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C. R. Mecanique 333 (2005) 453–458

COMPTES RENDUS MECANIQUE

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On nano-scale hydrodynamic lubrication models

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Received 1 December 2004; accepted 29 April 2005

Presented by Patrick Huerre

Abstract

Current magnetic head sliders and other micromechanisms involve gas lubrication flows with gap thicknesses in the nanometer range and stepped shapes fabricated by lithographic methods. In mechanical simulations, rarefaction effects are accounted for by models that propose Poiseuille flow factors which exhibit singularities as the pressure tends to zero or $+\infty$. In this Note we show that these models are indeed mathematically well-posed, even in the case of discontinuous gap thickness functions. Our results cover popular models that were not previously analyzed in the literature, such as the Fukui–Kaneko model and the second-order model, among others. ***To cite this article: G. Buscaglia et al., C. R. Mecanique 333 (2005).***

Résumé

À propos de modèles de lubrification hydrodynamique à l'échelle nanométrique. Actuellement, de nombreux microdispositifs tels que les têtes de lecture magnétiques mettent en oeuvre des écoulements lubrifiés compressibles d'une épaisseur de film à l'échelle nanométrique. Leurs géométries, obtenues par des méthodes lithographiques, sont pratiquement discontinues. Dans les simulations, les effets de raréfaction sont incorporés dans des modèles, couramment utilisés en lubrification (modèle de Fukui–Kaneko, modèle de deuxième ordre, etc.) qui font intervenir les facteurs de Poiseuille qui deviennent singuliers quand la pression tend vers zero ou vers $+\infty$. Dans cette Note nous montrons que ces modèles sont mathématiquement bien posés, même avec des fonctions d'épaisseur discontinues. ***Pour citer cet article : G. Buscaglia et al., C. R. Mecanique 333 (2005).***

Keywords: Computational fluid mechanics; Hydrodynamic lubrication models; Rarefaction effects; Numerical simulation; Magnetic storage

Mots-clés : Mécanique des fluides numérique ; Modèles de lubrification hydrodynamique ; Effets de raréfaction ; Simulation numérique ; Enregistrement magnétique

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1631-0721/$ – see front matter
doi:10.1016/j.crme.2005.04.006

1. Introduction

Several current technologies involve lubrication flows of gases with ultra-low gap thicknesses of a few nanometers. Typical examples are rigid disks used for magnetic storage and micromechanisms such as silicon accelerometers. Models for these flows are necessarily based on the kinetic theory of gases, so as to account for compressibility and rarefaction effects. Along the years, effective numerical methods have been proposed to deal with the resulting equations (e.g., [1]) and they have been applied to static and dynamic analyses and more recently incorporated into optimal-design methodologies [2–5,1].

The mathematics of kinetic-based lubrication equations, on the other hand, has received little attention in the literature. In the steady case the problem consists of a diffusion–advection-like elliptic equation (very similar to the usual Reynolds equation) with Dirichlet boundary conditions, but classical results are not applicable because the diffusion coefficient is a non-linear function of the pressure that diverges as p tends to either zero or $+\infty$. Modern (lithographic) fabrication technologies add another mathematical difficulty which is not considered in previous analyses of the Reynolds equation [6–9]: the gap thickness is practically *discontinuous*.

The purpose of this Note is thus to bridge the gap between theory and state-of-the-art applications in the area of rarefied compressible lubrication problems. We show below that the equations resulting from kinetic-based models indeed lead to well-posed problems in what concerns existence, uniqueness and positivity of solutions. The results are stated so as to encompass a wide family of models.

2. Governing equations

Let Ω be a regular bounded domain in $\mathbb{R}^2$. In this work we consider the Generalized Reynolds Equation

$$\begin{cases} \nabla \cdot \big(h^2(x) Q(h(x)p)\nabla p\big) = \nabla \cdot \big(\Lambda h(x) p\big) & \text{in } \Omega \\ p = p_a & \text{on } \partial\Omega \end{cases} \tag{1}$$

in which p represents the (unknown) pressure in the fluid film between two given surfaces in relative motion, $h : \Omega \to]0, +\infty[$ is the gap thickness, $Q :]0, \infty[\to]0, \infty[$ is the Poiseuille flow factor, $\Lambda \in \mathbb{R}^2$ is the bearing number and p_a is a positive constant (typically the atmospheric pressure).

The Poiseuille-flow factor Q is assumed to satisfy the following hypothesis

$$\begin{aligned} & Q \text{ is continuous on }]0, +\infty[\\ & \exists \alpha > 0 \text{ such that } Q(z) \geqslant \alpha \ \forall z > 0 \\ & Q(z) \to +\infty \text{ for } z \to 0 \text{ and for } z \to +\infty \end{aligned} \tag{H0}$$

The most popular kinetic-based model, due to Fukui and Kaneko [10], the second- and third-order models (see, e.g., [11] for a description, [2] for a recent application) and more recent variants such as that proposed by Peng et al. [12] indeed satisfy (H0).

We impose very weak regularity requirements on the gap-thickness function h, namely

$$\begin{aligned} & h \in L^\infty(\Omega) \\ & \exists h_m > 0,\ h_M > 0 \text{ such that } h_m \leqslant h(x) \leqslant h_M \ \forall x \in \Omega \end{aligned} \tag{H1}$$

so that discontinuous shapes, such as those obtained by lithographic fabrication methods, are indeed considered.

3. Existence and uniqueness

We begin with two preliminary results.

Proposition 3.1. *For any $s > 2$ and $u \in H_0^1(\Omega)$ we have*

$$\|u\|_{L^s(\Omega)} \leqslant \frac{s}{2\sqrt{2}}|\Omega|^{1/s}\left(\sum_{i=1}^{2}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(\Omega)}\right)$$

Proof. The demonstration is based upon the inequality [13, Theorem 7.10]

$$\|u\|_{L^{2p/(2-p)}(\Omega)} \leqslant \frac{p}{2-p}\frac{1}{\sqrt{2}}\left(\sum_{i=1}^{2}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}\right)$$

for any $p \in]1,2[$ and $u \in H_0^1(\Omega)$. Taking now $s = \frac{2p}{2-p}$ and using the Hölder's inequality the result follows.

Lemma 3.2. *Let $\tilde{b} \in L^\infty(\Omega)^2$ be a vector field and $\tilde{a} \in L^\infty(\Omega)$ be a scalar function with $\inf_{x\in\Omega}\tilde{a}(x) > 0$. Let $u \in H^1(\Omega)$ satisfy*

$$\nabla\cdot(\tilde{a}\nabla u) = \nabla\cdot\tilde{b} \quad \text{in } \Omega \tag{2}$$

and let

$$\omega_0 = \{x \in \Omega:\ u(x) > 0\}$$

Assume that ω_0 is a nonempty set such that $\overline{\omega_0} \subset \Omega$. Then the following L^∞-estimate holds

$$u(x) \leqslant K\frac{\|\tilde{b}\|_{L^\infty(\omega_0)}}{\tilde{a}_0} \quad \text{for a.e. } x \in \omega_0$$

where $K = \bar{s}\,2^{-(\bar{s}-3)/(\bar{s}-2)}\sqrt{|\Omega|}$ with $\bar{s} = 2 + \ln 2 + \sqrt{4\ln 2 + (\ln 2)^2}$ and $\tilde{a}_0 = \inf_{x\in\omega_0}\tilde{a}(x)$.

Proof. The proof is adapted from of Kinderlehrer–Stampacchia [14, Theorem B.2 (page 63)]. For $k > 0$ we define

$$\xi = \begin{cases} u-k & \text{if } u \geqslant k \\ 0 & \text{if } u \leqslant k \end{cases}$$

and

$$A(k) = \{x \in \Omega:\ u(x) \geqslant k\}$$

Taking ξ as test function in the variational form of (2), since $A(k) \subset \omega_0$ one gets the result following the same idea as in [14, Theorem B.2 (page 63)] and using also Proposition 3.1. □

Theorem 3.3. *Let $p_* \in]0, p_a[$ and $\gamma_* \in]0,1[$ be such that*

$$\inf_{s\in[h_m\gamma_* p_*, h_M p_*]} Q(s) \geqslant \frac{K|\Lambda|h_M}{h_m^2(1-\gamma_*)} \tag{3}$$

and $p^* > p_a$ and $\gamma^* > 1$ such that

$$\inf_{s\in[h_m p^*, h_M p^*\gamma^*]} Q(s) \geqslant \frac{K|\Lambda|h_M\gamma^*}{h_m^2(\gamma^*-1)} \tag{4}$$

Problem (1) *admits at least one solution satisfying*

$$\gamma_* p_* \leqslant p(x) \leqslant \gamma^* p^* \tag{5}$$

Proof. Notice first that the existence of p_*, γ_*, p^* and γ^* verifying (3) and (4) follows from hypothesis (H0) with no monotonicity requirements on Q. To prove the existence of p we use Schauder's fixed-point theorem. Let us define the set

$$B_* = \{u \in L^2(\Omega);\ \gamma_* p_* \leqslant u(x) \leqslant \gamma^* p^* \text{ a.e. in } \Omega\}$$

which is a closed subset of $L^2(\Omega)$. We introduce the operator

$$T : B_* \to H^1(\Omega)$$

defined by $q = Tp$, where q is the solution of

$$\begin{cases} \int_\Omega h^2(x) Q\big(h(x)p\big)\nabla q \cdot \nabla v = \int_\Omega \Lambda h(x) p \cdot \nabla v \quad \forall v \in H_0^1(\Omega) \\ q \in p_a + H_0^1(\Omega) \end{cases} \tag{6}$$

Existence and uniqueness of q follows from (H0) and (H1). We now introduce the truncation function $\theta : \mathbb{R} \to \mathbb{R}$ defined by

$$\theta(s) = \begin{cases} \gamma_* p_* & \text{if } s < \gamma_* p_* \\ s & \text{if } \gamma_* p_* \leqslant s \leqslant \gamma^* p^* \\ \gamma^* p^* & \text{if } s > \gamma^* p^* \end{cases} \tag{7}$$

and we consider the operator $S : B_* \to B_*$ given by

$$(Sp)(x) = \theta\big(q(x)\big) \quad \text{a.e. in } \Omega$$

where $q = Tp$. We show classically that S admits a fixed point, denoted $\bar{q}$. Denoting $q = T\bar{q}$, it remains to show that $\bar{q}(x) = q(x)$, that is, that $\gamma_* p_* \leqslant q(x) \leqslant \gamma^* p^*$. To show the first inequality let us take $\omega_* = \{x \in \Omega\colon\ q(x) < p_*\}$. Since $p_* < p_a$ we also have $\overline{\omega_*} \subset \Omega$. If $\omega_* = \emptyset$ the proof is finished, so that it remains to consider the case $\omega_* \neq \emptyset$. Let us set $u = p_* - q$, which is positive on ω_*. Then $u \in H^1(\Omega)$ and satisfies $\nabla \cdot (h^2 Q(h\bar{q})\nabla u) = -\nabla \cdot \Lambda h\bar{q}$. The inequality of Lemma 3.2 thus applies with $\omega_0 = \omega_*$. As $\bar{q}(x) \in [\gamma_* p_*, p_*]$ a.e. in ω_*, we obtain

$$h^2(x) Q\big(h(x)\bar{q}(x)\big) \geqslant h_m^2 \inf_{s \in [h_m \gamma_* p_*, h_M p_*]} Q(s)$$

and

$$|\Lambda h\bar{q}| \leqslant |\Lambda| h_M p_*$$

Lemma 3.2 and relation (3) then imply

$$u(x) \leqslant K \frac{|\Lambda| h_M p_*}{h_m^2 \inf_{s \in [h_m \gamma_* p_*, h_M p_*]} Q(s)} \leqslant (1 - \gamma_*) p_* \quad \text{for a.e. } x \in \omega_*$$

or, equivalently, $p_* - q \leqslant (1 - \gamma_*) p_*$ in ω_*, and we get $q \geqslant \gamma_* p_*$ a.e. in Ω as claimed. To show the other inequality, $q(x) \leqslant \gamma^* p^*$, we proceed in a similar manner by taking $\omega^* = \{x \in \Omega\colon\ q(x) > p^*\}$, $u = q - p^*$, and then use Lemma 3.2 and relation (4). □

Remark 1. In the frequent case in which there exist z_1, z_2, $0 < z_1 \leqslant z_2$, such that Q is strictly decreasing in $]0, z_1[$ and strictly increasing in $]z_2, +\infty[$ we have the following bounds on p

$$p^- \leqslant p(x) \leqslant p^+ \quad \text{a.e. } x \in \Omega$$

with

$$p^- = \sup_{\gamma_* \in]0,1[} \gamma_* p_*(\gamma_*) \quad \text{and} \quad p^+ = \inf_{\gamma^* > 1} \gamma^* p^*(\gamma^*)$$

with

$$p_*(\gamma_*) = \begin{cases} \dfrac{z_1}{h_M} & \text{if } \dfrac{K|\Lambda|h_M}{h_m^2(1-\gamma_*)} \leqslant Q(z_1) \\ \dfrac{1}{h_M} Q_1^{-1}\left(\dfrac{K|\Lambda|h_M}{h_m^2(1-\gamma_*)}\right) & \text{if } \dfrac{K|\Lambda|h_M}{h_m^2(1-\gamma_*)} > Q(z_1) \end{cases}$$

where $Q_1^{-1} : [Q(z_1), +\infty[\to]0, z_1]$ denotes the inverse of the restriction of Q to $]0, z_1]$; and

$$p^*(\gamma^*) = \begin{cases} \dfrac{z_2}{h_m} & \text{if } \dfrac{K|\Lambda|h_M\gamma^*}{h_m^2(\gamma^*-1)} \leqslant Q(z_2) \\ \dfrac{1}{h_m} Q_2^{-1}\left(\dfrac{K|\Lambda|h_M\gamma^*}{h_m^2(1-\gamma_*)}\right) & \text{if } \dfrac{K|\Lambda|h_M}{h_m^2(\gamma^*-1)} > Q(z_2) \end{cases}$$

where now $Q_2^{-1} : [Q(z_2), +\infty[\to [z_2, +\infty[$ denotes the inverse of the restriction of Q to $[z_2, +\infty[$.

We finally give the uniqueness theorem

Theorem 3.4. *Assuming in addition that Q is Lipschitzian on any compact set contained in $]0, +\infty[$, we have uniqueness among all positive bounded weak solutions of problem* (1)*. Further, suppose that p_i is a weak solution of* (1) *corresponding to the boundary data p_a^i, $i = 1, 2$. If $p_a^1 \geqslant p_a^2$, then $p_1 \geqslant p_2$ a.e. in Ω.*

Proof. The proof is similar to that of [8, Lemma 3.5]. □

4. Conclusions

We have shown that the nonlinearities introduced by Poiseuille-flow factors derived from the kinetic theory of gases lead to well-posed mathematical problems, even in the case of discontinuous gap-thickness functions. Explicit upper and lower bounds for the pressure have been introduced as part of the existence proof (Eq. (5) and Remark 1). These results not only provide rigorous support to numerical simulations performed with the most popular rarefied-lubrication models, but also tell modelers that *any* Lipschitzian Poiseuille-flow factor that diverges as p tends to zero and to $+\infty$ can be 'safely' proposed from the mathematical viewpoint.

Acknowledgement

GCB also belongs to CONICET, Argentina. This work was partially supported by grant A03E01 from ECOS-SECYT.

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