x), \quad x \in \Omega^\varepsilon, \qquad u_\varepsilon(x) = 0, \quad x \in \Gamma^\varepsilon = \partial\Omega^\varepsilon \tag{5}$$

In our setting $\Omega^0 \subset \Omega^\varepsilon$ and $\operatorname{meas}_2(\Omega^\varepsilon \setminus \Omega^0) = \varepsilon$, hence an admissible boundary variation requires some restrictions on the function h in (3), which can be introduced in the following way

$$h(s) \geqslant 0, \quad s \in \Gamma^0 \tag{6}$$

$$\operatorname{meas}_2(\Omega^\varepsilon \setminus \Omega^0) = \int_{\Omega^\varepsilon\setminus\Omega^0} \mathrm{d}x = \int_{\Gamma^0}\int_0^{h(s)} \left(1 + \varkappa(s)n\right)\mathrm{d}n\,\mathrm{d}s = \int_{\Gamma^0} \left(h(s) + \frac{1}{2}\varkappa(s)h(s)^2\right)\mathrm{d}s = \varepsilon \tag{7}$$

where $\varkappa$ stands for the curvature of Γ^0 and $(1+\varkappa(s)n)$ is the Jacobian in local coordinates. It is clear that the optimal function h, if it exists, depends on the small parameter $\varepsilon > 0$, which is not indicated in the sequel. In what follows, we consider h as a small functional parameter and derive some asymptotic formulae with respect to h. For brevity '$\cdots$' means the terms of higher order which can be neglected, however, here we do not precise the order of such terms.

In the classical theory of shape optimization [1] the following form of the first order variation of the energy shape functional for Dirichlet Laplacian is obtained

$$\Delta\mathcal{E} = -\frac{1}{2}\int_{\Gamma^0} h(s)\big|\partial_n u_0(0,s)\big|^2\,\mathrm{d}s + \cdots \tag{8}$$

However the problem of minimizing the right-hand side in (8) is ill-posed and leads *formally* to the solution $h(s) = \varepsilon\delta(s-s_*)$, where $\delta(s-s_*)$ denotes the Dirac mass concentrated at s_*, and $s_* \in \Gamma^0$ is a point at which the positive function $\Gamma^0 \ni s \mapsto |\partial_n u_0(0,s)|$ attains its maximum. Here we denote by $\partial_n u_0(0,s) = \partial_n u_0(x)$, the normal derivative on Γ^0.

The method of determination of the optimal domain applied in this note, is similar to the method used already in the fracture criteria [2–5], for a plane crack in the elastic space. Actually, the three terms asymptotic approximation of the potential energy is constructed using the following asymptotic ansatz for solutions to (5)

$$u^\varepsilon = u^0 + u' + u'' + \cdots \tag{9}$$

with the terms u', u'' linear and quadratic with respect to h. In particular, functional (4) can be replaced by its quadratic approximation which is minimized with respect to h and the procedure leads to a well-posed problem for determination of the function h for a given $\varepsilon > 0$. The resulting problem, which allows to evaluate the unknown boundary, takes the form of a variational inequality on the contour Γ^0. The bilinear form in the variational inequality is defined by a nonnegative integral operator. If the inequality constraint (6) is relaxed, and only the quadratic inequality constraint (7) is imposed, the variational inequality becomes an equation. Our approach is in fact different; we solve the variational inequality with the only sign constraints and then adapt the solution in such a way that the quadratic constraint is also verified. It is useful to point out that the asymptotic solution of the variational inequality gets a singular structure: the perturbation of the contour Γ^0 is supported in the $c\varepsilon^{1/4}$-neighbourhood of the point s_*, with magnitude of the order $\mathrm{O}(\varepsilon^{3/4})$. However, the difference in orders of the infinitesimal magnitudes $\varepsilon^{1/4}$ and $\varepsilon^{3/4}$ shows that the boundary perturbation although singular, it is still gently sloped.

3. Steklov–Poincaré operator

For the Dirichlet problem

$$-\Delta v(x) = 0, \quad x \in \Omega^0, \qquad v(x) = H(s), \quad x \in \Gamma^0 \tag{10}$$

we introduce a pseudo-differential operator, called *Steklov–Poincaré operator*, in the usual way [6]. First, we need the Poisson kernel $P(x;\sigma)$, a solution to the problem

$$-\Delta P(x;\sigma) = 0, \quad x \in \Omega^0; \qquad P(x;\sigma) = \delta(s-\sigma), \quad x \in \Gamma^0$$

with the asymptotics $P(x;\sigma) = -\pi^{-1}n(x-\sigma)^{-2} + \mathrm{O}(1)$, $x \to \sigma \in \Gamma^0$. Here, we use the same symbol σ for a point on Γ^0 and its coordinate (the length of the curve). Actually, after the regularization of hipersingular integral (see e.g., [2,7]) we obtain the following formulae

$$\mathcal{M}H(s) = \partial_n v(0,s) \tag{11}$$

$$\mathcal{M}H(s) = \int_{\Gamma^0}\big(H(s) - H(\sigma)\big)K(\sigma,s)\,\mathrm{d}\sigma + k(s)H(s) \tag{12}$$

where $K(\sigma,s)=-\partial_n P(0,s;\sigma)$, $k(s)=\mathcal{K}_0(s-0,s)-\mathcal{K}_0(s+0,s)$, with the symmetric and positive kernel $K(\sigma,s)$, which has the singularity

$$K(\sigma,s)=\pi^{-1}(x-\sigma)^{-2}+\mathrm{O}(1),\quad \sigma\to s\in\Gamma^0 \tag{13}$$

and $\mathcal{K}(\sigma,s)=\pi^{-1}(x-\sigma)^{-1}+\mathcal{K}_0(\sigma,s)$ is a primitive of the function $\Gamma^0\ni\sigma\mapsto K(\sigma,s)$.

The operator $\mathcal{M}$ is a symmetric, elliptic, pseudo-differential operator with the principal symbol $(2\pi)^{-1/2}|\xi|$. For our applications we set

$$H(s)=-h(s)\partial_n u_0(0,s)=h(s)\left|\partial_n u_0(0,s)\right| \tag{14}$$

and recall that by the maximum principle and under the restriction imposed on the function f the normal derivative of the solution to (1) is strictly negative on the boundary Γ_0.

4. Leading terms of the asymptotics

The solution u_0 is extended in a smooth way outside of Ω^0 and expanded in the Taylor series in the normal variable n. As a result, on the contour Γ^ε we have

$$u_0(h(s),s)=0+h(s)\partial_n u_0(0,s)+\frac{1}{2}h(s)^2\partial_n^2 u_0(0,s)+\cdots \tag{15}$$

Taking into account the form of the Laplace operator in the curvilinear coordinate system (n,s) and the equality $\Delta u=0$ on Γ^0, inherited from the assumption $f=0$ on Γ_0, we obtain that

$$\partial_n^2 u_0(0,s)=-\varkappa(s)\partial_n u_0(0,s) \tag{16}$$

The second term in right-hand side of (15) constitutes the main discrepancy in the boundary condition (5) on Γ^ε. The discrepancy is compensated by the solution of the problem

$$-\Delta u'(x)=0,\quad x\in\Omega^0,\qquad u'(x)=-H(s)=-h(s)\partial_n u_0(0,s),\quad x\in\Gamma^0 \tag{17}$$

We recognize here the so-called shape derivative of solution to the Dirichlet problem [1]. Integration by parts yields the leading part of the energy variation i.e., the term $-\frac{1}{2}\int_{\Omega^0} f(x)u'(x)\,\mathrm{d}x$ takes the form in the right-hand side of (8). Collecting second order terms in formula (15) and using the similar Taylor expansion for u' results in the boundary value problem for the function u'' in ansatz (9):

$$-\Delta u''(x)=0,\quad x\in\Omega^0,\qquad u''(x)=-h(s)\partial_n u'(0,s)-\frac{1}{2}h(s)^2\partial_n^2 u_0(0,s),\quad x\in\Gamma^0 \tag{18}$$

Therefore, taking into account equality (16) we find that

$$\Delta\mathcal{E}=-\frac{1}{2}\int\limits_{\Gamma^0} h(s)\left|\partial_n u_0(0,s)\right|^2\mathrm{d}s-\frac{1}{2}\int\limits_{\Gamma^0} h(s)\partial_n u_0(0,s)\left\{\partial_n u'(0,s)-\frac{1}{2}\varkappa(s)h(s)\partial_n u_0(0,s)\right\}\mathrm{d}s+\cdots \tag{19}$$

In addition, the normal derivative $\partial_n u'(0,s)$ can be expressed in terms of the right-hand side in the boundary conditions of (17) using the Steklov–Poincaré operator, namely, $\partial_n u'(0,s)=-\mathcal{M}[h(s)\partial_n u_0(0,s)]=\mathcal{M}[H(s)]$.

5. Variational inequality

In view of the expansion (19) we can consider the quadratic approximation of the energy functional

$$\mathcal{J}(h)=-\frac{1}{2}\int\limits_{\Gamma^0} h(s)\left|\partial_n u_0(0,s)\right|^2\mathrm{d}s-\frac{1}{2}\int\limits_{\Gamma^0} h(s)\partial_n u_0(0,s)\left\{\mathcal{M}\big[h(s)\partial_n u_0(0,s)\big]-\frac{1}{2}\varkappa(s)h(s)\partial_n u_0(0,s)\right\}\mathrm{d}s$$

The minimization of this functional over the cone $H_+^{1/2,2}(\Gamma^0)$ of nonnegative functions in the Sobolev–Slobodetskii space $H^{1/2,2}(\Gamma^0)$ leads to the variational inequality

$$H \in H_+^{1/2,2}(\Gamma^0)\colon\ \langle 2\mathcal{M}H + \Lambda\varkappa H - \Lambda\partial_n u_0, \psi - H\rangle \geqslant 0 \quad \forall\psi \in H_+^{1/2,2}(\Gamma^0) \tag{20}$$

where $\langle\,,\rangle$ denotes the duality pairing between $H^{1/2,2}(\Gamma^0)$ and $H^{-1/2,2}(\Gamma^0)$, as an extension of the scalar product in $L_2(\Gamma^0)$. We select

$$\Lambda = 1 - \left|\partial_n u_0(0,s)\right|^{-2}\lambda \tag{21}$$

so the solution H to (20) depends on the Lagrangian parameter λ which is to be chosen in such a way that H verifies the quadratic constraints (7) of the form

$$\int_{\Gamma^0} \left|\partial_n u_0(0,s)\right|^{-1} H(s)\left(1 + \frac{1}{2}\varkappa(s)\left|\partial_n u_0(0,s)\right|^{-1} H(s)\right)\mathrm{d}s = \varepsilon \tag{22}$$

In agreement with the properties of the kernel of $\mathcal{M}$ (see (13) and (20)), under the assumption

$$2k + \Lambda\varkappa > 0 \quad \text{on } \Gamma^0 \tag{23}$$

it turns out that quadratic term in $\mathcal{J}(h) = \frac{1}{2}a(h,h) - L(h)$ contains the bilinear form $a(\cdot,\cdot)$ which is symmetric and coercive in the space $H^{1/2,2}(\Gamma^0)$. The standard result for variational inequalities implies the existence of a unique solution to (20). In fact, more refined result was proved in [4,8,9].

Proposition 5.1. *Under condition* (23) *variational inequality* (20) *admits a unique solution* $H \in H_+^{1/2,2}(\Gamma^0)$ *for any right-hand side* $\Lambda\partial_n u_0 \in L_2(\Gamma^0)$. *The following estimate holds true*

$$\left\|H; H_+^{1/2,2}(\Gamma^0)\right\| \leqslant C\left\|\Lambda_+\partial_n u_0; L_2(\Gamma^0)\right\| \tag{24}$$

where $\Lambda_+ = (\Lambda + |\Lambda|)/2$. *If, in addition* $\Lambda\partial_n u_0 \in L_p(\Gamma^0)$ *for some* $p \in [2,\infty)$, *then the solution* H *belongs to the Sobolev space* $H^{1,p}(\Gamma^0)$ *and the following estimate is valid*

$$\left\|H; H^{1,p}(\Gamma^0)\right\| \leqslant C_p\left\|\Lambda_+\partial_n u_0; L_p(\Gamma^0)\right\| \tag{25}$$

We point out, that the exponent $p = \infty$ is excluded from the range. The reason is that in the case of $p = \infty$ the Hilbert transform: $L_p(\mathbb{R}) \mapsto L_p(\mathbb{R})$ changes properties and the argument of the proof used in the references does not apply. On the other hand, the asymptotic solution constructed in the next section belongs to the space $H^{1,\infty}(\Gamma^0)$.

6. Asymptotic solution

We assume that the positive function $\Gamma^0 \ni s \mapsto |\partial_n u_0(0,s)|$ attains the unique global maximum at the point s_*, which in addition is strong, i.e.,

$$\partial_n u_0(0,s) = A - a(s - s_*)^2 + \mathrm{O}\left(|s - s_*|^3\right) \tag{26}$$

with $A, a > 0$. From estimate (24), which contains the positive part Λ_+ of the quantity (21), we deduce that for the existence of a small nontrivial solution to the variational inequality (20) it is necessary that the Lagrangian multiplier takes the form

$$\Lambda = A(1 - \alpha) \tag{27}$$

where $\alpha > 0$ is a small parameter, which should be related with the parameter ε in the constraints (7) (=(22)). An asymptotic solution of variational inequality (20) is searched in the form

$$H(s) \sim (A\alpha)^{3/2}(2a)^{-1/2}\mathcal{H}(t), \quad t = (A\alpha)^{1/2}(2a)^{-1/2}(s - s_*) \tag{28}$$

The fast variable t and the multiplier at the unknown function $\mathcal{H}$ are chosen in such a way that after the corresponding changes and neglecting the higher order terms (with respect to $\alpha \to +0$) in variational inequality (20) depending on Λ, we obtain the variational inequality independent of α

$$\frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \big(\mathcal{H}(t)-\mathcal{H}(\tau)\big)\big(\Psi(t)-\mathcal{H}(t)\big)|t-\tau|^{-2}\,\mathrm{d}\tau\,\mathrm{d}t \geqslant \int_{\mathbb{R}} (1-t^2)\big(\Psi(t)-\mathcal{H}(t)\big)\,\mathrm{d}t$$
$$\forall \Psi \in C_0^\infty(\mathbb{R}),\ \Psi \geqslant 0 \tag{29}$$

which admits the unique solution

$$\mathcal{H}(t) = \begin{cases} 3^{-1}(2-t^2)^{3/2} & \text{for } t \leqslant \sqrt{2} \\ 0 & \text{for } t > \sqrt{2} \end{cases} \tag{30}$$

We note that the function $\mathcal{H}$ belongs to the space $H^{1,\infty}(\mathbb{R})$ and $H^{2-\mu,2}(\mathbb{R})$ for any $\mu > 0$.

Formulae (28), (14) and (30), (26) provide the asymptotic solution for arbitrarily small $\alpha > 0$. We insert the formula for h into (7) (or the expression for $\mathcal{H}$ into (22)) and taking into account only the leading term, we obtain the relation between the small parameters α and ε:

$$\varepsilon = \pi(2a)^{-1}A^2\alpha^2 \tag{31}$$

Assuming (23), which implies estimates (24) and (25), easily follows that the representation (28) effectively constitutes an asymptotic approximation of solutions to (20).

References

[1] J. Sokołowski, J.-P. Zolesio, Introduction to Shape Optimization. Shape Sensitivity Analysis, Springer-Verlag, 1992.

[2] S.A. Nazarov, Derivation of the variational inequality for small increase of mode-one crack, Mekh. Tverd. Tela 2 (1989) 152–160; English transl.: S.A. Nazarov, Mech. Solids 24 (1989) 145–152.

[3] S.A. Nazarov, O.R. Polyakova, On the equivalence of the fracture criteria for a mode-one crack in an elastic space, Mekh. Tverd. Tela 2 (1992) 101–113 (in Russian).

[4] L.H. Kolton, S.A. Nazarov, Quasistatic propagation of a mode-I crack in an elastic space, C. R. Acad. Sci. Paris. Sér. II 315 (1992) 1453–1457.

[5] M. Bach, S.A. Nazarov, W.L. Wendland, Stable propagation of a mode-1 crack in an isotropic elastic space. Comparison of the Irwin and the Griffith approaches, in: Problemi attuali dell'analisi e della fisica matematica, Aracne, Roma, 2000, pp. 167–189.

[6] V.I. Lebedev, V.I. Agoshkov, Poincare–Steklov Operators and Their Applications in Analysis, Akad. Nauk SSSR, Vychisl. Tsentr, Moscow, 1983, 18 pp. (in Russian).

[7] S.A. Nazarov, O.R. Polyakova, Deformation and tear-off of a thin gasket from hardly compressible material, Mekh. Tverd. Tela 5 (1993) 123–134 (in Russian).

[8] L.H. Kolton, S.A. Nazarov, Variation of the shape of the front of plane mode-one crack which is not in equilibrium locally, Mekh. Tverd. Tela 3 (1997) 125–133 (in Russian).

[9] M. Bach, S.A. Nazarov, Smoothness properties of solutions to variational inequalities describing propagation of mode-1 cracks, in: Mathematical Aspects of Boundary Element Method (Palaiseau, 1998), in: CRC Res. Notes Math., vol. 414, Chapman & Hall, London, 2000, pp. 23–32.