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The two-dimensional problem of steady waves on water of finite depth: regimes without waves of small amplitude

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Abstract

The two-dimensional problem of steady waves on water of finite depth is considered without assumptions about periodicity and symmetry of waves. A new form of Bernoulli's equation is derived, and it involves a new bifurcation parameter which is the product of the Froude number μ and the rate of flow ω. The main result obtained from this equation is the absence of waves, having sufficiently small amplitude, provided $|\mu\omega| > 1$. ***To cite this article: V. Kozlov, N. Kuznetsov, C. R. Mecanique 333 (2005).***

Résumé

Le problème bidimensionelle d'ondes de surface stationnaires en profondeur finie : le régime sans ondes de petite amplitude. Nous étudions le problème bidimensionnel d'ondes stationnaires sur la surface des eaux de profondeur finie sans hypothèse préalable concernant leur symétrie ou périodicité. Une nouvelle forme d'équations de Bernoulli est dérivée avec l'introduction d'un nouveau paramètre de bifurcation qui est le produit du nombre de Froude μ et le débit fluide ω. Il résulte de cette équation que les ondes de petite amplitude n'existent pas pour $|\mu\omega| > 1$. ***Pour citer cet article : V. Kozlov, N. Kuznetsov, C. R. Mecanique 333 (2005).***

Keywords: Fluid mechanics; Steady water waves; Bernoulli's equation; Small amplitude; Froude number; Rate of flow

Mots-clés : Mécanique des fluides ; Ondes de surface stationnaires ; Équation de Bernoulli ; Petite amplitude ; Nombre de Froude ; Débit fluide

1. Statement of the problem and formulation of the result

In its simplest form the surface-wave problem concerns two-dimensional motion of an inviscid, incompressible, heavy fluid, say water, bounded above by a free surface and below by a rigid horizontal bottom. The water motion is assumed to be irrotational, and so there exists a velocity potential ϕ, whose gradient gives the velocity field. The

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present Note is concerned with steady waves, in which case the free-surface profile (assumed to be the graph of an unknown $C^{1,\alpha}$-function η) and the velocity field are both stationary with respect to a frame of reference in uniform horizontal motion. In appropriate Cartesian coordinates (x, y), gravity acts in the negative y-direction and the bottom's equation is $y = -1$ (the dimensionless coordinates, unknowns $\phi(x, y)$ and $\eta(x)$, and parameters are used in the paper, whereas the dimensional depth $d > 0$, phase velocity c, and the acceleration due to gravity $g > 0$ are applied for obtaining dimensionless quantities). Furthermore, we assume that the free surface is confined to a certain horizontal strip of finite width and the origin is placed on this surface so that the x-axis coincides with the horizontal surface of the unperturbed flow. More precisely, we suppose that for our choice of the origin, implying that $\eta(0) = 0$, there exist constants m_- and m_+ such that $-1 < m_- < m_+$ and

$$m_- \leqslant \eta(x) \leqslant m_+ \quad \text{for all } x \in \mathbb{R} \tag{1}$$

Mathematically ϕ and η must satisfy the following free-boundary problem in the water domain $D = \{-\infty < x < +\infty,\ -1 < y < \eta(x)\}$:

$$\phi_{xx} + \phi_{yy} = 0, \quad (x, y) \in D \tag{2}$$

$$\phi_y = 0, \quad x \in \mathbb{R},\ y = -1 \tag{3}$$

$$\phi_y = \eta_x \phi_x, \quad x \in \mathbb{R},\ y = \eta(x) \tag{4}$$

$$|\nabla\phi|^2 + 2\mu^{-2}\eta = \kappa^2, \quad x \in \mathbb{R},\ y = \eta(x) \tag{5}$$

Here $\mu = \sqrt{c^2/(gd)} > 0$ and κ are constants known as the Froude number and the Bernoulli constant, respectively. It is clear that problem (1)–(5) always has a *trivial solution* representing the uniform flow; the corresponding η vanishes identically and $\phi = \text{const}$ in $\overline{D}$.

Mathematically rigorous studies of problem (1)–(5) were initiated by Nekrasov in 1921, when he derived an integral equation now called after him. Since then, this equation for waves on deep water and its analogue for water of finite depth were investigated in detail (see, for instance, [1] and [2], respectively). The points of bifurcation from the uniform flow were found and the families of non-trivial solutions that arise at these points were investigated. These solutions describe the so-called Stokes waves which are periodic, have exactly one crest per period, and are symmetric about the vertical line through the crest. Other analytical approaches to Stokes waves can be found in [3–7] (see also references cited therein). Besides, significant numerical evidence about the existence of steady waves that distinguish from Stokes waves had appeared during the past 25 years. Secondary branches of sub-harmonic bifurcations for deep water were first computed in [8]. In [7], a different method was applied for obtaining similar numerical results for finite depth. Taking into account these and other computations demonstrating the existence of various types of permanent, travelling waves (see references cited in [3] and [7]), it is of interest to develop a technique applicable to problem (1)–(5) without additional assumptions on the shape of the free surface like those for Stokes waves.

In this Note, we make a step in this direction and announce a result (it describes regimes for which no waves exist) obtained by means of a new approach that involves averaging of the velocity potential over vertical cross-sections of water having finite depth. An advantage of this averaging procedure lies in the fact that the dimensionless parameter arising in the transformed Bernoulli's equation (see the coefficient at η in the left-hand side of (8) below) has a transparent hydrodynamic meaning. Namely, this parameter is equal to the product of the Froude number μ and the so-called *rate of flow* ω defined as follows:

$$\omega = \int_{-1}^{\eta(x)} \phi_x(x, y)\,\mathrm{d}y$$

It is easy to show that ω is a non-zero constant for non-trivial solutions of (1)–(5). Now we are in a position to formulate our main theorem.

Theorem 1.1. *Let (ϕ, η) be a solution of problem (1)–(5), and let $|\mu\omega| > 1$. If $\eta \in C^{1,\alpha}(\mathbb{R})$, $0 < \alpha < 1$, and $\|\eta\|_{C^{1,\alpha}(\mathbb{R})} < \delta$ for a sufficiently small $\delta > 0$ which depends on $\mu\omega$, then this solution is trivial.*

Our proof of this theorem is based on Theorems 3.1 and 4.1 formulated below. It must be also emphasized that the form (8) of Bernoulli's equation plays an essential role in the proof as well as formula (11) concerning the solution of the linearized problem.

2. The rate of flow and Bernoulli's equation

In this section, we give another expression for ω and apply it for transforming the Bernoulli's equation (5) into a form that is suitable for proving Theorem 1.1. First, we introduce the following function

$$u(x) = \left[1+\eta(x)\right]^{-1} \int_{-1}^{\eta(x)} \phi(x,y)\,\mathrm{d}y$$

which is defined for all $x \in \mathbb{R}$ as follows from (1). This function u will be referred to as the *vertical average* of the velocity potential. The difference $v(x,y) = \phi(x,y) - u(x)$ will be called the *y-dependent component* of the velocity potential. An important property of v is that it satisfies the following orthogonality condition:

$$\int_{-1}^{\eta(x)} v(x,y)\,\mathrm{d}y = 0 \quad \text{for all } x \in \mathbb{R} \tag{6}$$

Proposition 2.1. *Let ϕ and η satisfy equations* (2)–(4), *then* $\omega = [1+\eta(x)]u_x(x) - \eta_x(x)v(x,\eta(x))$.

According to (1), this result can be written as follows:

$$u_x(x) = \frac{\omega + \eta_x(x)v(x,\eta(x))}{1+\eta(x)} \quad \text{for all } x \in \mathbb{R} \tag{7}$$

Substituting this into (5) and using the boundary condition (4), we get

$$\begin{aligned} &\left[(\mu\omega)^{-2} - 1\right]\eta + V_x = \frac{(\kappa/\omega)^2 - 1}{2} - \frac{\eta_x^2}{2} - \eta_x^2\left(V_x + \frac{\eta_x V - \eta}{1+\eta}\right) - \frac{\eta_x V + \eta^2}{1+\eta} - \frac{1+\eta_x^2}{2}\left(V_x + \frac{\eta_x V - \eta}{1+\eta}\right)^2 \\ &\quad y = \eta(x),\ x \in \mathbb{R} \end{aligned} \tag{8}$$

where $V = v/\omega$; note that only linear terms appear in the left-hand side. As we shall see in the next section, V and V_x may be considered as images obtained by applying some nonlinear operators to η. Therefore, Eq. (8) is a nonlinear equation for η involving $(\mu\omega)^2$ as a bifurcation parameter.

3. The nonlinear operator $\eta \mapsto V$

We introduce a weak formulation of the boundary value problem for the function V. For this purpose we consider a set $Z(\overline{D})$ of smooth trial functions $\zeta(x,y)$ that satisfy the following conditions: (i) ζ has a compact support in $\overline{D}$; (ii) the orthogonality condition (6) holds for ζ. By $L^2_{\mathrm{loc}}(D)$, $W^{1,2}_{\mathrm{loc}}(D)$, etc. we denote the spaces of functions belonging to $L^2(K)$, $W^{1,2}(K)$, etc. for every bounded open subset $K \subset D$. Similar definitions are applicable for the strip $S = \{-\infty < x < +\infty,\ 0 \leqslant z \leqslant 1\}$ instead of D.

The standard procedure based on the divergence theorem and using the Laplace equation, the boundary conditions, the orthogonality condition (6), and formula (7) leads to the weak formulation.

Problem $\mathbf{P}_V$. Find $V \in W^{1,2}_{\mathrm{loc}}(D)$ satisfying (6) for a.e. $x \in \mathbb{R}$ and such that the following integral identity

$$\int_D \nabla\zeta \cdot \nabla V\,\mathrm{d}x\,\mathrm{d}y = \int_{-\infty}^{+\infty} \zeta\big(x,\eta(x)\big)\frac{\eta_x(x)[1+\eta_x(x)V(x,\eta(x))]}{1+\eta(x)}\,\mathrm{d}x \tag{9}$$

holds for every $\zeta \in Z(\overline{D})$.

A drawback of problem P_V is that (9) involves the unknown water domain D in the left-hand side. Therefore, it is convenient to introduce a new vertical coordinate

$$z = (y+1)/\left[1+\eta(x)\right] \in [0,1] \quad \text{for all } x \in \mathbb{R}$$

and to put $w(x,z)=V(x,y(x,z))$ in $\bar{S}$. In order to give a weak formulation of the boundary value problem for w we redefine the set of trial functions as follows: (i) ζ has a compact support in the closed strip $\bar{S}$; (ii) the orthogonality condition

$$\int_0^1 \zeta(x,z)\,\mathrm{d}z = 0 \quad \text{holds for all } x\in\mathbb{R} \tag{10}$$

Such a set of trial functions will be denoted $Z(\bar{S})$, and $\zeta(x,y)=h(x)\cos\pi mz$, $m=1,2,\ldots$, where h is an arbitrary cut-off function, delivers an example of function belonging to $Z(\bar{S})$. Now the weak formulation of the problem for w is similar to problem P_V.

Problem $\mathbf{P}_w$. Find $w\in W^{1,2}_{\mathrm{loc}}(S)$ satisfying (10) for a.e. $x\in\mathbb{R}$ and such that the integral identity

$$\int_S \left[\left(\zeta_x - \frac{z\eta_x}{1+\eta}\zeta_z\right)\left(w_x - \frac{z\eta_x}{1+\eta}w_z\right) + \frac{\zeta_z w_z}{(1+\eta)^2}\right](1+\eta)\,\mathrm{d}x\,\mathrm{d}y = \int_{-\infty}^{+\infty} \zeta(x,1)\frac{\eta_x(x)[1+\eta_x(x)w(x,1)]}{1+\eta(x)}\,\mathrm{d}x$$

holds for every $\zeta\in Z(\bar{S})$.

For this problem we claim the following theorem.

Theorem 3.1. *Let $\eta\in C^{1,\alpha}(\mathbb{R})$ be such that $\|\eta\|_{C^{1,\alpha}(\mathbb{R})}<\delta$ for a sufficiently small $\delta>0$. Then problem P_w has a solution $w\in C^{1,\alpha}(\bar{S})$ such that $\|w\|_{C^{1,\alpha}(\bar{S})}\leqslant C\|\eta_x\|_{C^{0,\alpha}(\mathbb{R})}$. This solution is unique in $C^{1,\alpha}(\bar{S})$.*

Our proof of this theorem is based on the results formulated in Section 5. Theorem 3.1 shows, in particular, that V and V_x in the right-hand side of (8) are images obtained by applying some nonlinear operators to η.

4. An asymptotic representation for w

Since the operator $\eta\mapsto w$ is nonlinear, it is important to find the leading term with respect to η in the asymptotic representation of w. For this purpose we introduce the following linear problem.

Problem $\mathbf{P}_\ell$. Find $w^{(\ell)}\in W^{1,2}_{\mathrm{loc}}(S)$ satisfying (10) for a.e. $x\in\mathbb{R}$ and such that the integral identity

$$\int_S \left(\zeta_x w_x^{(\ell)} + \zeta_z w_z^{(\ell)}\right)\mathrm{d}x\,\mathrm{d}y = \int_{-\infty}^{+\infty}\zeta(x,1)\eta_x(x)\,\mathrm{d}x \quad \text{holds for every } \zeta\in Z\big(\bar{S}\big)$$

The orthogonality condition (10) allows us to find an explicit expression for the Fourier transform $\widehat{w^{(\ell)}}(\tau,z)=\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\tau x}w^{(\ell)}(x,z)\,\mathrm{d}x$. Namely,

$$\widehat{w^{(\ell)}}(\tau,z)=\mathrm{i}\hat{\eta}(\tau)\left(\frac{\cosh\tau z}{\sinh\tau}-\frac{1}{\tau}\right)$$

Then the inverse Fourier transform leads to a representation of $w^{(\ell)}$ in the form of a pseudo-differential operator applied to η. Moreover, we have that

$$\widehat{w_x^{(\ell)}}(\tau,1)=-\hat{\eta}(\tau)(\tau\coth\tau-1) \tag{11}$$

which is a consequence of the previous formula and plays an important role in the proof of Theorem 1.1.

The latter proof also involves an estimate of the difference between solutions of the nonlinear and linear problems P_w and P_ℓ, respectively. For estimating $w-w^{(\ell)}$ we note that this difference satisfies the following integral identity:

$$\int_S \left[\zeta_x \left(w - w^{(\ell)}\right)_x + \zeta_z \left(w - w^{(\ell)}\right)_z\right] \mathrm{d}x\,\mathrm{d}z = \int_S R(\zeta, w)\,\mathrm{d}x\,\mathrm{d}z + \int_{-\infty}^{+\infty} \zeta(x,1)\,\frac{\eta_x(x)[\eta_x(x)w(x,1) - \eta(x)]}{1+\eta(x)}\,\mathrm{d}x \tag{12}$$

where ζ is an arbitrary function from $Z(\overline{S})$ and

$$R(\zeta, w) = z\eta_x\,(\zeta_z w_x + \zeta_x w_z) - \eta\zeta_x w_x + \frac{\eta - z^2\eta_x^2}{1+\eta}\zeta_z w_z$$

Hence the right-hand side in (12) is nonlinear with respect to w and η. For problem (12) we claim the following theorem.

Theorem 4.1. *Let $\eta \in C^{1,\alpha}(\mathbb{R})$ be such that $\|\eta\|_{C^{1,\alpha}(\mathbb{R})} < \delta$ for a sufficiently small $\delta > 0$. Then the following estimate holds*: $\|w - w^{(\ell)}\|_{C^{1,\alpha}(\overline{S})} \leqslant C(\|\eta_x\|^2_{C^{0,\alpha}(\mathbb{R})} + \|\eta_x\|_{C^{0,\alpha}(\mathbb{R})}\|\eta\|_{C^{0,\alpha}(\mathbb{R})})$.

The meaning of this theorem is twofold: (i) it shows that $w^{(\ell)}$ is the required leading term in asymptotics of w; (ii) it provides an estimate for the discrepancy. Moreover, Theorem 4.1 combined with formula (11) and fixed point arguments provides the proof of Theorem 1.1.

5. A sketch of proofs of Theorems 3.1 and 4.1

In order to handle problem P_w and the problem for $w - w^{(\ell)}$, involving the integral identity (12), we need a model linear problem.

Problem P_a. Let $p, q, r \in L^2_{\mathrm{loc}}(S)$ and let $H \in L^2_{\mathrm{loc}}(\mathbb{R})$. Find $U \in W^{1,2}_{\mathrm{loc}}(S)$ satisfying (10) for a.e. $x \in \mathbb{R}$ and such that

$$\int_S (\zeta_x U_x + \zeta_z U_z)\,\mathrm{d}x\,\mathrm{d}y = \int_S (\zeta p + \zeta_x q + \zeta_z r)\,\mathrm{d}x\,\mathrm{d}z + \int_{-\infty}^{+\infty} \zeta(x,1) H\,\mathrm{d}x \tag{13}$$

holds for every $\zeta \in Z(\overline{S})$.

Let H satisfy more strong assumption than admitted in problem P_a and belong to $L^p_{\mathrm{loc}}(\mathbb{R})$ for some $p > 2$. This allows us to define the function

$$N(x) = \left\|p(x,\cdot)\right\|_{L^2(0,1)} + \left\|q(x,\cdot)\right\|_{L^2(0,1)} + \left\|r(x,\cdot)\right\|_{L^2(0,1)} + \|H\|_{L^p(x,x+1)}$$

for a.e. $x \in \mathbb{R}$. Then we claim the following lemma.

Lemma 5.1. *Let $\int_{-\infty}^{+\infty} \mathrm{e}^{-\pi|x|} N(x)\,\mathrm{d}x < \infty$, then problem P_a has a solution such that*

$$\left\|U(x,\cdot)\right\|_{L^2(0,1)} \leqslant C \int_{-\infty}^{+\infty} \mathrm{e}^{-\pi|x-\xi|} N(\xi)\,\mathrm{d}\xi \tag{14}$$

Moreover, if

$$\left\|U(x,\cdot)\right\|_{L^2(0,1)} = \mathrm{o}\left(\mathrm{e}^{\pi|x|}\right) \quad \text{as } |x| \to \infty \tag{15}$$

(*this is true when* (14) *holds*), *then this solution is unique.*

Now we suppose that $p, q, r \in C^{0,\alpha}_{\mathrm{loc}}(\overline{S})$, $0 < \alpha < 1$, $H \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R})$, and put

$$N_\alpha(\xi) = \|p\|_{C^{0,\alpha}(\Pi_\xi)} + \|q\|_{C^{0,\alpha}(\Pi_\xi)} + \|r\|_{C^{0,\alpha}(\Pi_\xi)} + \|H\|_{C^{0,\alpha}([\xi,\xi+1])}$$

where $\Pi_\xi = \overline{S} \cap \{\xi \leqslant x \leqslant \xi + 1\}$. Then Lemma 5.1 along with the results on local regularity of solutions to the boundary value problems for the Laplace equation imply the following

Proposition 5.2. *Let* $p, q, r \in C^{0,\alpha}_{\mathrm{loc}}(\bar{S})$ *and let* $H \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R})$, $0 < \alpha < 1$. *If* $\int_{-\infty}^{+\infty} \mathrm{e}^{-\pi|x|} N_\alpha(x)\,\mathrm{d}x < \infty$, *then there exists a weak solution* U *of problem* P_a. *Moreover,* $U \in C^{1,\alpha}_{\mathrm{loc}}(\bar{S})$ *and the following estimate holds*:

$$\|U\|_{C^{1,\alpha}(\Pi_x)} \leqslant C \int_{-\infty}^{+\infty} \mathrm{e}^{-\pi|x-\xi|} N_\alpha(\xi)\,\mathrm{d}\xi$$

Finally, this solution is unique in the class defined by (15).

The immediate consequence of Proposition 5.2 is the following

Corollary 5.3. *Let* $p, q, r \in C^{0,\alpha}(\bar{S})$ *and let* $H \in C^{0,\alpha}(\mathbb{R})$. *Then the weak solution* U *of problem* P_a *belongs to* $C^{1,\alpha}(\bar{S})$ *and the following estimate holds*:

$$\|U\|_{C^{1,\alpha}(\bar{S})} \leqslant C\big[\|p\|_{C^{0,\alpha}(\bar{S})} + \|q\|_{C^{0,\alpha}(\bar{S})} + \|r\|_{C^{0,\alpha}(\bar{S})} + \|H\|_{C^{0,\alpha}(\mathbb{R})}\big]. \tag{16}$$

Since problem P_V has the form (13) with certain p, q, r, and H that depend on η, Theorem 3.1 can be derived from Corollary 5.3 and the Banach's fixed point theorem.

Now we turn to the integral identity (12) for the difference $w - w^{(\ell)}$ between the solutions of the nonlinear and linear problems P_w and P_ℓ, respectively. We note that (12) and (13) coincide when

$$p = 0, \quad q = z\eta_x w_z - \eta w_x, \quad r = z\eta_x w_x + \frac{\eta - z^2\eta_x^2}{1+\eta} w_z, \quad \text{and} \quad H = \frac{\eta_x}{1+\eta}[\eta_x + w(x,1) - \eta]$$

Therefore, Theorem 4.1 follows from Corollary 5.3 and Theorem 3.1.

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