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Rigorous justification of the Reynolds equations for gas lubrication

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Abstract

The goal of this Note is to give a rigorous justification of the compressible Reynolds model for gas lubrication, via asymptotic analysis. We start from the equations of motion of compressible viscous fluid in a thin domain and study the limit as the domain thickness tends to zero. At the limit we find the known engineering model. The key of the proof is the strong convergence for the pressure obtained by its decomposition. ***To cite this article: E. Marušić-Paloka, M. Starčević, C. R. Mecanique 333 (2005).***

Résumé

Justification rigoureuse d'équation de Reynolds pour lubrification par gaz. Le but de cette Note est de donner une justification de modèle de Reynolds compressible via une analyse asymptotique. À partir des équations de mouvement d'un fluide visqueux compressible dans un domaine mince nous étudions la limite lorsque l'épaisseur du domaine tend vers 0. À la limite nous trouvons un modèle de Reynolds compressible. La clé de la preuve est la convergence forte de la pression obtenue par la décomposition. ***Pour citer cet article : E. Marušić-Paloka, M. Starčević, C. R. Mecanique 333 (2005).***

Keywords: Lubrication; Compressible fluid; Gas lubrication; Magnetic hard disk; Rigorous asymptotic analysis

Mots-clés : Lubrification ; Fluide compressible ; Lubrication par gaz ; Disque dur magnétique ; Analyse asymptotique rigoureuse

Version française abrégée

Un coussinet est un appareil consistant des deux surfaces en mouvement relatif et une échappée entre eux remplis par un fluide visqueux (lubrifiant). Les différences principales des coussinets lubrifiés par un fluide compressible (un gaz) par rapport à ceux lubrifiés par un fluide incompressible (un liquide) sont la viscosité et l'écarte entre des deux surfaces en question, qui sont tout les deux, en général, beaucoup plus petits. Les exemples le plus

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communs des appareils lubrifiés par un fluide compressible sont des disques durs dans des ordinateurs ou des bandes magnétiques des magnétophones, où le fluide est l'air pure. Dans cette Note nous cherchons des équations macroscopiques gouvernant l'écoulement de cette couche mince de fluide visqueux et compressible, lubrifiant un coussinet, par une analyse asymptotique rigoureuse.

À partir des équations de Navier–Stokes compressibles, isothermiques et linearisés dans un domaine mince : $\Omega_\varepsilon = \{x = (x', x_n) \in \mathbf{R}^n ; x' = (x_1, \dots, x_{n-1}) \in \mathcal{O},\ 0 < x_n < \varepsilon h(x')\}$ où $\mathcal{O} \subset \mathbf{R}^{n-1}$ est un domaine borné

$$-\mu \Delta u^\varepsilon - (\lambda + \mu)\nabla(\operatorname{div} u^\varepsilon) + \nabla p^\varepsilon = 0, \quad \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0 \quad \text{dans } \Omega_\varepsilon \tag{1}$$

$$u^\varepsilon = 0 \quad \text{pour } x_n = \varepsilon h(x'), \quad u^\varepsilon = \mathbf{V} \quad \text{pour } x_n = 0, \quad u^\varepsilon = w_\varepsilon \quad \text{pour } x' \in \partial\mathcal{O} \tag{2}$$

pour w_ε vérifiant des conditions de compatibilité décrit dans la version anglaise. Nous cherchons la limite lorsque ε, l'épaisseur de la couche Ω_ε tends vers 0. À la limite nous trouvons des équations de Reynolds compressible :

$$U = -\frac{1}{2\mu} y_n (h - y_n) \nabla_{x'} P + \left(1 - \frac{y_n}{h(x')}\right)\mathbf{V} \tag{3}$$

$$\operatorname{div}_{x'}\left(P \int_0^{h(x')} U(x', \xi)\, \mathrm{d}\xi\right) = 0 \quad \text{dans } \mathcal{O}, \qquad P\left(\int_0^{h(x')} U(x', \xi)\, \mathrm{d}\xi\right) \cdot \mathbf{n} = 0 \quad \text{sur } \partial\mathcal{O} \tag{4}$$

ce qui donne leur justification rigoureuse.

Nous écrivons, tout d'abord, le problème (1), (2) sur un domaine fixe

$$\Omega = \left\{(x', y_n) \in \mathbf{R}^n;\ x' = (x_1, \dots, x_{n-1}) \in \mathcal{O},\ 0 < y_n < h(x')\right\}$$

en utilisant une variable dilatée $y_n = x_n/\varepsilon$. Nous désignons par

$$U^\varepsilon(x', y_n) = u^\varepsilon(x', \varepsilon y_n), \qquad P^\varepsilon(x', y_n) = p^\varepsilon(x', \varepsilon, y_n)$$

et (1) prends la forme suivante

$$-\mu\left(\frac{\partial^2 U_\alpha^\varepsilon}{\partial y_n^2} + \varepsilon^2 \Delta_{x'} U_\alpha^\varepsilon\right) - (\lambda + \mu)\left(\varepsilon \frac{\partial^2 U_n^\varepsilon}{\partial y_n \partial x_\alpha} + \varepsilon^2 \frac{\partial}{\partial x_\alpha} \operatorname{div}_{x'} U^\varepsilon\right) + \varepsilon^2 \frac{\partial P^\varepsilon}{\partial x_\alpha} = 0, \quad \alpha = 1, \dots, n-1 \tag{5}$$

$$-\mu\left(\frac{\partial^2 U_n^\varepsilon}{\partial y_n^2} + \varepsilon^2 \Delta_{x'} U_n^\varepsilon\right) - (\lambda + \mu)\left(\frac{\partial^2 U_n^\varepsilon}{\partial y_n^2} + \varepsilon \frac{\partial}{\partial y_n} \operatorname{div}_{x'} U^\varepsilon\right) + \varepsilon \frac{\partial P^\varepsilon}{\partial y_n} = 0 \tag{6}$$

$$\frac{\partial (P^\varepsilon U_n^\varepsilon)}{\partial y_n} + \varepsilon \operatorname{div}_{x'}(P^\varepsilon U^\varepsilon) = 0 \quad \text{in } \Omega \tag{7}$$

où $\Delta_{x'} U = \sum_{\alpha=1}^{n-1} \frac{\partial^2 U}{\partial x_\alpha^2}$, $\operatorname{div}_{x'} U = \sum_{\alpha=1}^{n-1} \frac{\partial U_\alpha}{\partial x_\alpha}$ et $\nabla_{x'} P = \sum_{\alpha=1}^{n-1} \frac{\partial P}{\partial x_\alpha} \mathbf{e}_\alpha$.

Le résultat principal de cette Note peut s'écrire sous la forme suivante :

Théorème 0.1.

$$U^\varepsilon \rightharpoonup U, \quad \frac{\partial U^\varepsilon}{\partial y_n} \rightharpoonup \frac{\partial U}{\partial y_n} \quad \textit{dans } L^2(\Omega) \textit{ faible}$$

$$\varepsilon^2 p^\varepsilon \to P \quad \textit{dans } L^2(\Omega) \textit{ fort}$$

où (U, P) *est une solution unique du problème* (3).

Pour la preuve nous dérivons d'abord des éstimations a priori (13) ce qui donne la convergence faible de une sous-suite de $(U^\varepsilon, \varepsilon^2 P^\varepsilon)$. Ça ne suffit pas de passer à la limite dans l'équation de continuité. La clé de la preuve est la convergence forte de $\varepsilon^2 P^\varepsilon$ obtenue dans les Lemmes 4.2 et 4.3 par la décomposition (17).

1. Introduction

Fluid film bearings are the machine elements consisting of two (in our case rigid) surfaces in relative motion and a thin gap between them filled by a fluid (lubricant). We are interested in studying the equations governing that thin fluid film. In our model, one of those surfaces is rough and the shape of its asperities plays an important role in our study. Another important feature are the physical properties of the fluid. In this paper we are interested in case when the fluid is not a liquid but a gas, in most applications, a clean dry air. There are several differences in qualitative behavior of gases compared to liquids, mainly: compressibility and small viscosities. In general, gas bearing operates with higher velocity and smaller clearance ratio than the liquid one. Although the gas viscosity is small (typically of order 10^{-5}) we rarely have to consider the turbulence. In fact, due to the small typical length (gap thickness smaller than 1 μm is not uncommon) the Reynolds numbers are usually smaller than 100 (see e.g. [1]). The most common examples where the gas lubrication appears are computer hard discs, magnetic tapes and some high precision measuring devices. To fix the ideas we give some details in case of magnetic hard disc. The model describing such situation was formally derived in [2]. Two surfaces, in that case are the disc and the magnetic head. The hard disc surface is artificially roughened in order to control the interfacial static force. In order to get higher recording density and, therefore, improve the performance of the recording device, the gap between two surfaces (flying height) is very small. For a present hard-disc the distance between the disc and the head ranges between 5 and 20 nanometers.[1] The typical speed of such device is between 5000 and 10 000 rounds per minute (usually smaller for notebooks then for desktop computers). With disc radius of 5–10 cm it gives the characteristic velocity between 20 and 100 m/s. Hard discs have a small pressure-equalization port keeping the internal pressure equal to the external so that the characteristic pressure is between 1000 and 1020 m bar. The typical dry air density is 1.2 $\mathrm{kg/m^3}$. Dry air viscosity equals 1.8×10^{-5} kg/m s. Recommended operating temperature, for most drives, is from 35 to 40 Celsius. In the above situation the Reynolds number[2] would be of order 10^{-2}, i.e., deeply in the laminar regime. In fact, in such circumstances it would not be unreasonable to neglect the effects of inertia.

As in the case of incompressible fluids, the lower dimensional model for describing the process of lubrication by compressible fluid, called the compressible Reynolds equation, has been first derived in the engineering literature, as for instance [3,2,4,1].

The obtained heuristic model has been extensively studied in the mathematical literature [5].

The rigorous justification, by asymptotic analysis of the Navier–Stokes system, for the Reynolds model for incompressible fluids (liquids) has been done almost 20 years ago in [6]. However, no results of that kind for compressible fluids are known to us. The basic difference is that the weak convergence method used in [6] does not pass directly here due to the nonlinearity of the continuity equation.

The goal of this Note is to derive the isothermal Reynolds model for gas lubrication using the rigorous asymptotic analysis. We were inspired by methods from [7] where the asymptotic analysis of the compressible flow through a periodic porous medium was performed.

2. Position of the problem

To derive the model we start from the equations of motion governing the compressible, stationary flow through a thin domain with thickness ε described by the shape function h:

$$\Omega_\varepsilon = \left\{x = (x', x_n) \in \mathbf{R}^n;\ x' = (x_1, \ldots, x_{n-1}) \in \mathcal{O},\ 0 < x_n < \varepsilon h(x')\right\}$$

where $\mathcal{O} \subset \mathbf{R}^{n-1}$ is a bounded smooth domain and $h : \overline{\mathcal{O}} \to \mathbf{R}$ is a smooth function such that there exist two constants $h_M, h_m > 0$ satisfying $h_m \leqslant h(x') \leqslant h_M$. Let $\Gamma_\varepsilon = \{x = (x', x_n) \in \mathbf{R}^n;\ x' = (x_1, \ldots, x_{n-1}) \in \partial\mathcal{O},\ 0 < x_n < \varepsilon h(x')\}$ be the lateral boundary. We shall also need rescaled domain and it's lateral boundary

[1] 1000 to 5000 times thinner than a human hair.
[2] that obtained by taking the flying height as a characteristic distance

$$\Omega = \{(x', y_n) \in \mathbf{R}^n;\ x' = (x_1, \ldots, x_{n-1}) \in \mathcal{O},\ 0 < y_n < h(x')\}$$
$$\Gamma = \{(x', y_n) \in \mathbf{R}^n;\ x' = (x_1, \ldots, x_{n-1}) \in \partial\mathcal{O},\ 0 < y_n < h(x')\}$$

The unknowns in the model are u^ε – the velocity, p^ε – the pressure, ρ^ε – the density. We suppose that the fluid is viscous and compressible and that the flow is stationary. As usual, we use the ideal gas law $\rho = \frac{p}{RT}$ where T is the temperature [K] and R is the gas constant [J/kg K] (equal to 287.05 for dry air). Typically, in the engineering literature, the temperature variations in the thin film are treated as negligible (see e.g. [3,4,1]) and the temperature is supposed to be constant, i.e. equal to the ambient temperature. Thus, we suppose that the flow is isothermal and, consequently, verifying the simple pressure–density relation $p^\varepsilon = a_\varepsilon \rho^\varepsilon$, where $a_\varepsilon = T_\varepsilon R > 0$ is a constant. To get an idea, at the room temperature (between 20 and 25 °C) and typical atmospheric pressure between 1000 and 1020 mm bar, the value of a_ε would be of order 10^5. We also neglect the inertial term, i.e. we assume that the Reynolds number $Re_\varepsilon \ll 1$. The total quantity of the fluid in the domain is prescribed and equal to $M_\varepsilon > 0$, i.e. $M_\varepsilon = \int_{\Omega_\varepsilon} \rho^\varepsilon(x)\,\mathrm{d}x$. For the two constants a_ε and M_ε we assume that $\limsup_{\varepsilon\to 0} \varepsilon^2 a_\varepsilon M_\varepsilon / |\Omega_\varepsilon| < \infty$. In case of magnetic hard disc, described in the introduction, the above condition is fulfilled, since for $\varepsilon = 10^{-9}$ we have $\varepsilon^2 a_\varepsilon M_\varepsilon / |\Omega_\varepsilon| \equiv 10^{-13}$. The velocity of the relative motion of two surfaces is denoted by $\mathbf{V} \in H_0^1(\mathcal{O})^n$. Of course, we assume that $\mathbf{V} \perp \mathbf{e}_n$ with $\mathbf{e}_n = (0, \ldots, 0, 1)$. Our system can then be expressed:

$$-\mu\Delta u^\varepsilon - (\lambda + \mu)\nabla(\operatorname{div} u^\varepsilon) + \nabla p^\varepsilon = 0, \quad \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon \tag{8}$$
$$u^\varepsilon = 0 \quad \text{for } x_n = \varepsilon h(x'), \quad u^\varepsilon = \mathbf{V} \quad \text{for } x_n = 0, \quad u^\varepsilon = w_\varepsilon \quad \text{on } \Gamma_\varepsilon \tag{9}$$

On the lateral boundary we have prescribed the value of the velocity w_ε. We suppose, for simplicity of this Note, that the normal component $\mathbf{n} \cdot w_\varepsilon = 0$ on Γ_ε. We assume in addition that it has the form $w_\varepsilon(x) = w(x', x_n/\varepsilon)$, where w is an H^1 function on Ω satisfying the compatibility condition $w(x', 0) = \mathbf{V}$, $w(x', h(x')) = 0$ and that $\operatorname{div} w = \sum_{\alpha=1}^{n-1} \frac{\partial w_\alpha}{\partial x_\alpha} + \frac{\partial w_n}{\partial y_n} = 0$.

The problem is solvable and admits a solution $u^\varepsilon \in H^1(\Omega_\varepsilon)^n$, p^ε, $\rho^\varepsilon \in L^2(\Omega_\varepsilon)$ such that $\rho^\varepsilon \geqslant 0$ and $\int_{\Omega_\varepsilon} \rho^\varepsilon = M_\varepsilon$. The existence theorem for (8), (9) can be found in [8], Section 6.10, page 162 (except for the non-homogeneous boundary condition and non-smoothness of the domain, but that is easy to handle in the absence of inertial term, due to the a priori estimate from Theorem 3.1).

3. A priori bounds

Theorem 3.1. *There exists a constant $C > 0$ such that*

$$|u^\varepsilon|_{L^2(\Omega_\varepsilon)} \leqslant C\varepsilon |\nabla u^\varepsilon|_{L^2(\Omega_\varepsilon)} \leqslant C\sqrt{\varepsilon} \tag{10}$$
$$\left| p^\varepsilon - \frac{a_\varepsilon M_\varepsilon}{|\Omega_\varepsilon|} \right|_{L^2(\Omega_\varepsilon)} \leqslant C\varepsilon^{-3/2} \tag{11}$$

Proof. As usual, we assume that the solution u^ε is smooth and we use $u^\varepsilon - w^\varepsilon$ as a test function in (8). We obtain

$$\mu \int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^2 + (\lambda + \mu) \int_{\Omega_\varepsilon} |\operatorname{div} u^\varepsilon|^2 + \int_{\Omega_\varepsilon} \nabla p^\varepsilon u^\varepsilon = \mu \int_{\Omega_\varepsilon} \nabla u^\varepsilon \nabla w^\varepsilon$$

To eliminate the pressure term we use the standard trick $\int_{\Omega_\varepsilon} \nabla p^\varepsilon u^\varepsilon = \int_{\Omega_\varepsilon} \nabla(\log p^\varepsilon) u^\varepsilon p^\varepsilon = -\int_{\Omega_\varepsilon} \log p^\varepsilon \times \operatorname{div}(p^\varepsilon u^\varepsilon) = 0$. The integral on the right-hand side can be estimated directly using the form of w_ε. That implies (10). To prove (11), we define a function $z_\varepsilon \in H_0^1(\Omega_\varepsilon)$ such that

$$\operatorname{div} z_\varepsilon = p^\varepsilon - \frac{a_\varepsilon M_\varepsilon}{|\Omega_\varepsilon|} \quad \text{in } \Omega_\varepsilon, \qquad z_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon$$

We can prove (see e.g. [9]) that there exists at least one z_ε satisfying the estimate

$$|\nabla z_\varepsilon|_{L^2(\Omega_\varepsilon)} \leqslant C\varepsilon^{-1}\left|p^\varepsilon - \frac{a_\varepsilon M_\varepsilon}{|\Omega_\varepsilon|}\right|_{L^2(\Omega_\varepsilon)}$$

Using such z_ε as a test function in (8) and applying (10) proves the pressure estimate (11). □

4. Asymptotic analysis

We first rewrite the problem on the fixed domain Ω by change of variables. We define

$$U^\varepsilon(x', y_n) = u^\varepsilon(x', \varepsilon y_n), \qquad P^\varepsilon(x', y_n) = p^\varepsilon(x', \varepsilon y_n) \tag{12}$$

We can then write Eq. (8) in the form (5), (7).

We deduce from Theorem 3.1 and hypothesis on $a_\varepsilon M_\varepsilon$, the following estimates for U^ε and P^ε

$$|U^\varepsilon|_{L^2(\Omega)} \leqslant C, \quad \left|\frac{\partial U^\varepsilon}{\partial y_n}\right|_{L^2(\Omega)} \leqslant C, \quad |\nabla_{x'} U^\varepsilon|_{L^2(\Omega)} \leqslant C\varepsilon^{-1}, \quad \varepsilon^2 |P^\varepsilon|_{L^2(\Omega)} \leqslant C \tag{13}$$

4.1. Passing to the limit

Using the estimates (13), we conclude that there exist $U \in Y(\Omega) = \{W \in L^2(\Omega)^n;\ \frac{\partial W}{\partial y_n} \in L^2(\Omega)^n\}$ and $P \in L^2(\mathcal{O})$ and a subsequences, denoted for simplicity by the same symbol $\{U^\varepsilon\}_{\varepsilon>0}, \{P^\varepsilon\}_{\varepsilon>0}$ such that

$$U^\varepsilon \rightharpoonup U, \quad \frac{\partial U^\varepsilon}{\partial y_n} \rightharpoonup \frac{\partial U}{\partial y_n}, \quad \varepsilon^2 P^\varepsilon \rightharpoonup P \quad \text{weakly in } L^2(\Omega) \tag{14}$$

Furthermore

$$U(x', 0) = \mathbf{V}, \qquad U\big(x', h(x')\big) = 0 \tag{15}$$

We can now prove that

Proposition 4.1. *The limit functions U and P satisfy the Reynolds equation*

$$U = -\frac{1}{2\mu} y_n(h - y_n)\nabla_{x'} P + \left(1 - \frac{y_n}{h(x')}\right)\mathbf{V} \tag{16}$$

Furthermore $P \in H^1(\mathcal{O})$.

Proof. Multiplying (6) by ε and taking the limit implies that P does not depend on y_n. To prove (16) we use (14) to pass to the limit in (5). Let $\psi \in C_0^\infty(\Omega)^n$ be such that $\psi_n = 0$. Then

$$\begin{aligned}
\int_\Omega P \,\mathrm{div}_{x'} \psi \leftarrow \int_\Omega \varepsilon^2 P^\varepsilon \mathrm{div}_{x'} \psi \\
&= \mu \int_\Omega \frac{\partial U^\varepsilon}{\partial y_n}\frac{\partial \psi}{\partial y_n} + \mu \int_\Omega \nabla_{x'} U^\varepsilon \nabla_{x'} \psi + \varepsilon(\lambda+\mu)\int_\Omega \left(\frac{\partial U_n^\varepsilon}{\partial y_n} + \varepsilon \mathrm{div}_{x'} U^\varepsilon\right)\mathrm{div}_{x'}\psi \\
&\to \mu \int_\Omega \frac{\partial U}{\partial y_n}\frac{\partial \psi}{\partial y_n}
\end{aligned}$$

Thus $-\mu \frac{\partial^2 U}{\partial y_n^2} + \nabla_{x'} P = 0$. Taking into account boundary conditions (15), we prove (16). As $U \in L^2(\Omega)$ we have $\int_0^h U(\cdot, \xi)\,\mathrm{d}\xi \in L^2(\mathcal{O})$ and we get $\nabla P \in L^2(\mathcal{O})$.

The estimates (10) and (11) imply the weak convergence (14) of (sub)sequences of solutions $(U^\varepsilon, \varepsilon^2 P^\varepsilon)$ permitting the passage to the limit in the momentum equation (5). However, that is not enough to pass to the limit in the continuity equation (7) since we have there a product of two weakly convergent sequences. The idea is to prove the strong $L^2(\Omega)$ convergence for the pressure. To do so, we decompose it in two parts:

$$\overline{P}^\varepsilon = \frac{1}{h}\int_0^h P^\varepsilon(x',\xi)\,\mathrm{d}\xi, \qquad \Pi^\varepsilon = P^\varepsilon - \overline{P}^\varepsilon \qquad \square \tag{17}$$

Lemma 4.2. *There exists a constant $C > 0$, depending on Ω but not on ε, such that*

$$\left|\varepsilon^2 \int_\Omega (P^\varepsilon - \overline{P}^\varepsilon)\varphi\right| \leqslant C\big(\varepsilon|\varphi|_{L^2(\Omega)} + \varepsilon^3|\nabla_{x'}\varphi|_{L^2(\Omega)}\big)$$

for any $\varphi \in H_0^1(\Omega)$.

Proof. Using (6) we have

$$\varepsilon^2\left|\left\langle\frac{\partial P^\varepsilon}{\partial y_n}\middle|\phi\right\rangle\right| \leqslant C\left(\varepsilon\left|\frac{\partial U^\varepsilon}{\partial y_n}\right|_{L^2(\Omega)} + \varepsilon^2|\nabla_{x'}U^\varepsilon|_{L^2(\Omega)}\right)\left|\frac{\partial \phi}{\partial y_n}\right|_{L^2(\Omega)} + C\varepsilon^3|\nabla_{x'}U^\varepsilon|_{L^2(\Omega)}|\nabla_{x'}\phi|_{L^2(\Omega)}$$
$$\leqslant C\varepsilon\left|\frac{\partial \phi}{\partial y_n}\right|_{L^2(\Omega)} + C\varepsilon^2|\nabla_{x'}\phi|_{L^2(\Omega)}$$

We now put $\phi(x', y_n) = \int_0^{y_n}\varphi(x', s)\,\mathrm{d}s$ so that

$$\varepsilon^2\left\langle\frac{\partial P^\varepsilon}{\partial y_n}\middle|\phi\right\rangle = -\varepsilon^2\int_\Omega (P^\varepsilon - \overline{P}^\varepsilon)\varphi$$

implying the claim. $\square$

It remains to prove:

Lemma 4.3. *Let $\{P^\varepsilon\}_{\varepsilon>0}$ be the weakly converging subsequence from Proposition* 4.1 *and let $P \in H^1(\mathcal{O})$ be the Reynolds pressure satisfying* (16). *Let $\overline{P}^\varepsilon$ be defined by* (17). *Then $\varepsilon^2\overline{P}^\varepsilon \to P$ strongly in $L^2(\mathcal{O})$.*

Proof. We prove that for any weakly converging sequence $\{\psi_\varepsilon\}_{\varepsilon>0},\ \psi_\varepsilon \rightharpoonup \psi$ in $L^2(\mathcal{O})$, we have

$$\int_{\mathcal{O}} \varepsilon^2\overline{P}^\varepsilon\psi_\varepsilon \to \int_{\mathcal{O}} P\psi.$$

In fact, we may assume, without loosing generality, that $\langle\psi_\varepsilon\rangle = |\mathcal{O}|^{-1}\int_{\mathcal{O}}\psi_\varepsilon(x')\,\mathrm{d}x' = 0$. That will imply the claim. So, let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be an arbitrary sequence in $L_0^2(\mathcal{O})$, where the subscript 0 stands for the zero mean value $\langle\cdot\rangle = 0$, such that

$$\psi_\varepsilon \rightharpoonup \psi \quad \text{weakly in } L_0^2(\mathcal{O})$$

For each $\varepsilon > 0$ there exists $w_\varepsilon \in H_0^1(\mathcal{O})$ such that

$$\operatorname{div} w_\varepsilon = \psi_\varepsilon$$

and that the following estimate holds (see e.g. [9])

$$|\nabla w_\varepsilon|_{L^2(\mathcal{O})} \leqslant C|\psi_\varepsilon|_{L^2(\mathcal{O})}$$

Thus, up to a subsequence,

$$\nabla w_\varepsilon \rightharpoonup \nabla w \quad \text{weakly in } L^2(\mathcal{O}), \qquad w_\varepsilon \to w \text{ strongly in } L^2(\mathcal{O})$$

where $w \in H_0^1(\mathcal{O})^{n-1}$ is such that $\operatorname{div} w = \psi$. Let $\varphi \in C^1(\overline{\Omega})$, $\varphi(x',0) = \varphi(x', h(x')) = 0$. Using $w_\varepsilon(x')\varphi(x', y_n)$ as a test function in (5), (6) (assuming that its last component equals zero), we obtain

$$\varepsilon^2\mu \int_\Omega \nabla_{x'} U^\varepsilon \nabla_{x'}(w_\varepsilon\varphi) + \mu \int_\Omega w_\varepsilon \frac{\partial U^\varepsilon}{\partial y_n}\frac{\partial \varphi}{\partial y_n} + \varepsilon^2(\lambda+\mu)\int_\Omega \operatorname{div}_{x'} U^\varepsilon \operatorname{div}_{x'}(w_\varepsilon\varphi)$$
$$-\varepsilon(\lambda+\mu)\int_\Omega U_n^\varepsilon \psi_\varepsilon \frac{\partial\varphi}{\partial y_n} - \varepsilon(\lambda+\mu)\int_\Omega U_n^\varepsilon w_\varepsilon \nabla_{x'}\frac{\partial \varphi}{\partial y_n} = \varepsilon^2 \int_\Omega P^\varepsilon \psi_\varepsilon \varphi + \varepsilon^2 \int_\Omega P^\varepsilon w_\varepsilon \nabla_{x'}\varphi$$

The left-hand side tends to

$$\mu\int_\Omega \frac{\partial U}{\partial y_n}\frac{\partial \varphi}{\partial y_n} w = -\mu \int_\Omega w\varphi \frac{\partial^2 U}{\partial y_n^2} = -\int_\Omega \nabla_{x'} P w \varphi = \int_{\mathcal{O}} P\psi\left(\int_0^h \varphi\right) + \int_{\mathcal{O}} Pw\left(\int_0^h \nabla_{x'}\varphi\right)$$

Thus, we conclude that

$$\varepsilon^2 \int_\Omega P^\varepsilon \psi_\varepsilon \varphi \to \int_{\mathcal{O}} P\psi\left(\int_0^h \varphi\right) \tag{18}$$

Let $\Omega_\delta^- = \mathcal{O} \times]0,\delta[$, $\Omega_\delta^+ = \{(x', y_n) \in \mathbf{R}^n;\ x' \in \mathcal{O}, h(x') - \delta < y_n < h(x')\}$ and let $\Omega_\delta = \Omega_\delta^+ \cup \Omega_\delta^-$. We choose $\eta_\delta \in C^1(\overline{\Omega})$ such that $\eta_\delta \leqslant 1$ on Ω, $\eta_\delta = 1$ on $\Omega \backslash \Omega_\delta$, $\eta_\delta = 0$ on $\Omega_{\delta/2}^+ \cup \Omega_{\delta/2}^-$. Now

$$\left|\int_\Omega \big(\varepsilon^2 P^\varepsilon \psi_\varepsilon - P\psi\big)(1-\eta_\delta)\right| \leqslant \varepsilon^2 |P^\varepsilon|_{L^2(\Omega_\delta)} |\psi_\varepsilon|_{L^2(\Omega_\delta)} + |P|_{L^2(\Omega_\delta)}|\psi|_{L^2(\Omega_\delta)}$$

Since

$$|\psi_\varepsilon|^2_{L^2(\Omega_\delta)} = \int_{\mathcal{O}} \psi_\varepsilon(x')^2 \left(\int_0^\delta \mathrm{d}y_n + \int_{h-\delta}^h \mathrm{d}y_n\right) \leqslant 2\delta |\psi_\varepsilon|_{L^2(\mathcal{O})} \leqslant C\delta$$

we obtain

$$\left|\int_\Omega \big(\varepsilon^2 P^\varepsilon \psi_\varepsilon - P\psi\big)(1-\eta_\delta)\right| \leqslant C\sqrt{\delta}$$

That estimate is uniform in ε so that

$$\left|\int_\Omega \big(\varepsilon^2 P^\varepsilon \psi_\varepsilon - P\psi\big)\right| \leqslant \left|\int_\Omega \big(\varepsilon^2 P^\varepsilon \psi_\varepsilon - P\psi\big)\eta_\delta\right| + C\sqrt{\delta}$$

Since δ is an arbitrary positive number, using (18) we conclude that

$$\lim_{\varepsilon\to 0}\left|\int_\Omega \big(\varepsilon^2 P^\varepsilon \psi_\varepsilon - P\psi\big)\right| = 0$$

Finally we conclude $\int_{\mathcal{O}} \varepsilon^2 \overline{P^\varepsilon} h \psi^\varepsilon \to \int_{\mathcal{O}} Ph\psi$ so that $\varepsilon^2 \overline{P}_\varphi^\varepsilon \to P$ strongly in $L^2(\mathcal{O})$. □

Theorem 4.4. *Let* $(u^\varepsilon, p^\varepsilon)$ *be the solution of the equations of motion* (8), (9) *and let* U^ε, P^ε *be defined from it by change of variables* (12)*. Then*

$$(U^\varepsilon, \varepsilon^2 P^\varepsilon) \to (U, P) \quad \textit{weakly in } Y(\Omega) \times L^2(\Omega) \tag{19}$$

where (U, P) *is a unique solution of the compressible Reynolds equations*

$$U = -\frac{1}{2\mu} y_n(h - y_n)\nabla_{x'} P + \left(1 - \frac{y_n}{h(x')}\right)\mathbf{V} \tag{20}$$

$$\operatorname{div}_{x'}(P\overline{U}) = 0 \quad \textit{in } \mathcal{O}, \qquad P\overline{U} \cdot \mathbf{n} = 0 \quad \textit{on } \partial\mathcal{O} \tag{21}$$

and $\overline{U}(x') = \int_0^{h(x')} U(x', \xi)\,\mathrm{d}\xi$.

Proof. It remains only to pass to the limit through the continuity equation. That will prove that the limit (U, P) is a solution to (20). Since (20) has a unique solution (see [5]), the whole sequences converge and not only the subsequences. Using (19) we obtain for $\varphi \in C_0^\infty(\mathcal{O})$

$$0 = \int_\Omega \varepsilon^2 P^\varepsilon U^\varepsilon \cdot \nabla_{x'}\varphi = \int_\Omega \varepsilon^2 \overline{P}^\varepsilon U^\varepsilon \cdot \nabla_{x'}\varphi + \int_\Omega \nabla_{x'}\varphi(\varepsilon^2 P^\varepsilon - \varepsilon^2 \overline{P}^\varepsilon) U^\varepsilon$$

where the first integral on the right-hand side converges to $\int_\Omega PU \cdot \nabla_{x'}\varphi$ and the second to zero, since, due to Lemma 4.2

$$\int_\Omega \nabla_{x'}\varphi \varepsilon^2 (P^\varepsilon - \overline{P}^\varepsilon) U^\varepsilon \leqslant C\varepsilon |U^\varepsilon|_{L^2(\Omega)} + C\varepsilon^3 |\nabla_{x'} U^\varepsilon|_{L^2(\Omega)} \to 0$$

Thus $\operatorname{div}_{x'}(P\overline{U}) = 0$. □

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