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Subgrid models preserving the symmetry group of the Navier–Stokes equations

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Abstract

In order to preserve the physical properties of the flow (scaling laws, conservation laws, ...) during the simulation, a class of subgrid models respecting the symmetry group of the Navier–Stokes equations is built. The class is then refined such that models satisfy the second law of thermodynamics and are suited to take into account the inverse energy cascade. A simple model belonging to the class is tested and a better result than those provided by Smagorinsky and dynamic models is obtained. ***To cite this article: D. Razafindralandy, A. Hamdouni, C. R. Mecanique 333 (2005).***

Résumé

Modèles de sous-maille préservant le groupe de symétrie des équations de Navier–Stokes. Dans le but de respecter les propriétés physiques de l'écoulement (lois d'échelle, lois de conservation, ...) lors de la simulation, on construit une classe de modèles de sous-maille conservant le groupe de symétrie des équations de Navier–Stokes. On raffine ensuite cette classe de telle sorte que les modèles satisfassent le second principe de la thermodynamique et qu'ils soient capables de prendre en compte la cascade inverse d'énergie. Un modèle simple de la classe est testé dans le cas d'un écoulement dans une chambre ventilée. Les premiers calculs donnent un résultat nettement meilleur que ceux obtenus avec le modèle de Smagorinsky et le modèle dynamique. ***Pour citer cet article : D. Razafindralandy, A. Hamdouni, C. R. Mecanique 333 (2005).***

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Mots-clés : Mécanique des fluides numérique ; Turbulence ; Modèles de sous-maille ; Groupe de symétrie

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Considérons un fluide newtonien incompressible de densité ρ et de viscosité cinématique ν. Le mouvement de ce fluide est régi par les équations de Navier–Stokes (1). L'ensemble des solutions de ces équations est invariant par les transformations (2)–(6) qui forment un groupe dénommé groupe de symétrie des équations de Navier–Stokes. Ce groupe joue un rôle important à plusieurs titres. D'abord, d'aprés le théorème de Nœther [1], les grandeurs physiques conservées sont issues des éléments de ce groupe. Ensuite, différentes solutions des équations peuvent s'en déduire [4]. En particulier, le groupe des transformations d'échelle est fondamental pour les lois de paroi [2]. En quelque sorte, le groupe de symétrie contient la physique des équations.

Notre approche consiste, d'une part, à imposer aux modèles LES d'être invariants par ce groupe de symétrie et, d'autre part, de respecter le second principe de la thermodynamique qui requiert ici que la dissipation générée par le modèle soit positive.

Pour construire une classe de modèles LES respectant le principe énnoncé ci-dessus, on postule dans un premier temps que la contrainte de sous-maille τ_s est fonction du taux de déformation filtré $\overline{S}$; une telle forme satisfait les symétries (2), (3) et (5), ainsi que l'objectivité. En imposant aux modèles de conserver la rotation (4) et la transformation d'échelle (6), on construit la forme générale du modèle en fonction des invariants de $\overline{S}$, ce qui est donnée par l'Éq. (8).

Afin d'exprimer le second principe dans un cadre semblable à celui utilisé dans les lois de comportement de la mécanique des milieux continus, on postule que τ_s dérive d'un pseudo-potentiel (9) par rapport à $\overline{S}$. Ceci permet d'écrire les modèles en fonction d'une seule application g et des invariants de $\overline{S}$. On a alors la relation (11). Enfin, en exprimant que la dissipation induite par le modèle doit être positive, on obtient que :

$$\nu + g \geqslant 0$$

Nous testons ensuite un modèle simple où $g = C\nu$, la constante C étant une fonction de la constante de Smagorinsky. Le test numérique porte sur la configuration d'une piéce ventilée (Fig. 1) qui nous intéresse pour des applications dans l'étude des phénoménes de transfert dans les espaces habitables. Les résultats du test montre que le modèle proposé, que l'on a dénommé modèle invariant, améliore nettement le résultat par rapport aux modèles d'usage courant, qui sont les modèles de Smagorinsky et le modèle dynamique.

Au delà des validations, cette approche permet la construction de modèles LES consistants avec les propriétés des équations de Navier–Stokes. On pourra envisager une modélisation plus fine en utilisant dans la liste des arguments de τ_s, outre $\overline{S}$, des grandeurs comme la pseudo-dissipation ε ou la fonction de structure introduite dans [8].

1. Introduction

Navier–Stokes equations admit transformations (on time, space, velocity and pressure variables) which transform solutions to other solutions. The set of such transformations is a group called the Navier–Stokes symmetry group. This group plays an important role in the existence of conservation laws [1] and wall laws [2].

A turbulence modeling by large eddy simulation (LES) approach consists of filtering Navier–Stokes equations and modeling the arising additional terms. A consistent model respects the symmetry group of the original equations. Unfortunately, several subgrid models in literature (Smagorinsky, ...) violate this group, as shown by Oberlack in [3]. In addition, among the standard models, many do not satisfy the second law of thermodynamics because they induce negative dissipation. An a posteriori forcing becomes then unavoidable for these models.

In this Note, a class of models which both preserve the symmetry group of Navier–Stokes equations and naturally respect the second law of thermodynamics is proposed. This document is then organized as follow. In Section 2, the symmetry transformations of Navier–Stokes equations are reminded. A class of models preserving the symmetry group is then build in Section 3. This class is refined in Section 4 to models which satisfy the second law of thermodynamics. A numerical validation of a simple model of the class will conclude the document.

2. The symmetry group of Navier–Stokes equations

Consider an incompressible Newtonian fluid, with density ρ and kinematic viscosity ν. Let t and x be respectively the time and space variables, u the velocity field and p the pressure. The motion of the fluid is governed by Navier–Stokes equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \operatorname{div}(u \otimes u) + \dfrac{1}{\rho}\nabla p = \operatorname{div}\tau \\ \operatorname{div} u = 0 \end{cases} \tag{1}$$

where τ is the viscous constraint tensor,[1] which can be linked to the strain rate tensor $S = (\nabla u + \nabla u^T)/2$ according to the relation:

$$\tau = \frac{\partial \psi}{\partial S}$$

ψ being a positive and convex potential defined by:

$$\psi = \nu \operatorname{tr} S^2$$

Let $y = (t, x, u, p)$. We will call a one-parameter transformation a transformation

$$T_a : y \mapsto \hat{y} = \hat{y}(y, a)$$

which depends continuously on the parameter a. We will then say that T_a is a symmetry of (1) if it transforms the solutions of (1) to other solutions.

Thanks to Lie group theory [4], we can compute all one-parameter symmetries of Navier–Stokes equations. They are generated by:

– *time translation*:

$$(t, x, u, p) \mapsto (t + a, x, u, p) \tag{2}$$

– *pressure translation*:

$$(t, x, u, p) \mapsto \big(t, x, u, p + \zeta(t)\big) \tag{3}$$

– *rotation*:

$$(t, x, u, p) \mapsto (t, Rx, Ru, p) \tag{4}$$

– *generalized Galilean transformation*:

$$(t, x, u, p) \mapsto \big(t, x + \alpha(t), u + \alpha'(t), p - \rho x \cdot \alpha''(t)\big) \tag{5}$$

– and *scaling transformation*:

$$(t, x, u, p) \mapsto (a^2 t, ax, a^{-1}u, a^{-2}p) \tag{6}$$

[1] It is an abusive language because the real viscous constraint tensor is $\rho\tau$.

In these expressions, a is a parameter, ζ and α are arbitrary functions of time and R a rotation matrix. The symmetry group of Navier–Stokes equations is the group generated by all these transformations.

Symmetries have an important role when describing physical phenomena. They traduce the existence of conservation laws via Nœther's theorem [1]. They are also useful to find scaling laws [2] and exact solutions [5], or even to recover Kolmogorov spectra [6]. To some extent, the symmetry group contains the physics of the equations. It is then essential that turbulence models (RANS or LES) preserve the symmetry group of Navier–Stokes equations. However, many models in literature do not respect this group, as shown by Oberlack [3]. In the next section, we then build a class of subgrid models which preserve all the symmetries of (1) cited previously.

3. A class of invariant models

After filtering, Navier–Stokes equations becomes:

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + \operatorname{div}(\bar{u} \otimes \bar{u}) + \dfrac{1}{\rho} \nabla \overline{p} = \operatorname{div}(\bar{\tau} + \tau_s) \\ \operatorname{div} \bar{u} = 0 \end{cases} \tag{7}$$

where $\tau_s = -\overline{u \otimes u} + \bar{u} \otimes \bar{u}$ is the subgrid stress tensor. Now, our goal is to build models for τ_s such that each symmetry T_a of (1) is a symmetry of (7). When a model verifies this property, we say that it is invariant under the transformation T_a.

Suppose that τ_s is a function of $\overline{S}$ alone: $\tau_s = \mathcal{F}(\overline{S})$. Then, the invariance of τ under transformations (2), (3) and (5) are ensured.

Next, Cayley–Hamilton theorem and rotation invariance (4) impose to τ_s the following form: $\tau_s^d = A(\chi, \zeta)\overline{S} + B(\chi, \zeta) \operatorname{Adj}^d \overline{S}$ where $\chi = \operatorname{tr} \overline{S}^2$ and $\zeta = \det \overline{S}$ are the fundamental invariants of $\overline{S}$, Adj is the operator defined by $(\operatorname{Adj} \overline{S})\overline{S} = (\det \overline{S}) I_d$ and M^d denotes the deviatoric part $M^d = M - \frac{1}{3}(\operatorname{tr} M) I_d$ of an endomorphism M.

Finally, τ_s is invariant under scale transformation (6) only if $\hat{\tau}_s = a^{-2}\tau_s$. Rewritten for A and B, this condition becomes

$$A\left(\frac{1}{a^4}\chi, \frac{1}{a^6}\zeta\right) = A(\chi, \zeta), \qquad B\left(\frac{1}{a^4}\chi, \frac{1}{a^6}\zeta\right) = a^2 B(\chi, \zeta)$$

Deriving according to a and taking $a = 1$, it follows:

$$-4\chi \frac{\partial A}{\partial \chi} - 6\zeta \frac{\partial A}{\partial \zeta} = 0, \qquad -4\chi \frac{\partial B}{\partial \chi} - 6\zeta \frac{\partial B}{\partial \zeta} = 2B$$

In order to satisfy these equalities, we can take:

$$A(\chi, \zeta) = A_1\left(\frac{\zeta}{\chi^{3/2}}\right), \qquad B(\chi, \zeta) = \frac{1}{\sqrt{\chi}} B_1\left(\frac{\zeta}{\chi^{3/2}}\right)$$

Consequently, if we note $v = \zeta / \chi^{3/2}$, we get

$$\tau_s^d = A_1(v)\overline{S} + \frac{1}{\sqrt{\chi}} B_1(v) \operatorname{Adj}^d \overline{S} \tag{8}$$

Thus, a subgrid model belonging to the class (8) captures necessarily all the symmetries of Navier–Stokes equations.

Let us return to considerations which are more specific to LES. We know that τ_s represents the energy exchange between resolved and subgrid scales. Then, it generates certain dissipation. To respect the second law of thermodynamics, we must ensure that the total dissipation remains positive, which is not always verified by models in the literature. In order to satisfy this condition, we refine the class (8).

4. Consequences of the second law of thermodynamics

At molecular scale, the viscous constraint is $\tau = \frac{\partial\psi}{\partial S}$. The potential $\psi = \nu \operatorname{tr} S^2$ is convex and positive, which ensure a positive dissipation: $\Phi = \operatorname{tr}(\tau S) \geqslant 0$. τ_s can be considered as a constraint, inducing a dissipation $\Phi_s = \operatorname{tr}(\tau_s \overline{S})$. To remain compatible with Navier–Stokes equations, we suppose that the subgrid stress tensor has the same form as τ:

$$\tau_s = \frac{\partial\psi_s}{\partial\overline{S}} \tag{9}$$

where ψ_s is a potential depending on the invariants χ and ζ. This hypothesis restricts the class (8) in the following way.

Since $\operatorname{tr}\overline{S} = 0$, it can be deduced from (9) that

$$\tau_s^d = 2\frac{\partial\psi_s}{\partial\chi}\overline{S} + \frac{\partial\psi_s}{\partial\zeta}\operatorname{Adj}^d\overline{S}$$

This form is compatible with (8) only if

$$\frac{\partial}{\partial\zeta}\left(\frac{1}{2}A_1(v)\right) = \frac{\partial}{\partial\chi}\left(\frac{1}{\sqrt{\chi}}B_1(v)\right)$$

If g is a primitive of B_1, a solution of this equation is:

$$A_1(v) = 2g(v) - 3vg'(v) \quad \text{and} \quad B_1(v) = g'(v) \tag{10}$$

Then, the hypothesis (9) involves the existence of a function g such that:

$$\tau_s^d = \left[2g(v) - 3vg'(v)\right]\overline{S} + \frac{1}{\sqrt{\chi}}g'(v)\operatorname{Adj}^d\overline{S} \tag{11}$$

Let us now designate the total dissipation by Φ_T. We have: $\Phi_T = \operatorname{tr}\left[(\bar{\tau} + \tau_s)\overline{S}\right]$. Using (8) and (10), we then prove that:

$$\Phi_T \geqslant 0 \Leftrightarrow \nu + A_1(v) + 3vB_1(v) \geqslant 0 \Leftrightarrow \nu + g(v) \geqslant 0$$

In summary, as soon as

$$\nu + g \geqslant 0 \tag{12}$$

a model of the class (11) is a model which respects the symmetry group of Navier–Stokes and conforms to the second law of thermodynamics.

Notice that Φ_s may have negative values. The models are then able to take into account the inverse energy cascade. Secondly, it can be shown that v is bounded. In consequence, (12) must be satisfied only on a bounded interval.

5. Numerical test and conclusion

For the numerical test, we choose a simple linear function for g: $g(v) = Cv$ where $C = \nu(C_s\bar{\delta}/\ell)^2$ C_s the Smagorinsky constant, $\bar{\delta}$ the filter width and ℓ is a scale length related to the size of the domain. The corresponding model, which we call invariant model, is used to simulate the air flow inside a tridimensional ventilated room (Fig. 1). ℓ is set to 1m. A staggered $72 \times 52 \times 26$ grid is used.

The velocity field provided by the invariant model, at $x_3/W = 1/2$, is presented in Fig. 2. Fig. 3 shows the velocity profiles given by Smagorinsky, dynamic (see [7]) and invariant models, at $x_1/L = 2/3$. It can be observed

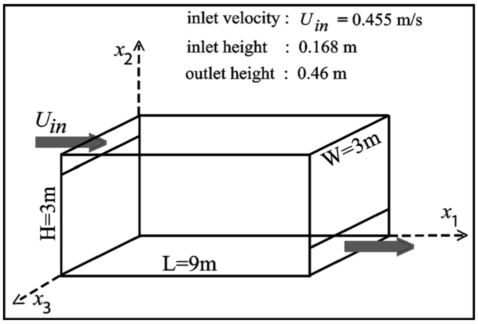

Fig. 1. Geometry of the ventilated room.

Fig. 1. Géométrie de la chambre ventilée. Hauteur d'entrée : 0,168 m, hauteur de sortie : 0,46 m.

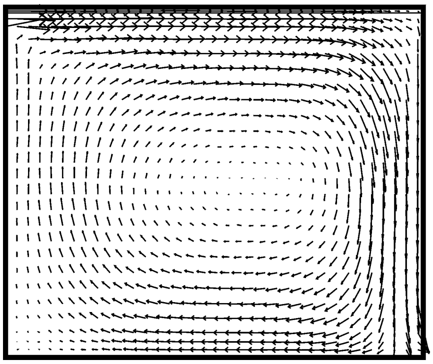

Fig. 2. Mean velocity field at $x_3/W = 1/2$.

Fig. 2. Champ de vitesse moyen en $x_3/W = 1/2$.

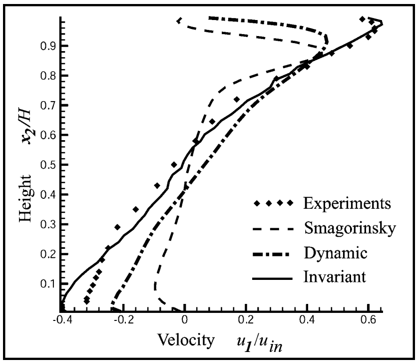

Fig. 3. Mean velocity profile at $x_1/L = 2/3$.

Fig. 3. Profil de vitesse moyen en $x_1/L = 2/3$.

on it that the invariant model gives a result in good agreement with experimental data, except in a small region near the floor. In all cases, the result is better than those provided by the two other models. Notice that neither a grid refinement nor a wall model is used.

Beyond the numerical validation, the present approach permitted the construction of models which are consistent with the properties of Navier–Stokes equations. Other parameters, such as the pseudo-dissipation or the structure function [8] could also be introduced in the model.

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