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On the three-dimensional alternative to the Blasius boundary-layer solution

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Abstract

Theoretical investigations have shown that there is a three-dimensional alternative to the Blasius solution for laminar boundary layer flow past a semi-infinite flat plate at zero incident. In this Note we show numerically that there exists a class of three-dimensional solutions confined to almost zero incident conditions. ***To cite this article: A. Ridha, C. R. Mecanique 333 (2005).***

Résumé

Sur l'alternative tri-dimensionnelle à la solution de couche limite de Blasius. Des investigations théoriques ont montré qu'une alternative tridimensionnelle à la solution de Blasius existe pour l'écoulement en couche limite laminaire sur une plaque plane semi-infinie à incidence nulle. Dans cette Note, on montre numériquement qu'il existe une classe des solutions tridimensionnelles confinée au cas correspondant à une plaque plane presque à incidence nulle. ***Pour citer cet article : A. Ridha, C. R. Mecanique 333 (2005).***
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1. Introduction

External flows in practical applications are generally three-dimensional in nature. Yet in view of the difficulties they present we are usually inclined to make approximations enabling us to reduce the problem to a two-dimensional one. One of the best known examples is that of a laminar boundary-layer flow on a flat plate at zero incident which leads to the well celebrated Blasius profile; the plate is finite in its dimensions but often considered to be semi-infinite and so attention is mainly focused on regions sufficiently far from its sides edges. However, as shown first by Ridha [1–3] and later by Dhanak and Duck [4], there exists a three-dimensional alternative (a similarity-type boundary-layer solution) to the two-dimensional Blasius solution. This occurs under the same external flow conditions when a cross-

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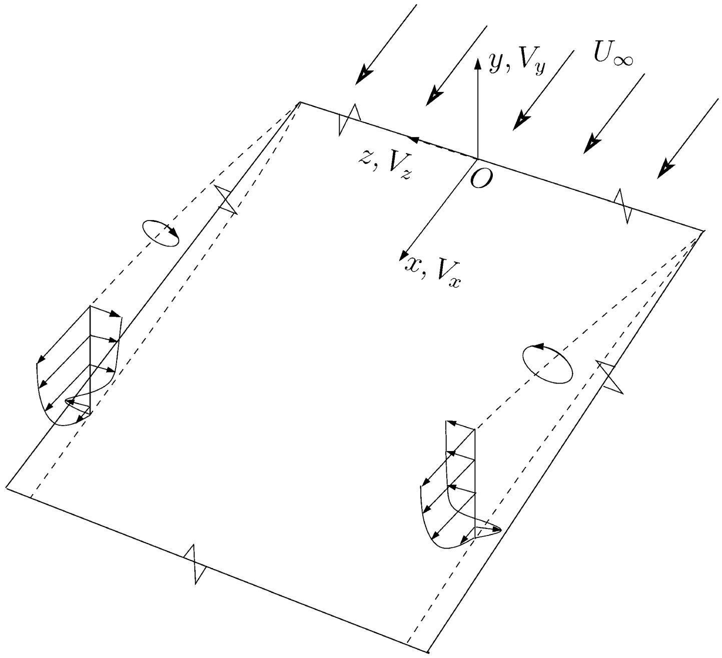

Fig. 1. Flow configuration and coordinates system frame.

Fig. 1. Shématisation de l'écoulement avec le système des coordonnées.

flow component growing linearly in the cross-flow direction is allowed for while the other two components remaining independent of this direction.

A further twist steps into this picture due to the existence of streamwise vortices (see Ridha, [5]) originating at the apex of the side-edge of 'quarter-infinite' flat plates (see Fig. 1). To these two aspects one may add of course the effect of the wind tunnel side-walls (on experiments involving *finite* flat plates). One or more of these factors may be considered 'adequate' to raise the question that (in experiments) secondary flow motion may well persist sufficiently far from the side edges and need, therefore, be taken into account. It is this question that constitute the origin of this Note in which we consider boundary-layer solutions for the viscous flow near the plane of symmetry of a finite flat plate at 'almost' zero-incident. The external flow is assumed of the form $(V_x, V_z) = U_\infty(x/l)^m(1, \lambda z/x)$, (U_∞, l) representing velocity and length scales with m being an arbitrary constant. In this frame work, boundary-layer solutions are sought in the *admissible* space of m when $\lambda(m)$, assumed as a function of m, is taken close to zero. This enables us to find (numerically) three-dimensional solutions confined to narrow bands in the admissible space of m. In some cases such a band is so narrow that in practice one would be tempted to assume it pertaining to the external conditions of a two-dimensional flow past a flat plate at zero incident. That is why it is believed that such results deserve communicating, particularly for experimental investigations of instability questions where the base state solutions could very well be non-parallel under conditions usually believed to lead to parallel flows.

2. Boundary-layer equations

We consider boundary layer flow (specifically in the vicinity of the symmetry plan) past a finite flat plate in the spanwise direction but semi-infinite in the streamwise direction. The flow configuration and the Coordinate system are as shown in Fig. 1, the plate being defined by $y = 0$ and the leading edge by $x = 0$. Then the inviscid flow velocity vector in the vicinity of the symmetry plan ($z = 0$) can be written [3] in the following form

$$(V_x, V_z) = U_\infty(x/l)^m(1, \lambda z/x) \tag{1}$$

Under these conditions we can look for steady laminar incompressible boundary-layer flow in the form

$$\left.\begin{aligned} V_x(x,y,z) &= u(x,y) + z^2u_1(x,y) + \cdots \\ V_y(x,y,z) &= v(x,y) + z^2v_1(x,y) + \cdots \\ V_z(x,y,z) &= zw(x,y) + z^3w_1(x,y) + \cdots \\ p(x,y,z) &= p_0(x,y) + \frac{1}{2}z^2p_2(x,y) + \cdots \end{aligned}\right\} \tag{2}$$

where p is the fluid's pressure. Applying the usual boundary-layer approximations to the continuity and Navier–Stokes equations the following governing equations are found

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + w = 0 \tag{3}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p_0}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{4}$$

$$\frac{\partial p_i}{\partial y} = \mathrm{O}(\nu), \quad i = 0, 2, \ldots \tag{5}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w^2 = -\frac{1}{\rho}p_2 + \nu\frac{\partial^2 w}{\partial y^2} \tag{6}$$

where ρ and ν designate the density and kinematic viscosity of the fluid, assumed constant.

3. Similarity solutions

Eqs. (3)–(6) admit similarity solutions which can be found by introducing two stream-like-functions $\psi(x,y)$ and $\chi(x,y)$ together with the following definitions

$$\left.\begin{aligned} & u = \frac{\partial\psi}{\partial y}, \quad w = \frac{\partial\chi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x} - \chi \\ & \psi = \sqrt{\frac{2l\nu U_\infty}{1+m}}\left(\frac{x}{l}\right)^{(m+1)/2} f(\eta), \quad \chi = \sqrt{\frac{2\nu U_\infty/l}{m+1}}\left(\frac{x}{l}\right)^{(m-1)/2} h(\eta) \\ & \eta = \sqrt{\frac{1+m}{2l\nu}U_\infty}\left(\frac{x}{l}\right)^{(m-1)/2} y, \quad \beta = \frac{2m}{m+1} \end{aligned}\right\} \tag{7}$$

which after substituting into the boundary-layer equations and using the outer flow conditions yield

$$f''' + \big[(2-\beta)h + f\big]f'' + \beta[1 - f'^2] = 0 \tag{8}$$

$$h''' + \big[(2-\beta)h + f\big]h'' + \big[2(1-\beta)f' - (2-\beta)h'\big]h' = \lambda\big[2(1-\beta) - \lambda(2-\beta)\big] \tag{9}$$

Here a prime denotes differentiation with respect to η. Note that the pressure term has been determined in the usual manner upon using the outer flow boundary conditions. These equations are to be solved subject to

$$\left.\begin{aligned} & f(\eta=0) = h(\eta=0) = f'(\eta=0) = h'(\eta=0) = 0 \\ & f'(\eta=\infty) = 1, \quad h'(\eta=\infty) = \lambda \end{aligned}\right\} \tag{10}$$

In order to look for solutions in the vicinity of zero streamwise pressure gradient we look for solutions of (8)–(10) for λ-forms satisfying a Taylor-series

$$\lambda(\beta) = \lambda(0) + \beta\frac{\mathrm{d}\lambda}{\mathrm{d}\beta}(0) + \frac{1}{2}\beta^2\frac{\mathrm{d}^2\lambda}{\mathrm{d}\beta}^2(0) + \cdots \tag{11}$$

This enables us to recover the outer flow conditions of the Blasius solution upon setting $\lambda(\beta=0)=0$. We have found that for a wide range of $\mathrm{d}\lambda/\mathrm{d}\beta(\beta=0)$, $\mathrm{d}^2\lambda/\mathrm{d}\beta^2(\beta=0)$ and higher order derivatives (for example $\mathrm{d}\lambda/\mathrm{d}\beta(\beta=0) = \pm 0.001, \pm 0.01$, $\mathrm{d}^2\lambda/\mathrm{d}\beta^2(0) = 1, 10, 1000$) the β admissible space turns out to lie within a very narrow band: $-1 \ll \beta \ll 1$. Typical examples of wall shear stress evolution with the streamwise pressure gradient parameter β

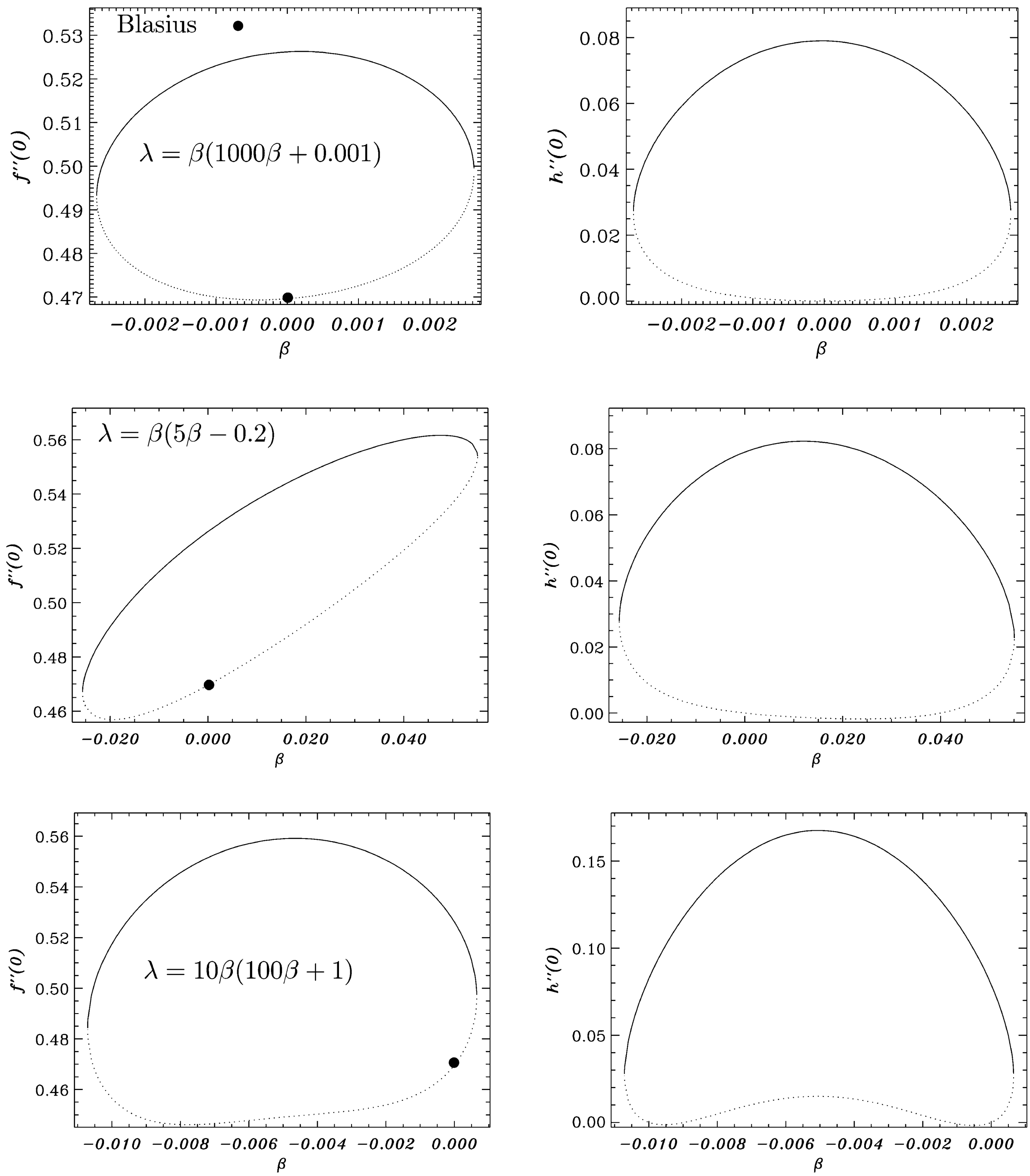

Fig. 2. Wall shear stress variation with the pressure gradient parameter β. Solid line solutions are associated with the zero pressure gradient ($\beta = 0$) three-dimensional alternative state and the dotted line ones are associated with Blasius solution obtained at $\beta = 0$.

Fig. 2. Evolution de contraintes pariétales suivant le paramètre de gradient de pression β. Les solutions en lignes continues sont associées à l'alternative tri-dimensionnelle obtenue lorsque $\beta = 0$ et celles en lignes pointillées sont associées à la solution de Blasius (obtenue quand $\beta = 0$).

are given in Fig. 2. Examples of velocity profiles are also depicted in Fig. 3. Observe how such a state of solutions could lead to three-dimensional alternative state with streamwise velocity profile close to the Blasius solution. These solutions are generally associated to cross flows having jet like profiles close to the wall as illustrated, oriented either towards or away from the plan of symmetry.

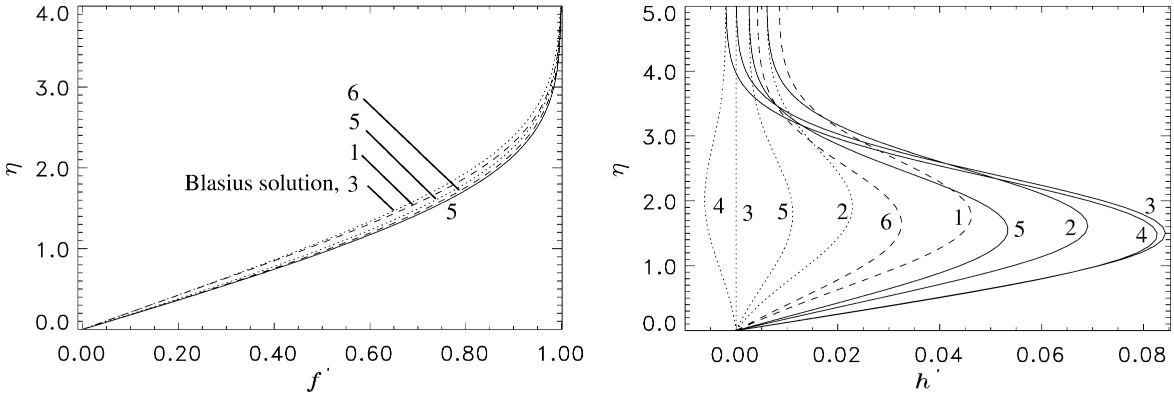

Fig. 3. Primary velocity f' profiles and secondary flow profiles h' when $\lambda = \beta(5\beta - 0.2)$. Dotted lines designate solutions associated to the Blasius solution and solid lines to the three-dimensional alternative solutions. Solutions at extreme values in β are given by the dashed lines. Curves 1–6 correspond to $\beta = -0.025625, -0.02, 0.0, 0.02, 0.05, 0.055075$.

Fig. 3. Profils de vitesse de l'écoulement principal f' et de l'écoulement secondaire h' lorsque $\lambda = \beta(5\beta - 0{,}2)$; les lignes continues désignent les solutions associées à l'alternative tridimensionnelle, celles en pointillées indiquent les solutions associées à la solution de Blasius et celles en traits discontinus désignent les solutions aux valeurs extrêmes de β. Les courbes 1–6 correspondent à $\beta = -0{,}025625; -0{,}02; 0{,}0; 0{,}02; 0{,}05; 0{,}055075$.

What these results tell us is that it is plausible to have three-dimensional flows arising under conditions seemingly leading 'uniquely' to two dimensional flows; measurements pertaining to the three-dimensional alternative flow state could (perhaps wrongly by inadvertence) be assumed to belong to a Blasius-profile or a Falkner–Skan base flow state. Note also that the smallness in the variation of the pressure gradient in these solutions may well be attributed in practice to experimental errors. Hence, in a study of flow stability this would amount to 'confusing' a non-parallel base flow state with a parallel flow base state. Now given that such three-dimensional solutions are associated to eigen solutions algebraically growing in the z direction [5] as well as in the streamwise direction [6,7], questions would naturally be raised with regards to the correctness of conclusions in practice and would require further care in experiments to discern as to which of the above possible base state (parallel or non-parallel) flows arise in practice.

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