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On localization modes in coupled thermo-hydro-mechanical problems

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Abstract

A perturbation approach is used to study localization phenomena in saturated porous media when thermo-mechanical loadings and thermo-hydro-mechanical couplings are fully taken into account. We show that various types of localization modes are possible depending on the constitutive behavior and loading conditions. Examination of the associated conditions in the light of the classical band approach reveals that the differences between these modes lie in their structure which may involve jumps in different variables (beside the velocity gradient) such as the gradients of heat and fluid fluxes, the temperature and the pressure rates. ***To cite this article: A. Benallal, C. R. Mecanique 333 (2005).***

Résumé

Sur les modes de localisation dans les problèmes thermo-hydro-mécaniques couplés. On étudie les phénomènes de localisation dans les milieux poreux saturés en prenant en compte de façon complète les couplages thermo-hydro-mécaniques. En utilisant une méthode de perturbation, on montre que plusieurs types de modes de localisation sont possibles et on donne les conditions associées. L'interprétation de ces differentes conditions dans le cadre classique d'analyse de la localisation en bandes révèle la possibilité d'émergence de diverses discontinuités (en plus de celle classique sur la vitesse de déformation). Celles-ci peuvent concerner les flux de chaleur ou de fluide, la vitesse de température ou encore la vitesse de pression ***Pour citer cet article : A. Benallal, C. R. Mecanique 333 (2005).***

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L'objectif de cette Note est l'obtention de conditions de localisation dans les milieux poreux saturés lorsque ceux-ci sont soumis à des chargements thermomécaniques et lorsque les couplages thermo-hydro-mécaniques sont entièrement pris en compte. Le comportement des matériaux considérés est représenté par la description de Biot [5] et Coussy [6]. Les variables cinématiques utilisées sont la déformation du squelette $\boldsymbol{\epsilon}$, la variation de volume fluide par unité de volume ζ auquelles seront associées la contrainte totale $\boldsymbol{\sigma}$ et la pression instersticielle du fluide p. Les mécanismes dissipatifs sont décrits par des variables internes $\boldsymbol{\alpha}$. Le comportement réversible est représenté par l'énergie libre Ψ. Le comportement irréversible est déterminé par la fonction de charge $f(\mathbf{A}, \boldsymbol{\alpha}, T)$, les lois d'évolution (3) contenant les relations de Kuhn–Tucker pour le multiplicateur inélastique $\dot{\lambda}$. Le transport du fluide est gouverné par la loi de Darcy (4) tandis que la conduction de la chaleur est régie par la loi de Fourier (5). Les équations de champ sont les équations d'équilibre quasi-statique du milieu et les relations de compatibilité (7), la conservation de la masse (8) et la conservation de l'énergie (9). L'analyse est faite en conditions quasi-statiques dans le cadre de l'hypothèse des petites perturbations (voir Coussy [6] pour une discussion détaillée). Cette hypothèse stipule en particulier les transformations infinitésimales et les petits déplacements pour le squelette solide, des petits apports de masse fluide, des petites variations des variables d'état du fluide, des petits vecteurs courant de masse fluide et des petits vecteurs courant de chaleur. Ces hypothèses permettent en particulier de simplifier l'écriture de la conservation de l'énergie (9) sous la forme (11) en négligeant les termes d'advection, de convection et la dissipation due au transport du fluide à travers le squelette. Dans ce cadre, la méthode de perturbation linéaire est utilisée ici pour l'analyse des phénomènes d'instabilité dans les milieux poreux saturés où seule la croissance illimitée des perturbations est considérée d'une part parce qu'elle est justifiée dans ce cas et d'autre part parce qu'elle peut servir à la définition de critère de localisation. La linéarisation des relations de comportement et des équations de champs autour de la solution de référence (associée ici à la déformation homogène et uniforme d'un massif infini) et la recherche de perturbations de la forme $\delta\mathbf{X} = \widetilde{\mathbf{X}}(\mathbf{t}) \exp(\mathrm{i}\xi \mathbf{n} \cdot \mathbf{x})$ conduisent à la condition d'instabilité (29) contenant l'amplitude de la perturbation, le taux de croissance η, la direction de polarisation $\mathbf{n}$ et le nombre d'onde ξ. L'examen de cette condition et ses conséquences quant à la croissance illimitée des perturbations conduisent aux trois conditions (33), (34) et (35). Ces trois conditions sont toutes associées au régime des longueurs d'ondes infiniment courtes ($\xi \to \infty$) et en ce sens elles peuvent être considérées comme des conditions de localisation. On montre alors que (33) correspond à des modes où en plus du gradient des vitesses, les gradients des flux de fluide $\mathbf{M}$ et de chaleur $\mathbf{q}$ sont discontinus à travers la bande de localisation. La condition (34) est associée à des modes où les vitesses de température et de pression sont discontinus.

1. Coupled thermo-hydro-mechanical constitutive and field equations for inelastic porous media

Isothermal poro-elastic-plastic models are widely used in various engineering fields including geomechanics and biomechanics. Localization phenomena are widely studied in this context and important contributions to the field are given in Loret and Prevost [1], Vardoulakis [2,3] and Rudnicki [4]. Other applications in geomechanics but also in buildings or nuclear waste repositories require the consideration of thermo-mechanical effects and their couplings with hydro-mechanical effects. The main objective of this note is to derive explicit localization conditions for inelastic saturated porous media when they are subjected to arbitrary thermomechanical loadings. This is carried out in the fully coupled thermo-hydro-mechanical situation.

1.1. Constitutive equations

A broad class of rate-independent, coupled thermo-irreversible constitutive equations is considered here for inelastic saturated porous media. They are given in a compact form in the framework of thermodynamics of irreversible processes and are essentially based on Biot's formulation [5] and Coussy's general presentation [6]; see also Coussy, Dormieux and Detourney [7]. In this context, the basic static variables are the total Cauchy stress $\boldsymbol{\sigma}$ in the combined solid and fluid mix and the pore fluid pressure p. The kinematic variables are the macroscopic strain $\boldsymbol{\epsilon}$ of the solid (skeleton) and the variation of fluid content ζ (i.e. the volume change of fluid per unit volume of mixture). In the poro-elastic-plastic context the kinematic variables are both partitioned into elastic and plastic parts:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p, \qquad \zeta = \zeta^e + \zeta^p \tag{1}$$

Beside the static and kinematic variables, a generic collection of supplementary internal variables of various tensorial nature (scalars, vectors or second-order tensors) represented by the vector $\boldsymbol{\alpha}$ are used to describe different physical mechanisms governing inelastic deformation and associated dissipative phenomena. Finally, the basic thermodynamical variables to be considered here are the absolute temperature T, the temperature gradient ∇T and the heat flux vector $\mathbf{q}$.

We denote by $\Psi = \Psi(\boldsymbol{\epsilon}, \boldsymbol{\alpha}, \zeta, T)$ ($\Psi = e - Ts$) the Helmholtz free energy per unit volume of the mixture (e and s being the internal energy and entropy per unit volume). It leads to the state laws giving the stress tensor $\boldsymbol{\sigma}$, the pore pressure p, the thermodynamical forces $\mathbf{A}$ associated to the internal variables $\boldsymbol{\alpha}$ and the entropy s by

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}}, \quad p = \frac{\partial \Psi}{\partial \zeta}, \quad \mathbf{A} = -\frac{\partial \Psi}{\partial \boldsymbol{\alpha}}, \quad s = -\frac{\partial \Psi}{\partial T} \tag{2}$$

The reversibility domain, defining the range in which inelastic processes are excluded, is defined through the yield function $f(\mathbf{A}, \boldsymbol{\alpha}, T) \leqslant 0$ and inelastic deformations are therefore possible only if $f = 0$, and during inelastic flow the evolution of the internal variables must satisfy Prager's consistency $\dot{f} = 0$. The evolution of internal variables is given by

$$\dot{\boldsymbol{\alpha}} = \dot{\lambda}\mathbf{P}, \quad \dot{\lambda} \geqslant 0, \quad f \leqslant 0, \quad \dot{\lambda} f = 0 \tag{3}$$

where the vector $\mathbf{P}$ gives the inelastic flow direction (that may derive from a flow potential function $F = F(\mathbf{A}, \boldsymbol{\alpha}, T)$ as $\mathbf{P} = \frac{\partial F}{\partial \mathbf{A}}$ though not necessarily) and the inelastic multiplier $\dot{\lambda}$ satisfies the Kuhn–Tucker conditions.

In the rest of the Note we use the notation Ψ_{ab} denoting the second derivative of Ψ with respect to a and b and define the gradient of the yield surface $\mathbf{Q} = \frac{\partial f}{\partial \mathbf{A}}$.

Remark 1. In the small strain regime and due to the partition assumption (1), the stress $\boldsymbol{\sigma}$ and the pore pressure p are the thermodynamical forces associated to the plastic strain $\boldsymbol{\epsilon}^p$ and ζ^p respectively. Therefore, they are omitted in the explicit dependence of functions f and F as they are both contained in $\mathbf{A}$.

To complete the description of porous media under thermomechanical loadings, one has to specify two more things: the relative movement of the fluid with respect to the solid and the heat conduction process.

Let $\mathbf{M}$ denote the fluid flux (M_i being the fluid volume crossing in the time unit the unit surface normal to the ith axis). The classical transport law for isotropic porous media, relating $\mathbf{M}$ to the gradient of the fluid pressure p is Darcy's law

$$\mathbf{M} = -\frac{K}{\nu}(\nabla p - \rho_f \mathbf{g}) \tag{4}$$

where K is the permeability, ν the viscosity of the fluid and $\mathbf{g}$ is the gravity acceleration. The permeability K and viscosity ν may be state dependent but are assumed here constant for simplicity.

Regarding heat conduction, we adopt the classical Fourier's law

$$\mathbf{q} = -k\nabla T \tag{5}$$

relating the heat flux $\mathbf{q}$ to the temperature gradient and in which k is the heat conduction coefficient that may also be state dependent.

The Clausius–Duhem inequality (Second Law of Thermodynamics, see e.g. Coussy [6]) reads in this context

$$\frac{\mathrm{d}s}{\mathrm{d}t} + s\,\mathrm{div}\,\mathbf{v} + \mathrm{div}(s_f\mathbf{w}) + \mathrm{div}\left(\frac{\mathbf{q}}{T}\right) - \frac{r}{T} \geqslant 0 \tag{6}$$

where s_f is the fluid entropy per unit mass, r the external heat supply, $\mathbf{w} = \rho_f\phi(\mathbf{v}_f - \mathbf{v})$ with ρ_f the fluid mass density, $\mathbf{v}$ the solid skeleton velocity and $\mathbf{v}_f$ the fluid velocity. Note that $\mathbf{M} = \frac{\mathbf{w}}{\rho_f}$. $\frac{\mathrm{d}}{\mathrm{d}t}$ is the material time derivative with respect to the solid motion, i.e. $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial x}{\partial t} + \mathbf{v}\cdot\mathrm{grad}\,x$.

1.2. Field equations

The quasi-static evolution of a poro-elastic-plastic body in the small strain range is described by the previous constitutive relations, supplemented by the field equations, the boundary and the initial conditions. The relevant field equations are here conservation of mass, balance of momentum, compatibility and energy conservation. Conservation of linear momentum and the geometrical compatibility conditions read respectively

$$\mathrm{div}\,\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}, \qquad \boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) \tag{7}$$

$\mathbf{u}$ being the skeleton displacement field and $\mathbf{b}$ the body force in the solid–fluid mixture. Mass conservation gives

$$\mathrm{div}\,\mathbf{M} + \dot{\zeta} = 0, \quad \text{or} \quad \dot{\zeta} - \frac{K}{\nu}(\nabla^2 p - \mathrm{div}\,\rho_f\mathbf{g}) = 0 \tag{8}$$

the second equation being the result of the substitution of (4) in the first one. Finally, local conservation of energy, i.e. the First Law of Thermodynamics is (see Coussy [6] for details)

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \left[\boldsymbol{\sigma} : \frac{\mathrm{d}\boldsymbol{\epsilon}}{\mathrm{d}t} + r - \mathrm{div}\,\mathbf{q}\right] + \left[p\frac{\mathrm{d}\zeta}{\mathrm{d}t} - \mathrm{div}(e_f\mathbf{w}) - \mathbf{M}\cdot\mathrm{grad}\,p\right] + \mathbf{Mb} - e\,\mathrm{div}\,\mathbf{v} \tag{9}$$

where e_f is the fluid internal energy per unit mass and $\mathbf{b}$ the volume force per unit volume. Notice that the first bracket is what one would have classically obtained for a solid on its own and that the last bracket is due to the fluid and its interaction with the solid skeleton.

Using the free energy Ψ and the state laws (2), we obtain an alternative, equivalent form of the local conservation of energy (9).

$$\begin{aligned} -T\Psi_{TT}\frac{\mathrm{d}T}{\mathrm{d}t} = {} & T\Psi_{\epsilon T}\cdot\frac{\mathrm{d}\boldsymbol{\epsilon}}{\mathrm{d}t} + (T\Psi_{\alpha T} + \mathbf{A})\frac{\mathrm{d}\alpha}{\mathrm{d}t} + (T\Psi_{\zeta T} - \rho_f e_f)\frac{\mathrm{d}\zeta}{\mathrm{d}t} - \mathrm{div}\,\mathbf{q} + r \\ & - e\,\mathrm{div}\,\mathbf{v} - \mathbf{M}\cdot[\mathrm{grad}\,p - \mathrm{grad}\,\rho_f e_f - \mathbf{b}] \end{aligned} \tag{10}$$

2. Perturbation and localization analysis

2.1. Basic assumptions

The perturbation and localization analysis to follow will be based on some assumptions. Beside neglecting inertia, small strains and small displacements are assumed for the solid skeleton. For the fluid, only small fluid fluxes (or small pressure gradients through Darcy's law) and small state variables changes (pressure, mass density)are considered while for both constituents small temperature changes and heat fluxes are assumed. The significance and consequences of these assumptions are thoroughly discussed in Coussy [6]. It should be emphasized that the fluid displacements are not assumed to be small and this allows to apply the results to various practical applications. These assumptions allow one to simplify the energy conservation relation (10) (and in particular to neglect the advective and convective terms but also the dissipation due to the motion of the fluid across the solid) to the following form

$$-T\Psi_{TT}\dot{T} = T\Psi_{\epsilon T}\cdot\dot{\boldsymbol{\epsilon}} + (T\Psi_{\alpha T} + \mathbf{A})\dot{\alpha} + T\Psi_{\zeta T}\dot{\zeta} - \operatorname{div}\mathbf{q} + r \tag{11}$$

2.2. *Incremental constitutive laws*

Incremental constitutive relations can be derived by time differentiation of (2) together with (3). However it is not possible in general to derive a relation between the stress rate and strain rate because of diffusion and convection phenomena. Such relations can be found in some particular but important circumstances that will be shown to play a crucial role in the rest of the note. More precisely, in the framework of the above assumptions, this is possible when in the thermomechanical setting, one assumes either *isothermal* ($\dot{T} = 0$) or *local adiabatic* ($r = \mathbf{q} = 0$) conditions while in the hydro-mechanical problem one assumes either *drained* ($\dot{p} = 0$) or *undrained* ($\dot{\zeta} = 0$) conditions. This leads to four types of incremental relations: isothermal drained, isothermal undrained, adiabatic drained and finally adiabatic undrained.

2.2.1. *Incremental constitutive laws under isothermal drained conditions*

This is actually the behavior of the porous skeleton under isothermal conditions and corresponds to $\dot{T} = 0$ and $\dot{p} = 0$. We have under elastic or inelastic straining the following relations

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}_i^d : \dot{\boldsymbol{\epsilon}}, \quad \text{or} \quad \dot{\boldsymbol{\sigma}} = \mathbf{H}_i^d : \dot{\boldsymbol{\epsilon}} \tag{12}$$

$$\mathbf{E}_i^d = \Psi_{\epsilon\epsilon} - \frac{\Psi_{\zeta\epsilon}\otimes\Psi_{\epsilon\zeta}}{\Psi_{\zeta\zeta}}, \quad \mathbf{H}_i^d = \mathbf{E}_i^d + \frac{[\Psi_{\alpha\epsilon}[\mathbf{P}] - \frac{1}{\Psi_{\zeta\zeta}}(\Psi_{\alpha\zeta}\cdot\mathbf{P})\Psi_{\zeta\epsilon}]\otimes[\Psi_{\alpha\epsilon}[\mathbf{Q}] - \frac{1}{\Psi_{\zeta\zeta}}(\Psi_{\zeta\alpha}\cdot\mathbf{Q})\Psi_{\epsilon\zeta}]}{H_i^d} \tag{13}$$

$$H_i^d = \frac{\partial f}{\partial\alpha}\cdot\mathbf{P} - \mathbf{Q}\cdot\Psi_{\alpha\alpha}[\mathbf{P}] + \frac{1}{\Psi_{\zeta\zeta}}(\Psi_{\zeta\alpha}\cdot\mathbf{Q})(\Psi_{\alpha\zeta}\cdot\mathbf{P}) \tag{14}$$

2.2.2. *Incremental constitutive laws under isothermal undrained conditions*

This corresponds to $\dot{T} = 0$ and $\dot{\zeta} = 0$. The relevant elastic and plastic isothermal undrained tangent moduli to be put in (12) are now

$$\mathbf{E}_i^u = \Psi_{\epsilon\epsilon}, \quad \mathbf{H}_i^u = \mathbf{E}_i^u + \frac{\Psi_{\alpha\epsilon}[\mathbf{P}]\otimes\Psi_{\alpha\epsilon}[\mathbf{Q}]}{H_i^u} \quad \text{with } H_i^u = \frac{\partial f}{\partial\alpha}\cdot\mathbf{P} - \mathbf{Q}\cdot\Psi_{\alpha\alpha\cdot\mathbf{P}} \tag{15}$$

2.2.3. *Incremental constitutive laws under adiabatic drained conditions*

In this situation, we have $\dot{p} = 0$ and $\mathbf{q} = r = 0$. The elastic and plastic adiabatic drained tangent moduli involved in (12) read

$$\mathbf{E}_a^d = \Psi_{\epsilon\epsilon} - \frac{\Psi_{\zeta\epsilon}\otimes\Psi_{\epsilon\zeta}}{\Psi_{\zeta\zeta}} + \frac{\Psi_{\zeta\zeta}}{\Psi_{\zeta T}^2 - \Psi_{\zeta\zeta}\Psi_{TT}}\left(\Psi_{T\epsilon} - \frac{\Psi_{T\zeta}}{\Psi_{\zeta\zeta}}\Psi_{\zeta\epsilon}\right)\otimes\left(\Psi_{\epsilon T} - \frac{\Psi_{\zeta T}}{\Psi_{\zeta\zeta}}\Psi_{\epsilon\zeta}\right) \tag{16}$$

$$\mathbf{H}_a^d = \mathbf{E}_a^d + \frac{\mathbf{K}\otimes\mathbf{L}}{H_a^d}, \quad \mathbf{K} = \Psi_{\alpha\epsilon}[\mathbf{P}] - \frac{\mathbf{P}\cdot\Psi_{\zeta\alpha}}{\Psi_{\zeta\zeta}}\Psi_{\epsilon\zeta} + \frac{\Psi_{\zeta\zeta}[\chi - \frac{\Psi_{\zeta T}}{\Psi_{\zeta\zeta}}(\Psi_{\alpha\zeta}\cdot\mathbf{P})]}{\Psi_{\zeta T}^2 - \Psi_{\zeta\zeta}\Psi_{TT}}\left(\Psi_{T\epsilon} - \frac{\Psi_{T\zeta}}{\Psi_{\zeta\zeta}}\Psi_{\zeta\epsilon}\right) \tag{17}$$

$$\mathbf{L} = \Psi_{\epsilon\alpha}[\mathbf{Q}] - \frac{\mathbf{Q}\cdot\Psi_{\zeta\alpha}}{\Psi_{\zeta\zeta}}\Psi_{\epsilon\zeta} - \frac{\Psi_{\zeta\zeta}[\frac{\partial f}{\partial T} - (\mathbf{Q}\cdot\Psi_{T\alpha}) + \frac{\Psi_{T\zeta}}{\Psi_{\zeta\zeta}}(\Psi_{\zeta\alpha}\cdot\mathbf{Q})]}{\Psi_{\zeta T}^2 - \Psi_{\zeta\zeta}\Psi_{TT}}\left(\Psi_{\epsilon T} - \frac{\Psi_{\zeta T}}{\Psi_{\zeta\zeta}}\Psi_{\epsilon\zeta}\right) \tag{18}$$

$$H_a^d = \frac{\partial f}{\partial \alpha} \cdot \mathbf{P} - \mathbf{Q} \cdot \Psi_{\alpha\alpha}[\mathbf{P}] + \frac{(\mathbf{Q} \cdot \Psi_{\zeta\alpha}) \cdot (\Psi_{\alpha\zeta} \cdot \mathbf{P})}{\Psi_{\zeta\zeta}} + \frac{\Psi_{\zeta\zeta}[\chi - \frac{\Psi_{\zeta T}}{\Psi_{\zeta\zeta}}(\Psi_{\alpha\zeta} \cdot \mathbf{P})][\frac{\partial f}{\partial T} - (\mathbf{Q} \cdot \Psi_{T\alpha}) + \frac{\Psi_{T\zeta}}{\Psi_{\zeta\zeta}}(\Psi_{\zeta\alpha} \cdot \mathbf{Q})]}{\Psi_{\zeta T}^2 - \Psi_{\zeta\zeta}\Psi_{TT}} \tag{19}$$

2.2.4. *Incremental constitutive laws under adiabatic undrained conditions*

We have now $\dot{\zeta} = 0$ and $\mathbf{q} = r = 0$ the elastic and plastic adiabatic undrained tangent moduli are

$$\begin{aligned} \mathbf{E}_a^u &= \Psi_{\epsilon\epsilon} - \frac{1}{\Psi_{TT}} \Psi_{T\epsilon} \otimes \Psi_{\epsilon T} \\ \mathbf{H}_a^u &= \mathbf{E}_a^u + \frac{(\Psi_{\alpha\epsilon}[\mathbf{P}] - \frac{\chi}{\Psi_{TT}}\Psi_{T\epsilon}) \otimes (\Psi_{\alpha\epsilon}[\mathbf{Q}] + \frac{1}{\Psi_{TT}}(\frac{\partial f}{\partial T} - \mathbf{Q} \cdot \Psi_{T\alpha})\Psi_{\epsilon T})}{H_a^u} \end{aligned} \tag{20}$$

$$H_a^u = \frac{\partial f}{\partial \boldsymbol{\alpha}} \cdot \mathbf{P} - \mathbf{Q} \cdot \Psi_{\alpha\alpha}[\mathbf{P}] - \frac{\chi}{\Psi_{TT}} \left(\frac{\partial f}{\partial T} - \mathbf{Q} \cdot \Psi_{T\alpha} \right) \tag{21}$$

Remark 2. Algebraic manipulations, too long to be reported here show that each of the moduli $\mathbf{H}_i^d$, $\mathbf{H}_i^u$, $\mathbf{H}_a^d$ and $\mathbf{H}_a^u$ are a rank-one update of the other. This remark will help in reducing the growth condition (29).

2.3. *Perturbation analysis, growth condition and localization*

Let us consider the evolution problem described by Eqs. (1)–(11) for an infinite poro-elastic-plastic medium with uniform physical properties. This body is assumed to be remotely and uniformly loaded in such a way that a homogeneous solution in terms of stresses and strains prevails throughout it. We denote by a superscript 0 all the fields corresponding to this solution. This solution is such that $\mathbf{M}^0 = \nabla p^0 = \mathbf{q}^0 = \mathbf{0} = \nabla T^0$. To investigate its stability a perturbation approach is used; thus we superpose to the homogeneous solution at a generic instant an infinitesimal perturbation denoted by δ and we analyze the behavior of the perturbed fields $\mathbf{u} = \mathbf{u}^0 + \delta\mathbf{u}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \delta\boldsymbol{\sigma}$, etc. Stability is assured if small perturbations produce only limited changes in the solution.

The perturbation fields satisfy a nonlinear system of partial differential equations. This nonlinear initial boundary value problem is not differentiable due to the Kuhn–Tucker conditions. To cope with this difficulty, the reference solution whose stability is in question will be assumed in total loading during the whole loading process. Moreover, unloading from the perturbed solution is neglected. Using the fact that the reference solution satisfies all field and constitutive equations, this nonlinear system is fully linearized around the reference solution

Because of space homogeneity and linearity of the obtained system it is convenient to apply the (space) Fourier transform. For a generic perturbation field $\delta\mathbf{X}$, this is $\widetilde{\mathbf{X}}(\mathbf{n}, t) = \int_{\mathbf{R}^3} \delta\mathbf{X}(\mathbf{x}, t) \exp(-\mathrm{i}\xi\mathbf{n} \cdot \mathbf{x})\,\mathrm{d}\mathbf{x}$. One eliminates then all the spatial derivatives and gets only algebraic or ordinary differential equations (in time) for the Fourier transforms $\widetilde{\mathbf{X}}(\mathbf{n}, t)$. This is actually equivalent to seeking solutions of the system in the form $\delta\mathbf{X} = \widetilde{\mathbf{X}}(\mathbf{t}) \exp(\mathrm{i}\xi\mathbf{n} \cdot \mathbf{x})$ where $\mathbf{n}$ is a polarization direction, ξ a wave number. Doing so one obtains a (time-dependent) linear system of differential equations. For stability analysis, the eigenvalues η of this system are required. These are obtained by seeking solutions in the form $\delta\mathbf{X}(\mathbf{t}) = \widetilde{\mathbf{X}} \exp(\mathrm{i}\xi\mathbf{n} \cdot \mathbf{x}) \exp(\eta t)$ and now η is related to the *local rate of growth* (*in time*) *of the perturbation*. The eigenvalues η satisfy the following algebraic system (from now on, to simplify the notation, we omit here the subscript 0 related to the reference solution):

$$\tilde{\boldsymbol{\sigma}} = \Psi_{\epsilon\epsilon}[\tilde{\boldsymbol{\epsilon}}] + \tilde{\lambda}\Psi_{\alpha\epsilon}[\mathbf{P}_0] + \widetilde{T}\Psi_{T\epsilon} + \tilde{\zeta}\Psi_{\zeta\epsilon} + \cdots \tag{22}$$

$$\tilde{p} = \Psi_{\epsilon\zeta}[\tilde{\boldsymbol{\epsilon}}] + \tilde{\lambda}\Psi_{\alpha\zeta}[\mathbf{P}_0] + \widetilde{T}\Psi_{T\zeta} + \tilde{\zeta}\Psi_{\zeta\zeta} + \cdots \tag{23}$$

$$\tilde{\mathbf{A}} = -\Psi_{\epsilon\alpha}[\tilde{\boldsymbol{\epsilon}}] - \tilde{\lambda}\Psi_{\alpha\alpha}[\mathbf{P}_0] - \widetilde{T}\Psi_{T\alpha} - \tilde{\zeta}\Psi_{\zeta\alpha} + \cdots \tag{24}$$

$$\frac{\partial f}{\partial \mathbf{A}} \cdot \tilde{\mathbf{A}} + \tilde{\lambda} \frac{\partial f}{\partial \boldsymbol{\alpha}} \cdot \mathbf{P} + \frac{\partial f}{\partial T} \cdot \widetilde{T} = 0 \tag{25}$$

$$\mathrm{i}\xi \tilde{\boldsymbol{\sigma}} \cdot \mathbf{n} = \mathbf{0}, \qquad \tilde{\epsilon} = \frac{\mathrm{i}\xi}{2}(\tilde{\mathbf{u}} \otimes \mathbf{n} + \mathbf{n} \otimes \tilde{\mathbf{u}}) \tag{26}$$

$$\eta \tilde{\zeta} + \frac{K}{\nu} \xi^2 \tilde{p} = 0, \qquad -T \Psi_{TT} \eta \widetilde{T} = \eta T \Psi_{\epsilon T} \cdot \tilde{\boldsymbol{\epsilon}} + \eta \tilde{\lambda} T \chi + \eta \tilde{\zeta} T \Psi_{\zeta T} - k \xi^2 \widetilde{T} \tag{27}$$

To obtain this system, we have dropped all terms in $\frac{\dot{\lambda}}{\eta}$ in the eigenvalue problem as we are concerned only with unbounded growth of perturbation. The three first equations are the linearized versions of the constitutive relations. The fourth is the linearization of the yield condition. Relations (26) represent the linearization of the equilibrium equations and the compatibility relations (7). Relations (27) are the linearization of the balance of mass (8) and the energy conservation (11) respectively. This system can be reduced further. Indeed Eq. (23) gives explicitly the pore pressure p as a function of the strain, the plastic multiplier, the temperature and the fluid content ζ. Putting this result in the mass conservation $(27)_1$ allows to obtain now the fluid content as a function of the strain, the plastic multiplier and temperature only. Again, reporting the last result in the energy conservation $(27)_2$ gives the temperature as a function of the strain and the plastic multiplier only. Using now the consistency condition (25) gives the plastic multiplier as a function of the strain only. One can then compute successively the temperature, the fluid content and finally the pore pressure only in terms of the strain. This allows to obtain the stress perturbation (using (22)) as a function of strain (or displacement) only $\tilde{\boldsymbol{\sigma}} = \mathcal{H}(\eta, \xi) : \tilde{\boldsymbol{\epsilon}}$ where the moduli $\mathcal{H}$ are given by

$$\mathcal{H}(\eta, \xi) = \eta[-\eta T \Psi_{TT} H_a^u \mathbf{H}_a^u + k\xi^2 H_i^u \mathbf{H}_i^u] + \frac{K\xi^2}{\nu}\big[\eta(T\Psi_{\zeta T}^2 - T\Psi_{TT}\Psi_{\zeta\zeta}) H_a^d \mathbf{H}_a^d + k\xi^2 \Psi_{\zeta\zeta} H_i^d \mathbf{H}_i^d\big] \tag{28}$$

Inserting (28) in the equilibrium equation $(26)_1$ taking into account the compatibility relation $(26)_2$ one obtains the growth condition for perturbations as $[\mathbf{n} \cdot \mathcal{H}(\eta, \xi) \cdot \mathbf{n}] \cdot \tilde{\mathbf{u}} = \mathbf{0}$ which has nontrivial solutions if and only if

$$\det\big[\mathbf{n} \cdot \mathcal{H}(\eta, \xi) \cdot \mathbf{n}\big] = 0 \tag{29}$$

Now (see Remark 2), due to the fact that each of the moduli $\mathbf{H}_i^d$, $\mathbf{H}_i^u$, $\mathbf{H}_a^d$ and $\mathbf{H}_a^u$ is a rank one update of the other, algebraic manipulations allow one to write the instability condition (29) in the more convenient form

$$A\eta^2 + B\xi^2\eta + C\xi^4 = 0 \tag{30}$$

$$A = H_a^u \det(\mathbf{n} \cdot \mathbf{H}_a^u \cdot \mathbf{n}), \quad C = \frac{kK}{\nu} H_i^d \det(\mathbf{n} \cdot \mathbf{H}_i^d \cdot \mathbf{n}) \tag{31}$$

$$B = k H_i^u \det(\mathbf{n} \cdot \mathbf{H}_i^u \cdot \mathbf{n}) + \frac{K}{\nu}(-T\Psi_{TT}\Psi_{\zeta\zeta} + T\Psi_{\zeta T}^2) H_a^d \det(\mathbf{n} \cdot \mathbf{H}_a^d \cdot \mathbf{n}) \tag{32}$$

The rate of growth η is then easily computed as $\eta = \xi^2 \frac{-B \pm \sqrt{\Delta}}{2A}$ where $\Delta = B^2 - 4AC$.

When Δ is negative, η is complex and unbounded growth is marked by $\mathrm{Re}(\eta) = -\xi^2 \frac{B}{2A}$ becoming infinite. This happens when $\frac{B}{2A}$ changes its sign in a loading process. This may occur in two different ways: either through the change of sign of B and in this case $\xi \to \infty$ or through the change of sign of A and in this case ξ is arbitrary.

When Δ is positive, η is real and unbounded growth of perturbation corresponds to $\eta \to \infty$. Here again two possibilities arise: for an arbitrary wavenumber ξ if $\frac{-B+\sqrt{\Delta}}{2A} \to \infty$ which may happen again when A passes through zero or for ξ infinite when at least one of the roots root $\frac{-B \pm \sqrt{\Delta}}{2A}$ is positive. As the sum of the roots is $S = -\frac{B}{A}$ and their product is $P = \frac{C}{A}$, this is possible whenever $P < 0$ (in which case one of the roots is positive) or when in the same time $P > 0$ and $S < 0$ (in which case the two roots are positive).

Summing up, one can identify three situations that mark transition to unbounded growth of perturbations: $A = 0$, $B = 0$ and $C = 0$. The associated conditions are

$$\det[\mathbf{n} \cdot \mathbf{H}_a^u \cdot \mathbf{n}] = \mathbf{0} \tag{33}$$

$$\det[\mathbf{n}\cdot\mathbf{H}_i^d\cdot\mathbf{n}]=\mathbf{0} \tag{34}$$

$$kH_i^u\det(\mathbf{n}\cdot\mathbf{H}_i^u\cdot\mathbf{n})+\frac{K}{\nu}(-T\Psi_{TT}\Psi_{\zeta\zeta}+T\Psi_{\zeta T}^2)H_a^d\det(\mathbf{n}\cdot\mathbf{H}_a^d\cdot\mathbf{n})=\mathbf{0} \tag{35}$$

2.4. Discussion

We have exhibited above three conditions corresponding to unbounded growth of perturbations. Let us first recall and emphasize that these three conditions are associated the infinitely short wavelength ($\xi\to\infty$) regime though the first one (33) is associated to the full wavelength regime. In this sense the three conditions can be viewed as localization conditions. While the given presentation has the advantage of unifying these conditions, one could have followed the classical band approach to localization [8]. The obtained conditions can be highlighted in this framework.

It can be shown that condition (33) which states the singularity of the adiabatic undrained acoustic tensor corresponds to a localization mode involving beside a jump in the velocity gradient, jumps on temperature and pore pressure rates (temperature and pressure localization), and where the gradients of heat and fluid fluxes are continuous across the band.

In a similar way, when condition (34) is met, stating the singularity of the isothermal drained acoustic tensor, a localization mode with jumps on the gradients of heat and fluid fluxes are available where the temperature and pore pressure rates are now continuous.

The interpretation of condition (35) is more difficult. However, let us notice first that this condition is trivially satisfied when both the isothermal undrained and adiabatic drained acoustic tensors are simultaneously singular. This situation corresponds to a localization mode where the only jump is associated to the velocity gradient and where the gradients of heat and fluid fluxes together with the temperature and pore pressure rates are continuous across the band. It remains now the interpretation of (35) when both the isothermal undrained and adiabatic drained acoustic tensors are not singular and further when the adiabatic undrained and isothermal drained acoustic tensor are also not singular as these conditions correspond to (33) and (34). In this case, jumps are possible for all variables (fully localized mode). Let us also remark that condition (35) involves the nondimensional number ($P_r=\frac{K(-T\Psi_{TT}\Psi_{\zeta\zeta}+T\Psi_{\zeta T}^2)}{k\nu}$) that determines the relative strength of the two dissipation phenomena present in the problem, i.e. fluid and thermal diffusion.

References

[1] B. Loret, J.H. Prevost, Dynamic strain localization in elasto (visco-) plastic solids, Part 1. General formulation and one-dimensional examples, Comput. Methods Appl. Mech. Engrg. 83 (1990) 247273.
[2] I. Vardoulakis, Uniform undrained deformation and shear banding: Part I, Géotechnique 46 (1996) 441–456.
[3] I. Vardoulakis, Uniform undrained deformation and shear banding: Part II, Géotechnique 46 (1996) 457–472.
[4] J.W. Rudnicki, Diffusive instabilities in dilating and compacting geomaterials and structures, in: J.J. Chang, J.W. Rudnicki (Eds.), The James R. Rice 60th Anniversary Volume, Kluwer Academic, 2000.
[5] M.A. Biot, Theory of elasticity and consolidation for a porous anisotropic solid, J. Appl. Phys. 26 (1955) 182–185.
[6] O. Coussy, Mechanics of Porous Continua, Wiley, Chichester, 1995.
[7] O. Coussy, L. Dormieux, E. Detourney, From mixture theory to Biot's approach for porous media, Int. J. Solids Structures 35 (34–35) (1998) 4619–4635.
[8] J.R. Rice, The localization of plastic deformation, in: W.T. Koiter (Ed.), Theoretical and Applied Mechanics, North-Holland, Amsterdam, 1976, pp. 207–220.