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First order models and closure of the mass conservation equation in the mathematical theory of vehicular traffic flow

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Abstract

This article deals with a review and critical analysis of first order hydrodynamic models of vehicular traffic flow obtained by the closure of the mass conservation equation. The closure is obtained by phenomenological models suitable to relate the local mean velocity to local density profiles. Various models are described and critically analyzed in the deterministic and stochastic case. The analysis is developed in view of applications of the models to traffic flow simulations for networks of roads. Some research perspectives are derived from the above analysis and proposed in the last part of the paper. ***To cite this article: N. Bellomo, V. Coscia, C. R. Mecanique 333 (2005).***

Résumé

Modèles du premier ordre et fermeture de l'équation de conservation de masse dans la théorie mathématique du trafic routier. Ce article propose une revue et une analyse critique des modèles hydrodynamiques du trafic routier obtenus par fermeture de l'équation de conservation de la masse. La fermeture est obtenue à partir de modèles phénoménologiques reliant les profils des densités locales à la vitesse moyenne locale. Différents modèles sont décrits et discutés aussi bien dans le cas déterministe que stochastique. L'analyse est développée en vue d'applications aux simulations des modèles de trafic pour les réseaux routiers. Des perspectives de recherche, basées sur notre analyse, sont proposées dans la dernière partie de ce travail. ***Pour citer cet article : N. Bellomo, V. Coscia, C. R. Mecanique 333 (2005).***

Keywords: Continuum mechanics; Traffic flow models; Mass conservation; Continuum models; Nonlinear sciences

Mots-clés : Milieux continus ; Modèles de trafic routier ; Équation de conservation de la masse ; Modèles continus ; Nonlinéarité

1. Introduction

Mathematical modelling of vehicular traffic flow can be developed by different approaches corresponding to different scales of observation and representation of the system. Specifically, microscopic modelling corresponds to the

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dynamics of each single vehicle under the action of the surrounding vehicles; a statistical description, in a framework close to the one of the kinetic theory of gases, consists in the derivation of an evolution equation for the probability distribution function on the position and velocity of the vehicles along the road; while the macroscopic description, analogous to that of fluid dynamics, refers to the derivation of evolution equations for the mass density, linear momentum and energy, which are regarded as macroscopic observable quantities of the flow of vehicles assumed to be continuous. Conservation equations are closed by phenomenological models related to the material behavior of the mechanical system regarded as a continuum.

Despite the great complexity of the system which is well documented in various papers by Kerner and co-workers [1–3], the need for designing models valid to reproduce at least a part of the variety of phenomena related to traffic flow is motivated by the fast growing number of vehicles on networks of roads, either highways or urban streets, and the related economical and social implications, e.g. pollution and energy control, prevention of car crashes, etc. The review papers [4–7] provide a detailed account of the various research contribution available in the literature corresponding to all above modelling scales.

Modelling traffic flow phenomena by macroscopic hydrodynamic equations has to be regarded as an approximation of physical reality, however, useful for the applications. Certainly this type of representation can be criticized considering that the mean distances among vehicles are large enough to be in contrast with the paradigms of continuum mechanics [8]. On the other hand, at least for some particular applications, relatively simple models can be effectively useful, also considering that all models are somehow characterized by parameters to be identified by suitable comparisons between theory and experiments. The above mentioned identification is necessary toward an effective use of the model, while it appears to be possible, as documented in [9], only in the case of very simple ones.

Indeed, this is the case of models obtained by the equation of conservation of mass closed by a phenomenological relation suitable to link the local mean velocity to the local density profiles. It is a problem of closure of the mass conservation equation proposed as an alternative to the use of both mass and momentum equations. This alternative skips over the technical difficulty of closing momentum equation by phenomenological models which describe the acceleration applied to the vehicles in the elementary volume by all surrounding vehicles.

The closure of mass conservation equation leads to first order models, which may provide a relatively less accurate description of physical reality with respect to second order models. On the other hand, this relatively simpler class of models appears to be useful in the analysis of complex traffic flow conditions, such as those related to variable road conditions [10], or to network of roads [11–13].

This article deals with a review and critical analysis of the various models available in the literature concerning first order models and indicates some research perspective ideas. The main difficulty consists in attempting to take into account the very particular feature of the system: the vehicle is not simply a mechanical system, but it has to be thought of as a driver-vehicle system, where the ability of the driver to organize the dynamics cannot be neglected. The contents are organized through four more sections which follow this introduction. Section 2 describes the general mathematical setting, Section 3 deals with the mass closure problem and with the derivation of evolution equations for the density, Section 4 develops the same analysis for the flux, and finally Section 5 develops a critical analysis with special attention to research perspectives.

2. Mathematical setting

This section provides the mathematical setting related to the hydrodynamic modelling of a one-lane flow of vehicles on a road. The conservation equations will be given, following [7], in terms of dimensionless variables:

- $t = t_r/T$ is the dimensionless time variable referred to the characteristic time T, where t_r is the real time;
- $x = x_r/\ell$ is the dimensionless space variable referred to the characteristic length of the road ℓ, where x_r is the real dimensional space;
- $u = n/n_M$ is the dimensionless density referred to the maximum density n_M of vehicles corresponding to bump-to-bump traffic jam;
- $v = v_R/v_M$ is the dimensionless velocity referred to the maximum mean velocity v_M, where v_R is the real velocity of the single vehicle;
- q is the dimensionless linear mean flux referred to the maximum admissible mean flux $q_M = n_M v_M$.

In what follows the characteristic time T will be assumed according to the condition $v_M T = \ell$, that means that T is the time necessary to cover the whole road length at the maximum mean velocity.

Macroscopic models are obtained by conservation equations corresponding to mass and linear momentum, referring, for each lane, to the variables $u = u(t,x) \in [0,1]$, and $v = v(t,x) \in [0,1]$. Still referring to [7], the mathematical framework is the one concerning conservation of mass, and linear momentum

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(uv) = 0 \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = f[u,v] \end{cases} \tag{1}$$

where f defines the acceleration referred to the vehicles in the elementary volume. The word acceleration is used, when dealing with traffic flow models, to avoid the use of the term force for a system where the mass cannot be properly defined.

The analysis developed in what follows takes advantage of the experimental information delivered by the analysis of steady uniform flow conditions. The phenomenological behavior of the system shows that the mean velocity of the car decays with increasing density from the maximum value $v = 1$ when $u \cong 0$ to $v = 0$ when $u = 1$. Experiments are visualized in the so-called velocity diagram where v is related to u, in steady uniform flow, or in the fundamental diagram, where the flux q is related again to u. The following fit of experimental data:

$$v = v_e(u) = \exp\left\{-\alpha\frac{u}{1-u}\right\}, \quad \alpha > 0; \qquad q_e(u) = u v_e(u) = u\exp\left\{-\alpha\frac{u}{1-u}\right\} \tag{2}$$

was proposed in [9], where α is a parameter related to the specific features of the road and environmental conditions. Comparisons with experimental data show the following range of variability of the above parameter: $\alpha \in [1,3]$, where relatively larger values of α denote strong decay of the mean velocity with local density and hence relatively less favorable road–weather conditions. The above results can be used, as we shall see, to close the mass conservation equation.

Other models have been proposed in the literature to simulate experimental data by simple mathematical equations. A well known model, reported in [4], is the following:

$$q = u(1-u^{1+c})^{1+d}, \quad c, d > 0 \tag{3}$$

which needs, however, two constants. Model (2) only needs one parameter which allows, as shown in [9], a non ambiguous identification. Occasionally, simply for practical purposes, the following model is adopted:

$$v_e = 1 - u, \qquad q_e = u(1-u) \tag{4}$$

which has the advantage of generating, as we shall see, relatively simpler first order hydrodynamic models.

It is worth mentioning that paper [9] also introduces an additional parameter: the critical density u_c such that for $u \leqslant u_c$ the flow is free: $v = 1$, while for $u > u_c$ the decay of the velocity with the density is that shown by Eq. (2).

3. On the closure of mass conservation

The problem of the closure of the mass conservation equation consists in looking for a phenomenological model to link the mean velocity to the density profiles by a suitable analytic or functional equation. This results in avoiding the use of the momentum equation and the difficult task of identifying, at a practical level, the acceleration term.

Various models are available in the literature, based on different interpretation of the phenomenology of the system. A survey and a critical analysis of the above model is proposed in this section, while some developments are proposed in the next one. Technical calculations are developed not only for the variable u, but also for the flux q. Indeed, the statement of mathematical problems is relatively more efficient if the flux is involved since technical measurements of flux are relatively more accurate than those of the velocity.

3.1. Closure by velocity diagram

The closure of the mass conservation equation simply means substituting the expression of v delivered by the velocity diagram into the mass conservation equation. Using the model given by Eq. (2) yields:

$$\frac{\partial u}{\partial t} + f(u)\frac{\partial u}{\partial x} = 0, \quad \text{where } f(u) = \frac{1-(2+\alpha)u+u^2}{1-u}\exp\left\{-\alpha\frac{u}{1-u}\right\} \tag{5}$$

If the relatively simpler relation (4) is adopted, then the following model is obtained:

$$\frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = 0 \tag{6}$$

The above hyperbolic model shows unrealistic shock wave phenomena which are not experimentally observed. This inconsistency is due to the fact that conditions which correspond to steady uniform flow conditions are instantaneously imposed in unsteady conditions. Indeed, no driver is effectively, and luckily, able to adapt the vehicle to the steady flow conditions.

3.2. Linear and nonlinear diffusion models

Various authors suggested a closure based on the addition of a small diffusion term related to the assumption that the mean velocity is given by a small diffusion velocity related to local gradients which is added to the one given by the velocity diagram.

In general:

$$q = q_e(u) + q_d(u, u_x) = u\exp\left\{-\alpha\frac{u}{1-u}\right\} - \varepsilon k(u)\frac{\partial u}{\partial x} \tag{7}$$

A linear closure was proposed by Lightill, Payne and Witham [14,15], with $k(u) = 1$, which generates the following model:

$$\frac{\partial u}{\partial t} + f(u)\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2} \tag{8}$$

where $f = f(u)$ was defined in Eq. (5).

Later De Angelis [16] remarked that the assumption of linear diffusion may be reasonable in the case of small density gradients, while when u spans in the whole range $[0,1]$ the diffusion phenomenon depends on u and should tend to zero when u goes to 0 and to 1. A natural model is the following:

$$k = k(u) = u^{1+a}(1-u)^{1+b}, \quad a, b > 0 \tag{9}$$

while in general the model is as follows:

$$\frac{\partial u}{\partial t} + f(u)\frac{\partial u}{\partial x} = \varepsilon k_u(u)\left(\frac{\partial u}{\partial x}\right)^2 + \varepsilon k(u)\frac{\partial^2 u}{\partial x^2} \tag{10}$$

where k_u denotes the partial derivative of k with respect to u. For instance, if the phenomenological model (9) is taken with $a = 1$ and $b = 0$, then k_u takes the expression $k_u = u(2-3u)$.

Of course, different, hopefully more realistic, diffusion models can be proposed on the basis of experimental data which are not easy to recover. Nonlinear diffusion models provide a partial answer to the criticisms raised by Daganzo [17], but not yet to those in the paper by Aw and Rascle [18], where consistency of hyperbolic models is correctly claimed in contrast with the parabolic structure of the above models. Indeed, perturbations in traffic flow have a finite speed. Possibly a deeper analysis of diffusion coefficients related to porous media behavior may improve the above mathematical description.

3.3. Apparent density models

An additional phenomenological model to be taken into account was proposed in the paper by De Angelis [16], technically modified by Bonzani [19], which introduces the interesting concept of apparent local density related to the fact that the driver does not measure exactly the local density, but simply feels it. Specifically, the driver feels a density u^* which is larger than the real one if the local density gradient is positive (trend to jam conditions), while it is smaller than the real one if the gradient is negative (trend to vacuum). In addition, the above multiplicative effect increases with decreasing density. A conceivable expression of the apparent density is the following:

$$u^* = u^*(u, u_x) = u\left[1 + \eta(1-u)\frac{\partial u}{\partial x}\right] \tag{11}$$

where η is a positive parameter. This means that the equilibrium velocity which is felt by the driver is obtained substituting in (2) u with u^*: $v_e = v_e(u^*(u, u_x))$ and $q_e = q_e(u^*(u, u_x))$. Using the phenomenological model (4) yields

$$\frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = \eta u^2(1-u)\frac{\partial^2 u}{\partial x^2} + \eta u(2-3u)\left(\frac{\partial u}{\partial x}\right)^2 \tag{12}$$

which shows a remarkable analogy with the nonlinear diffusion model. The apparent density (11) generates a nonlinear diffusion model corresponding to $k(u) = u^2(1-u)$.

On the other hand, the use of models (2) and (11) generates the following evolution equation:

$$\frac{\partial u}{\partial t} = g(u, u_x)\frac{\partial u}{\partial x} + p(u, u_x)\frac{\partial^2 u}{\partial x^2} \tag{13}$$

where

$$g(u, u_x) = \frac{\alpha u[1 + \eta(1-2u)u_x][1 + 2u^*(u, u_x)] - [1 - u^*(u, u_x)]^2}{[1 - u^*(u, u_x)]^2} v_e^*\big(u^*(u, u_x)\big) \tag{14}$$

and

$$p(u, u_x) = \alpha \frac{\eta u(1-u)}{[1 - u^*(u, u_x)]^2} q_e^*\big(u^*(u, u_x)\big) \tag{15}$$

where u^* is given by Eq. (11) and v_e^*, q_e^* by Eq. (2), where u is replaced by u^*. Still the structure of second order terms is the one of parabolic equations.

3.4. Delay models

All models described in the previous subsections are based on the assumption that the driver instantaneously adapts the velocity of his vehicle to a certain velocity obtained through suitable phenomenological models such as those reported above. These models depend on the local density and density gradients. On the other hand it is reasonable to assume that the driver's reaction time is finite, so that the velocity at which car travels is appropriate to the density earlier in time, as:

$$v(\bar{u}) = v\big(u(x, t-\tau)\big) \tag{16}$$

where τ is a parameter small with respect to one, that corresponds to a relatively large time, thus introducing a retarded adaptation of the driver to the actual traffic conditions.

This type of modelling was proposed in [20], where a qualitative analysis on the stability of solutions was proposed, too. Indeed, some interesting features of the latter model can be exploited using relation (16) to close the continuity equation in (1) in case of small retardation time. We find:

$$\frac{\partial u}{\partial t} + q'(u)\frac{\partial u}{\partial x} = \tau\frac{\partial}{\partial x}\left(u v'(u)\frac{\partial u}{\partial t}\right) \tag{17}$$

where:

$$q'(u) = v(u) + u v'(u) \tag{18}$$

and where u without a bar means the density evaluated in x at time t. In the case of nearly uniform traffic flow, the density u can be considered 'almost' constant: $u(x, t) = U + w(x, t)$. Substituting back into (18) and retaining terms up to the first order in w, we have that the density 'perturbations' obey the following linearized equation:

$$\frac{\partial w}{\partial t} + q'(U)\frac{\partial w}{\partial x} = \tau U v'(U)\frac{\partial^2 w}{\partial x\,\partial t} \tag{19}$$

The above equation admits solutions in the form of normal modes $w(x,t) = W e^{ikx+\omega t}$, with the growth-rate parameter ω depending on the perturbation wavelength as:

$$\omega = \frac{-ikq'(U)}{1 - ikUv'(U)\tau} \tag{20}$$

For small values of the retardation parameter τ the function $\omega(\tau)$ can be approximated as:

$$\omega = -ikq'(U) + k^2 U q'(U) v'(U)\tau, \quad \text{for } \tau \ll 1 \tag{21}$$

Finally, the absolute value of perturbations goes as:

$$\left|w(x,t)\right| = \left|W \exp\left\{ik\left(x - q'(U)t\right)\right\} e^{k^2 U q'(U) v'(U)\tau t}\right| \leqslant |W| e^{k^2 U q'(U) v'(U)\tau t} \tag{22}$$

While the quantities k^2 and U are certainly positive, we must consider the sign of the product $q'(U)v'(U)$. The term $v'(U)$ is always less than zero. On the other hand, it is easy to verify, assuming the velocity diagram (2), that the quantity $q'(U) = v(U) + Uv'(U)$ is negative when $U > U_{\max}$, where:

$$U_{\max} = \frac{1}{2}\left(2 + \alpha - \sqrt{\alpha(4+\alpha)}\right) \tag{23}$$

is the value of density at which the flux attains its maximum. This means that, in case of heavy traffic, the exponential term in Eq. (23) grows for large t. As a consequence, uniform flows are (linearly, and then also nonlinearly) exponentially unstable, that is, small density perturbations increase in time, possibly leading to one of the observed instabilities of congested traffic flow such as ghost queues and 'stop and go' phenomena. These phenomena, which are documented and analyzed in [1,2], are described by this model in terms of instability.

4. Evolution equations for the flux

The various models presented in the preceding section refer to the local density of vehicles as dependent variable. The related statement of initial-boundary value problems needs the measurements of the densities at $x = 0$ and $x = 1$ to implement the boundary conditions. On the other hand it is well known [9] that the measurement of the flux is relatively more precise than the density. Therefore it is convenient to develop simulations by solution of the mathematical problems deriving equations for the flux following some technical indications already given in [21].

Consider the various models reported in Sections 3.1–3.3 based on closures derived by local flow properties. For all of them the flux can be regarded as a function of the local density and density gradients: $q = q(u, u_x)$. Therefore deriving q with respect to time provides the following formal expression:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial q}{\partial x} = 0 \\ \dfrac{\partial q}{\partial t} = -A(u,q)\dfrac{\partial q}{\partial x} + B(u,q)\dfrac{\partial^2 q}{\partial x^2} \end{cases} \tag{24}$$

where

$$A(u,q) = \frac{\partial q}{\partial u}, \qquad B(u,q) = \frac{\partial q}{\partial u_x} \tag{25}$$

and where the argument u_x does not appear in A and B as it has been technically expressed in terms of u and q through the phenomenological models (2), (4), (6), (12). The general structure (25) can be particularized for each of the above models. In particular, model (2) generates the following equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial q}{\partial x} \\ \dfrac{\partial q}{\partial t} = -f(u)\dfrac{\partial q}{\partial x} \end{cases} \tag{26}$$

The linear diffusion model can be rewritten as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial q}{\partial x} \\ \dfrac{\partial q}{\partial t} = -f(u)\dfrac{\partial q}{\partial x} + \varepsilon\dfrac{\partial^2 q}{\partial x^2} \end{cases} \tag{27}$$

The nonlinear diffusion model can be rewritten as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial q}{\partial x} \\ \dfrac{\partial q}{\partial t} = -f(u)\dfrac{\partial q}{\partial x} + \varepsilon\dfrac{k_u(u)}{k(u)}\left[\exp\left\{-\alpha\dfrac{u}{1-u}\right\} - q\right]\left(\dfrac{\partial q}{\partial x}\right)^2 + \varepsilon k(u)\dfrac{\partial^2 q}{\partial x^2} \end{cases} \tag{28}$$

De Angelis' model [16] can be regarded, as above, as a particular case of the diffusion model. If model (4) is adopted, then the apparent-density model can be written in term of flux as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial q}{\partial x} \\ \dfrac{\partial q}{\partial t} = \left[(1-2u) + \eta u(2-3u)\dfrac{\partial q}{\partial x}\right]\dfrac{\partial q}{\partial x} + \eta u(1-u)\dfrac{\partial^2 q}{\partial x^2} \end{cases} \tag{29}$$

Again the above model can be regarded as a nonlinear diffusion model corresponding to an appropriate choice of the diffusion coefficient, where $k(u) = u^2(1-u)$.

The above models are based on the assumption that the driver adapts instantaneously the mean velocity to the speed v_e as it is defined by the various models we have seen in Section 2. On the other hand, the driver can only attempt to reach the above velocity. Therefore, following the reasoning suggested in [22], the model below is obtained:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial q}{\partial x} \\ \dfrac{\partial q}{\partial t} = \mu\big[q_e(u) - q\big] \end{cases} \tag{30}$$

where μ is a positive constant. An interesting problem consists in analyzing the qualitative behavior of the solution to the above model in comparison with those of the delay model.

The model introduced in Section 3.4 relates the flux to the density earlier in time, so that the correspondent closure relation is expressed as follows:

$$q = \varphi(\bar{u}) = uv\big(u(x, t-\tau)\big) = u\exp\left\{-\alpha\frac{\bar{u}}{1-\bar{u}}\right\} \tag{31}$$

where $\bar{u} = u(x, t-\tau)$. When the retardation parameter is small, from Eq. (31) we find:

$$\varphi(\bar{u}) = uv(u) - \tau uv'(u)\frac{\partial u}{\partial t} = u\left(1 + \tau\alpha\frac{1}{(1-u)^2}\frac{\partial u}{\partial t}\right)\exp\left\{-\alpha\frac{u}{1-u}\right\}, \quad u = (\bar{u})_{\tau=0} \tag{32}$$

as $\tau \to 0$.

Computing the time derivative of $\varphi(\bar{u})$ for small τ, we finally get the following model:

$$\begin{cases} \dfrac{\partial q}{\partial x} = -\dfrac{\partial u}{\partial t} \\ \dfrac{\partial q}{\partial t} = \bar{\Phi}(u)\dfrac{\partial q}{\partial x} + \tau\bar{\Psi}(u)\dfrac{\partial^2 q}{\partial t\,\partial x} + \tau\bar{\Gamma}(u)\left(\dfrac{\partial q}{\partial x}\right)^2 \end{cases} \tag{33}$$

where

$$\bar{\Phi}(u) = -\exp\left\{-\alpha\frac{u}{1-u}\right\}, \qquad \bar{\Psi}(u) = -\frac{\alpha u}{(1-u)^2}\exp\left\{-\alpha\frac{u}{1-u}\right\}$$

and

$$\bar{\Gamma}(u) = -\frac{\alpha}{(1-u)^3}\left(2 + \frac{\alpha}{1-u}\right)\exp\left\{-\alpha\frac{u}{1-u}\right\}.$$

5. Research perspectives

First order models have the great advantage of being relatively simple, and then applicable to implement complex traffic situations such as urban environments and road networks [24]. In addition, they rely on a firm basis, that is, the equation of mass conservation suitably 'closed' with a velocity–density (or flux–density) relation that plays the role of a constitutive law of classical continuum mechanics. Such a closure relation, on its own, is derived through a suitable analysis of experimental traffic data, leading to different models, as analyzed in the previous section. On the other hand, one of the most interesting features of the traffic flow is the arising of collective phenomena such as stop-and-go waves and phantom queues. It appears that a necessary condition for a macroscopic model to allow for such situations is a kind of instability of the flow at high density, as described in Section 3.4. A key point in the search for a 'good' first order model should be the possibility to describe at least qualitatively all known features of traffic flow and in the meanwhile to contain only few parameters, intuitive and easy to measure, to be theoretically consistent and, last but not least, to allow for a fast numerical simulation. In this section we mention some possible pathways to improve the above mentioned closures of the mass conservation equation in order to possibly include more traffic aspects without loosing analytic robustness. We limit ourselves to simply sketch the ideas, inasmuch at present time they need deeper mathematical analysis.

A possible model is obtained by adding a space dislocation in the velocity–density closure relation, in a way similar to that exploited in Section 3.4, where a time dislocation (delay) was used. Specifically, we assume the velocity at point x and at time t is related to the density at an earlier time $t-\tau$ and in a spatial neighborhood $x-|\delta|$:

$$v(\bar{u}) = v\big(u\big(x-|\delta|, t-\tau\big)\big) \tag{34}$$

where, in principle, the parameters τ and $|\delta|$ can be assumed to be independent each other. Working as in Section 3.4, we suppose the dislocation parameters to be 'small', in such a way that the mass conservation equation together with Eq. (34) can be expressed:

$$\frac{\partial u}{\partial t} + q'(u)\frac{\partial u}{\partial x} = |\delta| v'(u)\left(\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right) + \tau v'(u)\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} + u\frac{\partial^2 u}{\partial x\,\partial t}\right) \tag{35}$$

as $\tau, |\delta| \to 0$, where $q'(u)$ is the same as in Eq. (18). It is worth to check the effect of the space dislocation with respect to long time behavior of perturbations $w(x,t)$ to a uniform traffic flow U. Limiting ourselves to small perturbations' norms, from Eq. (35) we get the linear evolution equation for the perturbations:

$$\frac{\partial w}{\partial t} + q'(U)\frac{\partial w}{\partial x} = U v'(U)\left(|\delta|\frac{\partial^2 w}{\partial x^2} + \tau\frac{\partial^2 w}{\partial x\,\partial t}\right) \tag{36}$$

A Fourier transformation of Eq. (36) shows that it has solutions in the form of modes $w(x,t) = W\mathrm{e}^{\mathrm{i}kx+\omega t}$ provided:

$$\omega = -\mathrm{i}kq'(U) + q'(U)Uv'(U)k^2\tau - Uv'(U)k^2|\delta| \tag{37}$$

as $\tau, |\delta| \to 0$. It clearly appears that the presence of a space dislocation has *always* a destabilizing effect, irrespectively to the density value. The combined effect of the space and the time dislocations result in a decrease of the critical density for linear (and a fortiori nonlinear) instability of steady uniform traffic flows. The values of the additional parameters $|\delta|$ and τ can be found matching the above computations with the experimental data, in particular with the threshold for the onset of congested flow. To summarize, some kind of dislocation appears to be a keynote for a first order model to allow for the instability phenomena characteristic of high density flow.

All different types of closure which have been reported in this paper are based on a deterministic approach at the macroscopic scale. On the other hand, while the equation of mass conservation is deterministic, fluctuations of the velocity at the microscopic scale are observed [1,2]. A conceivable way to model the above phenomena consists in attempting a stochastic closure by considering the velocity $V = V(\omega)$ in a suitable probability space (Ω, ω, σ).

Suppose that experiments in uniform equilibrium conditions are able not only to identify the mean velocity v_e, as given by Eq. (2), but also a suitable probability density $P_e = P_e(V; u)$, over V, parameterized with respect to the density u, such that

$$\forall u \in [0,1]: \quad \int P_e(V;u)\,\mathrm{d}V = 1, \quad \int V P_e(V;u)\,\mathrm{d}V = v_e(u) \tag{38}$$

Then, the closure is stated by the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(uV) = 0 \\ P_e = P_e(V;u) \end{cases} \tag{39}$$

Dealing with the above equations means sampling the solutions to obtain suitable moments of the density along time and space. Of course, technical developments are possible. For instance P can be parameterized, as in Section 3, with respect to u^* instead of u. Or, following the model (30) it can take into account the attempt of the driver to reach the equilibrium distribution:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(uV) = 0 \\ \dfrac{\partial P}{\partial t} = \mu_c(P_e - P) \end{cases} \tag{40}$$

Suitable experiments may possibly lead to a reliable choice of the above density functions, so that such a stochastic closure will significantly enrich the descriptive properties of the resulting model, especially in congested flow where highly correlated collective phenomena are particularly relevant [23]. On the other hand, a direct modelling of an evolution equation for P, for instance following [24], may lead to a relatively more accurate description of the complex system we are dealing with. The idea of averaging dynamical systems, i.e. models at the microscopic scale, was introduced by Darbha and Rajagopal [25,26]. Recent developments of this idea are proposed in the already cited paper [8].

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