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Asymptotic analysis of layered elastic beams

Michele Serpilli

Dipartimento di Architettura, Costruzioni e Strutture, Università Politecnica delle Marche, via Brecce Bianche, 60131 Ancona, Italy

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Abstract

We consider an elastic beam formed by three layers, fixed at one end and loaded at the free end. We call adherents the upper and lower layers Ω_+^ε and Ω_-^ε and an adhesive layer Ω_m^ε. We denote by $\varepsilon h_{\pm,m}$ the thickness of each layer and we suppose that the stiffness of the adhesive layer is ε^2, with respect to that of the adherents. By an asymptotic analysis we obtain the zeroth order limit problem and the form of the second order displacements. ***To cite this article: M. Serpilli, C. R. Mecanique 333 (2005).***

Résumé

Analyse asymptotique de poutres multicouche. Si l'on considère une poutre élastique constituée de trois couches, fixée à une extrémité. La strate supérieure et inférieure sont indiquées de la sorte Ω_+^ε et Ω_-^ε, alors que la partie centrale, ou adhésive Ω_m^ε. Nous prendrons $\varepsilon h_{\pm,m}$ comme l'épaisseur de chaque strate et supposerons que la rigidité du adhésive et de l'ordre de ε^2, par rapport avec les strates supérieures et inférieures. Par la méthode de l'expansion asymptotique on obtient le problème limite de l'ordre zéro et les fonctions de déplacement de l'ordre 2. ***Pour citer cet article : M. Serpilli, C. R. Mecanique 333 (2005).***

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Mots-clés : Mécanique des solides numérique ; Analyse asymptotique ; Poutres multicouche ; Collage

1. Introduction and statement of the problem

In this Note we consider a compound beam fixed at one end consisting of three layers, with the middle one being softer in comparison to the upper and lower ones. This situation corresponds, for instance, to two layers bonded together by an adhesive joint made of glue or other soft materials. Using the asymptotic expansion method, we

E-mail address: michele_serpilli@yahoo.co.uk (M. Serpilli).

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derive a mathematical one-dimensional model of a layered beam from two-dimensional linear elasticity equations. We refer to [1,2] for asymptotic methods and homogenization techniques and for an analysis of junctions for plates, to [4] for a complete analysis of rod models and to [3] for modelling of plates. Other important references are [5,6] which concern with models of adhesive joints, [7,9] about bonded joints with thin adhesive layer and [8] which deal with adhesively bonded nonlinearly elastic plates.

We consider the Euclidean space $\mathbb{E}^2$ with a Cartesian coordinate frame $(O,\mathbf{e}_1,\mathbf{e}_2)$. Let ε be a positive real 'small' parameter such that $0<\varepsilon<1$. Given the constant $L>0$ we define:

$$\begin{aligned}
&\overline{\Omega}^\varepsilon=\overline{\Omega}^\varepsilon_+\cup\overline{\Omega}^\varepsilon_m\cup\overline{\Omega}^\varepsilon_-\subset\mathbb{R}^2\\
&\Omega^\varepsilon_m=(-h^\varepsilon_m,h^\varepsilon_m)\times(0,L),\qquad \Omega^\varepsilon_+=(h^\varepsilon_m,h^\varepsilon_m+2h^\varepsilon_+)\times(0,L),\qquad \Omega^\varepsilon_-=(-h^\varepsilon_m,-h^\varepsilon_m-2h^\varepsilon_-)\times(0,L)\\
&\Gamma^{a,\varepsilon}=(h^{+\varepsilon},h^{-\varepsilon})\times\{a\},\quad a=0,L,\qquad \Sigma^{\pm\varepsilon}=\{h^{\pm\varepsilon}\}\times(0,L),\quad h^{+\varepsilon}=h^\varepsilon_m+2h^\varepsilon_+,\quad h^{-\varepsilon}=-h^\varepsilon_m-2h^\varepsilon_-\\
&\Sigma^\varepsilon_1=\Omega^\varepsilon_+\cap\Omega^\varepsilon_m=\{h^\varepsilon_m\}\times(0,L),\qquad \Sigma^\varepsilon_2=\Omega^\varepsilon_-\cap\Omega^\varepsilon_m=\{-h^\varepsilon_m\}\times(0,L)
\end{aligned}\tag{1}$$

We consider a two-dimensional three-layer strip of length L occupying the reference configuration $\overline{\Omega}^\varepsilon$. We study the physical problem corresponding to the mechanical behaviour of a two-dimensional compound three-layer beam with a soft core, supposed to be fixed at one end. The thicknesses $h^\varepsilon_\pm$ and h^ε_m are linearly dependent of ε: $h^\varepsilon_{\pm,m}=\varepsilon h_{\pm,m}$. The total height of the beam is $2h_++2h_m+2h_-=2h$. The beam is submitted to body forces $\boldsymbol{f}^{\pm,m\varepsilon}=(f_i^{\pm,m\varepsilon})$ in Ω^ε, to surface forces $\boldsymbol{g}^{\pm\varepsilon}=(g_i^{\pm\varepsilon})$ on boundaries $\Sigma^{\pm\varepsilon}$ and to a system of forces $\boldsymbol{h}^{L,\pm,m\varepsilon}=(h_i^{L,\pm,m\varepsilon})$ loading the free end $\Gamma^{L,\varepsilon}$. The material of each layer is homogeneous, isotropic and linearly elastic with Lamè's coefficients $\lambda^\varepsilon_\pm,\mu^\varepsilon_\pm$ and $\lambda^\varepsilon_m,\mu^\varepsilon_m$. The Lamè's constants of Ω^ε_+ and Ω^ε_- are independent of ε, $\lambda^\varepsilon_\pm=\lambda_\pm$ and $\mu^\varepsilon_\pm=\mu_\pm$, while the elastic moduli of Ω^ε_m depend on ε in this way: $\lambda^\varepsilon_m=\varepsilon^2\lambda_m,\ \mu^\varepsilon_m=\varepsilon^2\mu_m$.

We will employ the summation convention on repeated indices; moreover, we suppose Latin indices (except m) take their values in the set $\{1,2\}$. Indices m is only used to denote those functions related to the middle layer.

Let $V_0(\Omega^\varepsilon)$ be the following space of admissible displacements:

$$V_0(\Omega^\varepsilon):=\left\{(u_i^\varepsilon)\in\left[H^1(\Omega^\varepsilon)\right]^2\colon\ u_i^\varepsilon=0\text{ on }\Gamma^{0,\varepsilon}\right\}\tag{2}$$

The variational formulation of the two-dimensional linear elastic problem is given by:

$$\begin{aligned}
&\boldsymbol{u}^\varepsilon\in V_0(\Omega^\varepsilon)\quad\text{with }\boldsymbol{u}^\varepsilon=\left\{\boldsymbol{u}^\varepsilon_+\text{ in }\Omega^\varepsilon_+,\ \boldsymbol{u}^\varepsilon_m\text{ in }\Omega^\varepsilon_m,\ \boldsymbol{u}^\varepsilon_-\text{ in }\Omega^\varepsilon_-\right\}\\
&\int_{\Omega^\varepsilon_{\pm,m}}\sigma^\varepsilon_{ij}(\boldsymbol{u}^\varepsilon)e^\varepsilon_{ij}(\boldsymbol{v}^\varepsilon)\,\mathrm{d}\boldsymbol{x}^\varepsilon=\int_{\Omega^\varepsilon_{\pm,m}}f_i^{\pm,m\varepsilon}v_i^\varepsilon\,\mathrm{d}\boldsymbol{x}^\varepsilon+\int_{\Sigma^{\pm\varepsilon}}g_i^{\pm\varepsilon}v_i^\varepsilon\,\mathrm{d}x_2^\varepsilon+\int_{\Gamma^{L,\varepsilon}}h_i^{L,\pm,m\varepsilon}v_i^\varepsilon\,\mathrm{d}x_1^\varepsilon
\end{aligned}\tag{3}$$

for all $\boldsymbol{v}^\varepsilon\in V_0(\Omega^\varepsilon)$ with

$$\boldsymbol{v}^\varepsilon=\left\{\boldsymbol{v}^\varepsilon_+\text{ in }\Omega^\varepsilon_+,\ \boldsymbol{v}^\varepsilon_m\text{ in }\Omega^\varepsilon_m,\ \boldsymbol{v}^\varepsilon_-\text{ in }\Omega^\varepsilon_-\right\}$$

where $\boldsymbol{e}^\varepsilon(\boldsymbol{u}^\varepsilon)=(e^\varepsilon_{ij}(\boldsymbol{u}^\varepsilon))$ is the linearized strain tensor: $e^\varepsilon_{ij}(\boldsymbol{u}^\varepsilon)=(\partial^\varepsilon_i u^\varepsilon_j+\partial^\varepsilon_j u^\varepsilon_i)/2$ (where $\partial^\varepsilon_i:=\partial/\partial x^\varepsilon_i$) and $\boldsymbol{\sigma}^\varepsilon(\boldsymbol{u}^\varepsilon)=(\sigma^\varepsilon_{ij}(\boldsymbol{u}^\varepsilon))$ is the stress tensor, related to the displacement field $\boldsymbol{u}^\varepsilon=(u^\varepsilon_i)$ by Hooke's generalized law: $\sigma^\varepsilon_{ij}(\boldsymbol{u}^\varepsilon)=\lambda^\varepsilon e^\varepsilon_{pp}(\boldsymbol{u}^\varepsilon)\delta_{ij}+2\mu^\varepsilon e^\varepsilon_{ij}(\boldsymbol{u}^\varepsilon)$.

2. Equivalent formulation of the problem

In order to study the behaviour of $\boldsymbol{u}^\varepsilon$ when the thickness of the three layers goes to zero, we introduce the following change of variable from the reference configuration (dependent of ε) to a fixed one (independent of ε) [3,4]:

$$\begin{aligned}
&\Pi^\varepsilon_+:\boldsymbol{x}_+=(x_1,x_2)\in\overline{\Omega}_+\to\boldsymbol{x}^\varepsilon_+=\Pi^\varepsilon_+(\boldsymbol{x}_+)=(x^\varepsilon_1,x^\varepsilon_2)=(\varepsilon+\varepsilon x_1,x_2)\in\overline{\Omega}^\varepsilon_+\\
&\Pi^\varepsilon_m:\boldsymbol{x}_m=(x_1,x_2)\in\overline{\Omega}_m\to\boldsymbol{x}^\varepsilon_m=\Pi^\varepsilon_m(\boldsymbol{x}_m)=(x^\varepsilon_1,x^\varepsilon_2)=(\varepsilon x_1,x_2)\in\overline{\Omega}^\varepsilon_m\\
&\Pi^\varepsilon_-:\boldsymbol{x}_-=(x_1,x_2)\in\overline{\Omega}_-\to\boldsymbol{x}^\varepsilon_-=\Pi^\varepsilon_-(\boldsymbol{x}_-)=(x^\varepsilon_1,x^\varepsilon_2)=(-\varepsilon+\varepsilon x_1,x_2)\in\overline{\Omega}^\varepsilon_-
\end{aligned}\tag{4}$$

Then we suppose the following orders of magnitude for the forces:

$$\begin{aligned}
&f_1^{\pm\varepsilon}(\boldsymbol{x}^\varepsilon) = \varepsilon f_1^{\pm}(\boldsymbol{x}), \qquad f_2^{\pm\varepsilon}(\boldsymbol{x}^\varepsilon) = f_2^{\pm}(\boldsymbol{x})\\
&f_1^{m\varepsilon}(\boldsymbol{x}^\varepsilon) = \varepsilon^3 f_1^{m}(\boldsymbol{x}), \qquad f_2^{m\varepsilon}(\boldsymbol{x}^\varepsilon) = \varepsilon^2 f_2^{m}(\boldsymbol{x})\\
&g_1^{\pm\varepsilon}(\boldsymbol{x}^\varepsilon) = \varepsilon^2 g_1^{\pm}(\boldsymbol{x}), \qquad g_2^{\pm\varepsilon}(\boldsymbol{x}^\varepsilon) = \varepsilon g_2^{\pm}(\boldsymbol{x})\\
&h_1^{L,\pm\varepsilon}(\boldsymbol{x}^\varepsilon) = \varepsilon h_1^{L,\pm}(\boldsymbol{x}), \qquad h_2^{L,\pm\varepsilon}(\boldsymbol{x}^\varepsilon) = h_2^{L,\pm}(\boldsymbol{x})\\
&h_1^{L,m\varepsilon}(\boldsymbol{x}^\varepsilon) = \varepsilon^3 h_1^{L,m}(\boldsymbol{x}), \qquad h_2^{L,m\varepsilon}(\boldsymbol{x}^\varepsilon) = \varepsilon^2 h_2^{L,m}(\boldsymbol{x})
\end{aligned} \tag{5}$$

where the functions $f_i^{\pm,m}$, $g_i^{\pm}$ and $h_i^{L,\pm,m}$ are independent of the parameter ε. Hence, by introducing the scaling of unknowns and test functions as [3,4]:

$$u_1(\varepsilon)(\boldsymbol{x}) = \varepsilon u_1^\varepsilon(\boldsymbol{x}^\varepsilon), \quad u_2(\varepsilon)(\boldsymbol{x}) = u_2^\varepsilon(\boldsymbol{x}^\varepsilon) \quad \text{for all } \boldsymbol{u}^\varepsilon \in \left[H^1(\Omega^\varepsilon)\right]^2 \tag{6}$$

we can derive the following equivalent problem for the scaled displacements $\boldsymbol{u}(\varepsilon)$:

$$\begin{cases}
\boldsymbol{u}(\varepsilon) \in V_0(\Omega)\\
c_0\big(\boldsymbol{u}(\varepsilon), \boldsymbol{v}\big) + \varepsilon^2 c_2\big(\boldsymbol{u}(\varepsilon), \boldsymbol{v}\big) + \varepsilon^4 c_4\big(\boldsymbol{u}(\varepsilon), \boldsymbol{v}\big) + \varepsilon^6 c_6\big(\boldsymbol{u}_m(\varepsilon), \boldsymbol{v}_m\big)\\
\quad = \varepsilon^4 l_1(\boldsymbol{v}) + \varepsilon^6 l_2(\boldsymbol{v}_m) \quad \text{for all } \boldsymbol{v} \in V_0(\Omega)
\end{cases} \tag{7}$$

where the bilinear forms c_0, c_2, c_4 and c_6 and the linear forms l_1 and l_2 are defined in [10].

3. The asymptotic expansion

In order to obtain the characterization of the limit problem we assume the asymptotic expansion for the solution of the elastic problem (7):

$$\boldsymbol{u}_{\pm,m}(\varepsilon) = \boldsymbol{u}^0_{\pm,m} + \varepsilon^2 \boldsymbol{u}^2_{\pm,m} + \varepsilon^4 \boldsymbol{u}^4_{\pm,m} + h.o.t., \quad h.o.t. = \text{higher order terms} \tag{8}$$

By substituting these expressions into the scaled variational problem (7) and by identifying the terms with identical power of ε, we obtain, as customary, a sequence of equations, which are reported in [10].

By studying each variational problem we obtain the following characterization of the limit problem. First, we define the first and second order moments:

$$\begin{aligned}
&S_+ := \int_{h_m}^{h_m+2h_+} x_1\,\mathrm{d}x_1, \qquad S_- := \int_{-h_m-2h_-}^{-h_m} x_1\,\mathrm{d}x_1\\
&I_+ := \int_{h_m}^{h_m+2h_+} x_1^2\,\mathrm{d}x_1, \qquad I_m := \int_{h_m}^{-h_m} x_1^2\,\mathrm{d}x_1, \qquad I_- := \int_{-h_m-2h_-}^{-h_m} x_1^2\,\mathrm{d}x_1
\end{aligned} \tag{9}$$

Theorem 3.1. *If the system of applied forces verifies*

$$f_1^{\pm} \in L^2(\Omega_\pm), \quad g_1^{\pm} \in L^2(0,L), \quad f_2^{\pm} \in H^1\big(0,L; L^2(h_\pm^-, h_\pm^+)\big), \quad g_2^{\pm} \in H^1(0,L),$$

then

(a) *the limit displacement field $\boldsymbol{u}^0$ belongs to the space $\hat{V}_0(\Omega)$, that is*

$$
\begin{aligned}
&u^0_{+,1}(x_1,x_2)=u^0_{-,1}=u^0_{m,1}:=\zeta_1(x_2),\quad \zeta_1\in H^2(0,L)\\
&u^0_{+,2}(x_1,x_2)=\eta_2(x_2)-x_1\partial_2\zeta_1,\quad \eta_2\in H^1(0,L)\\
&u^0_{-,2}(x_1,x_2)=\xi_2(x_2)-x_1\partial_2\zeta_1,\quad \xi_2\in H^1(0,L)\\
&u^0_{m,2}(x_1,x_2)=\frac{\eta_2+\xi_2}{2}+x_1\left(\frac{\eta_2-\xi_2}{2h_m}-\partial_2\zeta_1\right)
\end{aligned}
\tag{10}
$$

where ζ_1,η_2,ξ_2 are solution of the coupled problem (*we denote with $\phi':=\partial_2\phi(x_2)$*)

$$
\begin{aligned}
&\int_0^L\big[(\widetilde{E}_+I_++\widetilde{E}_-I_-)\zeta_1''-\widetilde{E}_+S_+\eta_2'-\widetilde{E}_-S_-\eta_2'\big]\upsilon_1''\,\mathrm{d}x_2\\
&\quad=\int_0^L F_1\upsilon_1\,\mathrm{d}x_2-\int_0^L M_1\upsilon_1'\,\mathrm{d}x_2+F_1^{L,\pm}\upsilon_1(L)+M_1^{L,\pm}\upsilon_1'(L),\quad \forall\upsilon_1\in H^2(0,L)
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\int_0^L\big[K^2(\eta_2-\xi_2)\upsilon_2+(2\widetilde{E}_+h_+\eta_2'-\widetilde{E}_+S_+\zeta_1'')\upsilon_2'\big]\,\mathrm{d}x_2=\int_0^L F_2^+\upsilon_2\,\mathrm{d}x_2+F_2^{L,+}\upsilon_2(L),\\
&\quad\forall\upsilon_2\in H^1(0,L)
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&\int_0^L\big[-K^2(\eta_2-\xi_2)\underline{\upsilon}_2+(2\widetilde{E}_-h_-\xi_2'-\widetilde{E}_-S_-\zeta_1'')\underline{\upsilon}_2'\big]\,\mathrm{d}x_2=\int_0^L F_2^-\underline{\upsilon}_2\,\mathrm{d}x_2+F_2^{L,-}\underline{\upsilon}_2(L),\\
&\quad\forall\underline{\upsilon}_2\in H^1(0,L)
\end{aligned}
\tag{13}
$$

with $\widetilde{E}_{\pm,m}:=4\mu_{\pm,m}(\lambda_{\pm,m}+\mu_{\pm,m})/(\lambda_{\pm,m}+2\mu_{\pm,m})$, $K^2:=\mu_m/2h_m$ and F_1, M_1, $F_2^\pm$, $F_i^{L,\pm}$, $M_1^{L,\pm}$ are functions of the forces and are given in [10].

(b) *The limit variational problem* (11)–(13) *is equivalent to the following differential problem* (*coupled bending-stretching equations*):

$$(\widetilde{E}_+I_++\widetilde{E}_-I_-)\zeta_1''''-\widetilde{E}_+S_+\eta_2'''-\widetilde{E}_-S_-\xi_2'''=F_1+M_1'\quad \textit{in } (0,L) \tag{14}$$

$$K^2(\eta_2-\xi_2)+\widetilde{E}_+S_+\zeta_1'''-2\widetilde{E}_+h_+\eta_2''=F_2^+\quad \textit{in } (0,L) \tag{15}$$

$$-K^2(\eta_2-\xi_2)+\widetilde{E}_-S_-\zeta_1'''-2\widetilde{E}_-h_-\xi_2''=F_2^-\quad \textit{in } (0,L) \tag{16}$$

As we can clearly notice from the previous differential system, the central adhesive layer behaves as an elastic interphase of stiffness K^2 which reacts to the gap between the axial displacements of the two adherents. The fourth order equation (14), associated with the flexural behaviour of the beam, is strongly coupled with the axial displacements of the two adherents. Thus, the structural anisotropy and heterogeneity of the beam, due to the presence of the adhesive, lead to a strong coupling of the system.

4. Characterization of $\boldsymbol{u}^2$

Using the asymptotic development method, we characterize the second order displacements. We notice that, if the compatibility conditions (17) at the free end are satisfied, it is possible to characterize the term $\boldsymbol{u}^2$ and the well-known boundary layer phenomenon will not appear. Otherwise, a detailed analysis of boundary layer, to be developed by classical techniques [1,5], is required. This will be performed in [10].

Theorem 4.1. *If the system of applied forces verifies*

$$f_1^{\pm} \in L^2(\Omega_\pm), \quad g_1^{\pm} \in L^2(0,L), \quad f_2^{\pm} \in H^1\big(0,L;L^2(h_\pm^-,h_\pm^+)\big), \quad g_2^{\pm} \in H^1(0,L)$$

and the compatibility conditions in $x_2 = L$

$$\begin{aligned} h_2^{L,\pm,m} &= \big[\lambda_{\pm,m} e_{11}(\boldsymbol{u}^2) + (\lambda_{\pm,m} + 2\mu_{\pm,m}) e_{22}(\boldsymbol{u}^0)\big] \\ h_1^{L,\pm,m} &= \big[\lambda_{\pm,m} e_{12}(\boldsymbol{u}^2)\big] \end{aligned} \tag{17}$$

then the limit displacements $\boldsymbol{u}^2$ *can be written in the form*

$$u_{+,1}^2 = \underline{u}_{+,1}^2(x_2) - \tilde{\nu}_+\left(x_1 \partial_2 \eta_2 - \frac{x_1^2}{2}\partial_{22}\zeta_1\right) \tag{18}$$

$$u_{-,1}^2 = \underline{u}_{-,1}^2(x_2) - \tilde{\nu}_-\left(x_1 \partial_2 \xi_2 - \frac{x_1^2}{2}\partial_{22}\zeta_1\right) \tag{19}$$

$$\begin{aligned} u_{m,1}^2 = &\left[\frac{\underline{u}_{+,1}^2 + \underline{u}_{-,1}^2}{2} + \frac{1}{2}(\tilde{\nu}_+ + \tilde{\nu}_- + \tilde{\nu}_m) h_m^2 \partial_{22}\zeta_1 - \tilde{\nu}_+ h_m \partial_2\eta_2 + \tilde{\nu}_- h_m \partial_2 \xi_2 - \hat{\nu}_m h_m (\partial_2\eta_2 - \partial_2\xi_2)\right] \\ &+ x_1\left[\frac{\underline{u}_{+,1}^2 - \underline{u}_{-,1}^2}{2h_m} - \frac{\tilde{\nu}_+}{2}\partial_2\eta_2 - \frac{\tilde{\nu}_-}{2}\partial_2\xi_2 + \frac{1}{4}(\tilde{\nu}_+ - \tilde{\nu}_-) h_m \partial_{22}\zeta_1\right] \\ &+ \frac{x_1^2}{2}\left[\hat{\nu}_m \frac{\partial_2\eta_2 - \partial_2\xi_2}{2h_m} - \tilde{\nu}_m \partial_{22}\zeta_1\right] \end{aligned} \tag{20}$$

$$\begin{aligned} u_{+,2}^2 = \underline{u}_{+,2}^2(x_2) &+ x_1\left[4(\tilde{\nu}_+ + \hat{\nu}_+)\left(h\partial_{22}\eta_2 - \frac{h^2}{2}\partial_{222}\zeta_1\right) + \frac{q_2^+}{\mu_1} - \partial_2 \underline{u}_{+,1}^2\right] \\ &- (3\tilde{\nu}_+ + 4\hat{\nu}_+)\left[\frac{x_1^2}{2}\partial_{22}\eta_2 - \frac{x_1^3}{6}\partial_{222}\zeta_1\right] - \frac{\widetilde{F}_2^{*+}}{\mu_+} \end{aligned} \tag{21}$$

$$\begin{aligned} u_{-,2}^2 = \underline{u}_{-,2}^2(x_2) &+ x_1\left[4(\tilde{\nu}_- + \hat{\nu}_-)\left(-h\partial_{22}\xi_2 - \frac{h^2}{2}\partial_{222}\zeta_1\right) + \frac{q_2^-}{\mu_-} - \partial_2 \underline{u}_{-,1}^2\right] \\ &- (3\tilde{\nu}_- + 4\hat{\nu}_-)\left[\frac{x_1^2}{2}\partial_{22}\xi_2 - \frac{x_1^3}{6}\partial_{222}\zeta_1\right] - \frac{\widetilde{F}_2^{*-}}{\mu_-} \end{aligned} \tag{22}$$

$$u_{m,2}^2 = \underline{w}_{m,2}^2(x_2) - x_1\vartheta(x_2) - \frac{x_1^2}{2}\varphi(x_2) - \frac{x_1^3}{6}\psi(x_2) - \frac{\widetilde{F}_2^{*m}}{\mu_m} \tag{23}$$

where $\tilde{\nu}_{\pm,m} := \lambda_{\pm,m}/(\lambda_{\pm,m} + 2\mu_{\pm,m})$, $\hat{\nu}_{\pm,m} := \mu_{\pm,m}/(\lambda_{\pm,m} + 2\mu_{\pm,m})$, $F_2^{*\pm,m} := \int f_2^{\pm,m}\,\mathrm{d}x_1$, $\widetilde{F}_2^{*\pm,m} := \int F_2^{*\pm,m}\,\mathrm{d}x_1$, $q_2^{\pm} := (g_2^{\pm} + F_2^{*\pm})|_{x_1=\pm h}$.

The limit variational problem that allows us to find the unknown displacements $\underline{u}_{+,1}^2(x_2)$, $\underline{u}_{-,1}^2(x_2)$, $\underline{u}_{+,2}^2(x_2)$, $\underline{u}_{-,2}^2(x_2)$ *is expressed in* [10]; *the form of functions* ϑ, φ, ψ *and* $\underline{u}_{m,2}^2$ *is expressed in* [10] *and they only depend on* $\underline{u}_{+,1}^2(x_2)$, $\underline{u}_{-,1}^2(x_2)$, $\underline{u}_{+,2}^2(x_2)$, $\underline{u}_{-,2}^2(x_2)$, *and* η_2, ξ_2, ζ_1 *which are the solution of the limit problem* (11)–(13).

We decide to express explicitly the polynomial dependence with respect of x_1 of the second order displacements, instead of using liable functions of warping and of Timoshenko used by Trabucho–Viaño [4]. Although those functions have a significant mechanical meaning, we prefer to write more readable, practical and easily usable expressions.

In the paper [6] a similar model of a two-dimensional three-layer strip is reported. The main differences are that in [6] the compound beam consists of three layers with the middle one being thinner than the upper and lower ones: the order of magnitude of its thickness is ε^2 compared to that of the two adherents which is ε, while we considered the same magnitude ε for the thickness of each layer; the Young modulus of the middle layer is scaled with ε^3, while we scale it with ε^2 they suppose that the displacements satisfy the homogeneous Lamè systems $\mu_i \nabla^2 \boldsymbol{u}(\boldsymbol{x}) + (\lambda_i + 2\mu_i)\nabla\nabla \cdot \boldsymbol{u}(\boldsymbol{x}) = \mathbf{0}$, while we use a variational formulation of the problem; they consider the beam to be fixed on both ends, while we use a strong clamping just on one end. Then, Klarbring–Movchan characterize displacements at first and second orders. The first order longitudinal displacement is a piece-wise linear function of the through-the-thickness coordinate, while the transversal one is a function only of the axial coordinate. The second order transversal and axial displacements are respectively quadratic and cubic functions of the through-the-thickness coordinate. The equation which allows to characterize the transversal zeroth order displacement is substantially identical to the one we found: this means that by reducing of one order of magnitude both the thickness (from ε to ε^2) and the Young modulus of the adhesive (from ε^2 to ε^3), it is possible to obtain the same displacement field, thus the same beam model, which consequently seems to be valid in more general situations.

The work [8] deals with asymptotic method applied to adhesively bonded nonlinearly elastic plates. The assumptions concerning with the order of magnitude ratios of the thicknesses and the elastic moduli between adherents and adhesive are the same used in [6]. The two-dimensional plate model is similar to the one obtained in [6] and in my work, i.e., a piece-wise linear axial displacement with respect of the through-the-thickness coordinate, while the difference is in the governing equations of the limit problem which, indeed, are nonlinear according with the considered framework.

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