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Elastic perfectly plastic structures with temperature dependent elastic coefficients

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Abstract

We study the evolution of elastic perfectly plastic structures where the elastic coefficients depend on temperature, as they are subjected to classical loading and given variation of the temperature field. We prove variational theorems for the instantaneous fields of velocities and stress rates, and establish the generalized differential equation for the evolution of the stress field. ***To cite this article: B. Halphen, C. R. Mecanique 333 (2005).***

Résumé

Structures élastiques parfaitement plastiques dont les coefficients d'élasticité dépendent de la température. On étudie l'évolution des structures élastiques parfaitement plastiques dont les coefficients d'élasticité dépendent de la température, lorsqu'elles sont soumises à un trajet de chargement mécanique de type classique et à un champ de température donné variable dans le temps. On établit des théorèmes variationnels vérifiés par les champs de vitesse et de taux de contrainte à un instant donné, et l'équation différentielle généralisée vérifiée par l'évolution du champ de contrainte. ***Pour citer cet article : B. Halphen, C. R. Mecanique 333 (2005).***

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1. Introduction

Industrial structures or their elements sometimes undergo thermo-mechanical loadings where the amplitude of the temperatures is such that the variation of the elastic moduli with temperature may not be neglected. That is, for example, the case when one deals with the durability of parts of car or plane engines.

Under these conditions, we are going to investigate whether the time evolution of an elastic plastic system is similar to when the elastic coefficients are constant, i.e. variational theorems for the velocity and stress rate fields at a given time, and a generalized differential equation for the evolution of the stress field. We shall then emphasize the differences relative to the case of constant elastic coefficients.

2. Rate problem

2.1. Formulating the problem

We consider a volume V occupied by an elastic perfectly plastic standard material, undergoing infinitesimal transformation, assuming that the elastic coefficients depend on temperature. The thermo-elastic strain tensor of the material is written as:

$$\boldsymbol{\varepsilon}^e = \boldsymbol{\Lambda}(\theta) : \boldsymbol{\sigma} + \boldsymbol{A}(\theta) \tag{1}$$

where $\boldsymbol{\sigma}$ is the stress tensor, θ is the temperature, $\boldsymbol{\Lambda}(\theta)$ is the elastic compliance tensor and $\boldsymbol{A}(\theta)$ is the thermal strain tensor; we denote with: the double product.

The plastic potential $f(\boldsymbol{\sigma}, \theta)$, that we, in order to simplify the presentation, suppose to be differentiable, is convex and depends on temperature; the plastic strain rate is given by:

$$\boldsymbol{d}^{\mathrm{p}} = \lambda \frac{\partial f}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \theta), \quad \lambda \geqslant 0,\ f \leqslant 0,\ \lambda f = 0 \text{ and if } f = 0,\ \lambda \dot{f} = 0 \tag{2}$$

The loading of volume V is a classical one (given stress vectors and velocities on the surface ∂V at any time, and given mass forces, that may depend on time, in V) and a varying field of temperatures is given.

We want to determine the velocity and stress rate fields in volume V at a given instant.

2.2. Variational theorem for the velocity field

We investigate now how can we formulate the problem of determination of the velocity and strain rate fields in V at every moment.

At every time, and at a point in volume V, the total strain rate $\boldsymbol{d}$ is the sum of the thermo-elastic stain rate and of the plastic strain rate, which, following Eqs. (1) and (2), can be written as:

$$\boldsymbol{d} = \boldsymbol{d}^e + \boldsymbol{d}^{\mathrm{p}} = \boldsymbol{\Lambda} : \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\Lambda}} : \boldsymbol{\sigma} + \dot{\boldsymbol{A}} + \lambda \frac{\partial f}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \theta) \quad \text{with conditions (2) on } f, \dot{f}, \lambda \tag{3}$$

As $\boldsymbol{\Lambda}$ and $\boldsymbol{A}$ depend only on temperature, whose evolution is given, and as the stress state is known at the instant considered, we can introduce:

$$\boldsymbol{d}^0(t) = \dot{\boldsymbol{\Lambda}} : \boldsymbol{\sigma} + \dot{\boldsymbol{A}} \tag{4}$$

$\boldsymbol{d}^0(t)$ is known at the considered moment, and therefore plays the role of an 'initial' strain rate.

Reasoning similarly to the case when the elastic coefficients are constant, we now show that the stress rate $\dot{\boldsymbol{\sigma}}$ can be derived from the potential function $U(\boldsymbol{d})$ defined by:

$$U(\boldsymbol{d}) = \frac{1}{2}(\boldsymbol{d} - \boldsymbol{d}^0) : \boldsymbol{\Lambda}^{-1}(\theta) : (\boldsymbol{d} - \boldsymbol{d}^0) - Y(f)\frac{1}{2}\frac{\langle \partial f/\partial \boldsymbol{\sigma} : \boldsymbol{\Lambda}^{-1}(\theta) : (\boldsymbol{d} - \boldsymbol{d}^0) + (\partial f/\partial \theta)\dot{\theta}\rangle^2}{\partial f/\partial \boldsymbol{\sigma} : \boldsymbol{\Lambda}^{-1}(\theta) : \partial f/\partial \boldsymbol{\sigma}} \tag{5}$$

where $Y(f) = 0$ if $f < 0$ et $Y(f) = 1$ if $f = 0$, and $\langle x \rangle$ is the positive part of x:

$$\langle x \rangle = \frac{1}{2}\big(x + |x|\big)$$

It is easy to show that this potential function is convex, but not strictly convex. As in the case when the elastic coefficients are independent of temperature, we obtain the following minimum theorem for the velocity field at every given time:

Theorem 2.1. *Among all the velocity fields $\boldsymbol{v}^*$ that are kinematically admissible at instant t, an actual velocity field minimizes the functional*:

$$\int_V U(\boldsymbol{d}^*)\,\mathrm{d}V - (\boldsymbol{F}^{\mathrm{d}}, \boldsymbol{v}^*) \tag{6}$$

where $(\boldsymbol{F}^{\mathrm{d}}, \boldsymbol{v}^*)$ is the power of the applied forces at instant t.

2.3. Variational theorem for the stress rate field

Definition 2.2. Knowing the present stress and temperature fields in volume V, we say that a stress rate field $\dot{\tilde{\boldsymbol{\sigma}}}$ is *plastically admissible* at time t if:

$$\forall \boldsymbol{x} \in V,\ \text{ if } f(\boldsymbol{\sigma}, \theta) = 0, \quad \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\tilde{\boldsymbol{\sigma}}} + \frac{\partial f}{\partial \theta}\dot{\theta} \leqslant 0 \tag{7}$$

Then the following minimum theorem can be proved:

Theorem 2.3. *Among all stress rate fields $\dot{\tilde{\boldsymbol{\sigma}}}$ statically admissible with the given traction rates at instant t and plastically admissible at this instant, the actual stress rate field minimizes the functional*:

$$\int_V \left(\frac{1}{2}\dot{\tilde{\boldsymbol{\sigma}}} : \boldsymbol{\Lambda}(\theta) : \dot{\tilde{\boldsymbol{\sigma}}} + \dot{\tilde{\boldsymbol{\sigma}}} : \boldsymbol{d}^0\right)\mathrm{d}V - \big((\dot{\tilde{\boldsymbol{\sigma}}}\boldsymbol{n})\boldsymbol{v}^{\mathrm{d}}\big) \tag{8}$$

where $((\dot{\tilde{\boldsymbol{\sigma}}}\boldsymbol{n})\boldsymbol{v}^{\mathrm{d}})$ is the power of the traction rates at the surface of V against the field of given velocities.

This minimum theorem for stress rates can be established as the dual form of the preceding one for velocities, or we can build a direct proof, rather similar to the one that is used when elastic coefficients do not depend on temperature, and which rests specifically on the following property:

for a given state of stress and temperature, let $\dot{\boldsymbol{\sigma}}$ be a stress rate plastically admissible at time t and $\boldsymbol{d}^{\mathrm{p}}$ the associated plastic strain rate; let $\dot{\tilde{\boldsymbol{\sigma}}}$ be another stress rate plastically admissible at the same time; then:

$$\forall \boldsymbol{x} \in V, \quad (\dot{\boldsymbol{\sigma}} - \dot{\tilde{\boldsymbol{\sigma}}}) : \boldsymbol{d}^{\mathrm{p}} \geqslant 0 \tag{9}$$

As the elastic compliance tensor is positive definite, the functional (8) is strictly convex with respect to the field $\dot{\tilde{\boldsymbol{\sigma}}}$. The minimum that is looked for is unique. *The solution for the stress rate is unique at a given time*.

We see that, owing to the presence of $\boldsymbol{d}^0$ in the functional (8), and due to its form (4), the variational problem with respect to the stress rate field differs noticeably from the classical result of Greenberg [1].

Nevertheless, we can show that the minimization of the functional (8) is equivalent to the minimization of another, which can, as in the classical situation [2], be used to build numerical algorithms.

Let us define the fictitious elastic strain and stress rate fields, $\hat{\boldsymbol{d}}^{\mathrm{E}}$ and $\hat{\boldsymbol{\sigma}}^{\mathrm{E}}$ at time t as the solution of the following elasticity problem:

(a) $\hat{\boldsymbol{d}}^{\mathrm{E}}$ and $\hat{\boldsymbol{\sigma}}^{\mathrm{E}}$ are respectively kinematically and statically admissible with the given velocities and force rates;
(b) $\hat{\boldsymbol{d}}^{\mathrm{E}}$ and $\hat{\boldsymbol{\sigma}}^{\mathrm{E}}$ are locally linked by the fictitious elastic law:

$$\hat{\boldsymbol{d}}^{\mathrm{E}} = \boldsymbol{\Lambda}(\theta) : \hat{\boldsymbol{\sigma}}^{\mathrm{E}} + \boldsymbol{d}^0 \tag{10}$$

where θ is the temperature at time t and $\boldsymbol{d}^0$ is associated with the evolution of the real fields in the volume V following Eq. (4).

Then we prove the following theorem:

Theorem 2.4. *Among all stress rate fields $\dot{\tilde{\boldsymbol{\sigma}}}$ statically admissible with the given traction rates at instant t and plastically admissible at this instant, the actual stress rate field minimizes the functional*:

$$\int_V \frac{1}{2}(\dot{\tilde{\boldsymbol{\sigma}}} - \hat{\boldsymbol{\sigma}}^{\mathrm{E}}) : \boldsymbol{\Lambda}(\theta) : (\dot{\tilde{\boldsymbol{\sigma}}} - \hat{\boldsymbol{\sigma}}^{\mathrm{E}})\,\mathrm{d}V \tag{11}$$

Notice that (11) is the square value of the norm of $\dot{\tilde{\boldsymbol{\sigma}}} - \hat{\boldsymbol{\sigma}}^{\mathrm{E}}$ defined through a reduced elastic energy at time t.

3. Evolution of the stress field

The volume V is still subjected to a classical loading and a given temperature field which varies with time.

Given some initial conditions, let $\boldsymbol{\sigma}(t)$, $\boldsymbol{\varepsilon}(t)$, $\boldsymbol{\varepsilon}^{\mathrm{p}}(t)$, $\boldsymbol{d}(t)$ be the stress, strain, plastic strain, strain rate responses, respectively, of volume V. We denote by $\boldsymbol{\sigma}^{\mathrm{E}}(t)$, $\boldsymbol{\varepsilon}^{\mathrm{E}}(t)$ the purely elastic response of volume V at time t, that is, the stress and strain fields in volume V under the thermo-mechanical loading at time t if its behavior was purely elastic. In particular, at any point of V at time t:

$$\boldsymbol{\varepsilon}^{\mathrm{E}} = \boldsymbol{\Lambda}(\theta) : \boldsymbol{\sigma}^{\mathrm{E}} + \boldsymbol{A}(\theta) \tag{12}$$

Then, let $\tilde{\boldsymbol{\sigma}}(t)$ be a statically and plastically admissible stress field at time t, which means that it satisfies the plasticity criterion at any point of the volume and the conditions of equilibrium under the given forces at time t. From the virtual power principle, we obtain:

$$\int_V (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}) : \boldsymbol{d}\,\mathrm{d}V = \int_V (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{E}}\,\mathrm{d}V \tag{13}$$

Using the constitutive law of the material and Eq. (12), we can write:

$$\int_V (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}) : \left(\frac{\mathrm{d}}{\mathrm{d}t}\big(\boldsymbol{\Lambda}(\theta) : \boldsymbol{\sigma}\big) + \boldsymbol{d}^{\mathrm{p}}\right)\mathrm{d}V = \int_V (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}) : \frac{\mathrm{d}}{\mathrm{d}t}\big(\Lambda(\theta) : \boldsymbol{\sigma}^{\mathrm{E}}\big)\,\mathrm{d}V \tag{14}$$

Now from the flow rule (2) we can conclude that:

$$\int_V (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}) : \frac{\mathrm{d}}{\mathrm{d}t}\big(\boldsymbol{\Lambda}(\theta) : \boldsymbol{\sigma}\big)\,\mathrm{d}V \leqslant \int_V (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}) : \frac{\mathrm{d}}{\mathrm{d}t}\big(\boldsymbol{\Lambda}(\theta) : \boldsymbol{\sigma}^{\mathrm{E}}\big)\,\mathrm{d}V \tag{15}$$

Let us define the scalar product of two second order tensor fields on V by:

$$(\boldsymbol{a}, \boldsymbol{b}) = \int_V \boldsymbol{a} : \boldsymbol{b}\,\mathrm{d}V \tag{16}$$

As the temperature field is given as a function of time, and as $\tilde{\boldsymbol{\sigma}}$ is any statically and plastically admissible field at time t, inequality (15) leads to the following evolution equation for the stress field:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\big(\boldsymbol{\Lambda}(t):\boldsymbol{\sigma}\big) \in \partial\psi_{K(t)}(\boldsymbol{\sigma}) - \frac{\mathrm{d}}{\mathrm{d}t}\big(\boldsymbol{\Lambda}(t):\boldsymbol{\sigma}^{\mathrm{E}}\big) \tag{17}$$

where $\partial\psi_{K(t)}(\boldsymbol{\sigma})$ denotes the subgradient at point $\boldsymbol{\sigma}$, with respect to the scalar product (16), of the indicator function $\psi_{K(t)}$ of the convex set $K(t)$ of statically and plastically admissible fields at time t.

Introducing the residual stress field $\boldsymbol{\rho}(t) = \boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^{\mathrm{E}}(t)$, the evolution problem (17) can be written as:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\big(\boldsymbol{\Lambda}(t):\boldsymbol{\rho}(t)\big) \in \partial\psi_{K^0(t)}\big(\boldsymbol{\rho}(t)\big) \tag{18}$$

where $K^0(t) = K(t) - \boldsymbol{\sigma}^{\mathrm{E}}(t)$ is the convex set of self-stress fields $\tilde{\boldsymbol{\rho}}$, such that $\tilde{\boldsymbol{\rho}} + \boldsymbol{\sigma}^{\mathrm{E}}(t)$ is statically and plastically admissible at time t.

We see that the form of the evolution problem differs from the one obtained when the elastic coefficients do not depend on temperature [2,3]. The two modifications are that the scalar product which is used is not defined by the elastic energy, and that the elastic compliance appears in Eq. (17). As it could be foreseen with the variational theorems related to the rate problem, Eq. (17) shows that the numerical algorithms that work in the classical case do not work any more when elastic coefficients depend on temperature.

From Eq. (18) we also deduce that we cannot use the same arguments of convexity and convergence as in the classical case to prove uniqueness of the solution or to study the asymptotic behavior of the solution when time tends to infinity, although at least one attempt has been made in the literature [4].

Concerning the asymptotic behavior, we proposed a sufficient condition for shakedown as conjecture and we could check it on elementary numerical examples [5].

4. Conclusion

We have been able to establish results concerning the evolution of an elastic-plastic structure when its elastic coefficients depend on temperature. Having shown how the problem is posed, we leave the question open about the general properties of the stress field providing the solution of the problem.

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