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Fluid–solid interactions: modeling, simulation, bio-mechanical applications

Adaptive finite elements for the steady free fall of a body in a Newtonian fluid

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Abstract

The numerical simulation of the free fall of a solid body in a viscous fluid is a challenging task since it requires computational domains which usually need to be several order of magnitude larger than the solid body in order to avoid the influence of artificial boundaries. Toward an optimal mesh design in that context, we propose a method based on the *weighted* a posteriori error estimation of the finite element approximation of the fluid/body motion. A key ingredient for the proposed approach is the reformulation of the conservation and kinetic equations in the solid frame as well as the implicit treatment of the hydrodynamic forces and torque acting on the solid body in the weak formulation. Information given by the solution of an adequate dual problem allows one to control the discretization error of given functionals. The analysis encompasses the control of the free fall velocity, the orientation of the body, the hydrodynamic force and torque on the body. Numerical experiments for the two dimensional sedimentation problem validate the method. ***To cite this article: V. Heuveline, C. R. Mecanique 333 (2005).***

Résumé

Une méthode d'éléments finis adaptative pour la simulation de la sédimentation d'un corps solide dans un fluide Newtonien. La simulation numérique de la sédimentation d'un corps solide dans un fluide visqueux est un problème difficile car il exige, entre autres, l'emploi de domaines de calcul de plusieurs ordres de grandeur plus grands que le corps solide, ceci afin d'éviter l'influence des frontières artificielles. Dans le but de construire un maillage de calcul optimal, dans ce contexte, nous proposons un méthode basée sur des estimations d'erreur a posteriori avec poids pour l'approximation par éléments finis utilisée pour simuler le couplage fluide/solide. Un élément clé de l'approche proposée dans cet article est la reformulation des équations de l'écoulement, et du mouvement du corps solide, dans un repère mobile, rigidement attaché au solide ; par ailleurs, via une formulation variationelle bien choisie, nous évitons d'avoir à calculer, explicitement, la résultante et le moment des forces hydrodynamiques que le fluide exerce sur le solide. Les informations fournies par la solution d'un problème dual bien choisi, permettent de contrôler l'erreur de discrétisation pour des fonctionnelles données de la solution (traînée, par exemple). Notre analyse couvre le calcul de la vitesse de sédimentation, l'orientation du corps solide, la résultante et le moment des forces que le fluide exerce sur le solide. Des essais numériques, concernant la résolution d'un problème de sédimentation bi-dimensionnel, valident la méthode proposée. ***Pour citer cet article : V. Heuveline, C. R. Mecanique 333 (2005).***

Keywords: Computational fluid mechanics; Finite element method; A posteriori error estimation; Free steady fall problem; Particulate flow; Fluid–structure coupling

Mots-clés : Mécanaique des fluides numérique ; Méthode des élements finis ; Estimation d'erreur a posteriori ; Écoulements particulaires ; Couplage fluide–structure

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1. Introduction

Over the last decades, the study of the motion of small particles in viscous liquids has been the object of intensive research activities in fluid mechanics. The investigation topics range from the theoretical mathematical analysis (existence, uniqueness and stability of solution) (see e.g. [1–5] and references therein) to the numerical simulation of the liquid-particle interaction (see e.g. [6–13] and references therein). In the present article we focus on the numerical simulation of the steady free fall of a unique solid body in a viscous flow. Many aspects related to this problem are still not well understood. In particular, the issue of the stability of the *terminal states* in relation with the body geometry and orientation needs to be addressed. We propose in that context an a posteriori error estimator in order to control the discretization error and to design adequate mesh leading to an economical discretization for computing the physical quantities of interest. These features are of great importance since the numerical simulation of the free fall of a solid body in a viscous fluid requires computational domains which are usually several order of magnitude larger than the solid body.

The considered weighted a posteriori error estimator relies on the solution of an adequate dual problem which gives localized sensitivity factors with respect to the error measured by means of the quantity of interest. The key ingredients for the derivation of the proposed error estimators are the reformulation of the conservation and kinetic equations in the solid body frame as well as the implicit treatment of the hydrodynamic forces and torque acting on the body in the weak formulation. Our analysis encompasses the control of the free fall velocity, the orientation of the body, the hydrodynamic force and torque on the body.

The outline of the remainder of this article is as follows. In Section 2, we briefly derive the formulation of the stationary free fall problem. Special emphasis is put on the different special cases occurring in three and two dimensional problems. Section 3 deals with the weak formulation of the equations of the fluid-body motion and its discretization by means of the finite element method. Section 4 is dedicated to the derivation of the weighted a posteriori error estimators. In Section 5, numerical experiments for the two dimensional sedimentation problem are presented.

2. Problem formulation

2.1. General formulation of the fluid/body interaction

We consider the free fall of a solid body $\mathcal{S} \subset \mathbb{R}^d$ ($d = 2, 3$) in an incompressible liquid $\mathcal{L}$ filling the whole region $\mathcal{D} := \mathbb{R}^d \backslash \mathcal{S}$. The solid body $\mathcal{S}$ is assumed to be a bounded domain and the velocity of its mass center C (resp. its angular velocity) are denoted by $\mathcal{V}_C$ (resp. $\mathcal{O}$) in the inertial frame $\mathcal{F}$. The region occupied by $\mathcal{S}$ at time t is denoted by $S(t)$ and the corresponding attached frame is denoted by $\mathcal{R}(t)$. In the inertial frame $\mathcal{F}$ the equations of conservation of momentum and mass of $\mathcal{L}$ in their nonconservative form are given by

$$\left.\begin{aligned} &\rho\frac{\partial \mathrm{v}}{\partial t} + \rho(\mathrm{v}\cdot\nabla)\mathrm{v} = \rho g + \nabla\cdot\mathcal{T}(\mathrm{v},\mathrm{p}) \\ &\nabla\cdot\mathrm{v} = 0 \end{aligned}\right\} \quad \text{for } (x,t) \in \bigcup_{t>0}\bigl[\mathbb{R}^d\backslash S(t)\bigr]\times\{t\} \tag{1}$$

where ρ is the constant density of $\mathcal{L}$, v and p are the Eulerian velocity field and pressure associated with $\mathcal{L}$, $\mathcal{T}$ is the Cauchy stress tensor and ρg is the force of gravity which is assumed to be the only external force. We assume further a Navier–Stokes liquid model for which the Cauchy stress tensor is given by

$$\mathcal{T}(\mathrm{v},\mathrm{p}) := -\mathrm{p}\mathbf{1} + \mu\bigl(\nabla\mathrm{v} + (\nabla\mathrm{v})^{\mathrm{T}}\bigr) \tag{2}$$

where μ is the shear viscosity. The boundary conditions are given by

$$\mathrm{v}(x,0) = 0, \qquad \lim_{|x|\to\infty}\mathrm{v}(x,t) = 0 \quad \text{for } x \in \mathbb{R}^d\backslash S(t) \tag{3}$$

$$\mathrm{v}(x,t) = \mathcal{V}_C(t) + \mathcal{O}(t)\times\bigl(x - x_C(t)\bigr) \quad \text{for } x \in \partial S(t) \tag{4}$$

The fluid/body coupling occurs through the Dirichlet boundary condition (4). It relies on the determination of the body motion which is obtained by requiring the balance of the linear and angular momentum:

$$\begin{cases} m_S\dot{\mathcal{V}}_C = m_S g - \displaystyle\int_{\partial S(t)} \mathcal{T}(\mathrm{v},\mathrm{p})N\,\mathrm{d}\sigma \\ \dfrac{\mathrm{d}(J_{S(t)}\mathcal{O})}{\mathrm{d}t} = -\displaystyle\int_{\partial S(t)} (x - x_C)\times\big[\mathcal{T}(\mathrm{v},\mathrm{p})N\big]\,\mathrm{d}\sigma \end{cases} \tag{5}$$

where m_S is the mass of the body, N is the unit normal to $\partial S(t)$ oriented toward the body and J_S the inertia tensor with respect to the mass center C. Further we assume $\mathcal{V}_C(0) = 0$, $\mathcal{O}(0) = 0$.

The straightforward formulation (1)–(5) has the disadvantage that the region occupied by the liquid $\mathcal{L}$ is time dependent. This can be avoided by reformulating these equations in the body frame $\mathcal{R}(t)$. If y denotes the position of a point P in the frame $\mathcal{R}(t)$ and x is the position of the same point in $\mathcal{F}$, we have

$$x = Q(t)y + x_C(t), \quad Q(0) = \mathbf{1}, \quad x_C(0) = 0, \tag{6}$$

with Q orthogonal linear transformation. Considering the transformation (6) one can reformulate the system of Eq. (1) in the following form

$$\left.\begin{array}{l} \rho\left\{\dfrac{\partial v}{\partial t} + \big((v - V)\cdot\nabla\big)v + \omega\times v\right\} = \nabla\cdot T(v,p) + \rho G(t) \\ \nabla\cdot v = 0 \end{array}\right\} \quad \text{for } (y,t)\in\big[\mathbb{R}^d\backslash S(0)\big]\times(0,\infty) \tag{7}$$

where

$$\begin{array}{lll} v(y,t) := Q^{\mathrm{T}}\mathrm{v}(Qy + x_C, t), & p(y,t) := \mathrm{p}(Qy + x_C, t), & G := Q^{\mathrm{T}}g \\ V(y,t) := Q^{\mathrm{T}}\big(\mathcal{V}_C + \mathcal{O}\times(Qy)\big), & T(v,p) := Q^{\mathrm{T}}\mathcal{T}(Qv,p)Q, & \omega := Q^{\mathrm{T}}\mathcal{O} \end{array}$$

The additional term $\omega\times v$ in the momentum equation $(7)_1$ corresponds to the *Coriolis force* induced by the frame transformation (6). Correspondingly, system (5) describing the motion of the body is transformed into

$$\begin{cases} m_S\dot{V}_C + m_S(\omega\times V_C) = m_S G(t) - \displaystyle\int_{\partial S} T(v,p)n\,\mathrm{d}\sigma \\ I_S\dot{\omega} + \omega\times(I_S\omega) = -\displaystyle\int_{\partial S} y\times\big[T(v,p)n\big]\,\mathrm{d}\sigma \\ \dfrac{\mathrm{d}G}{\mathrm{d}t} = G\times\omega \end{cases} \tag{8}$$

where

$$V_C := Q^{\mathrm{T}}\mathcal{V}_C, \quad n := Q^{\mathrm{T}}N, \quad I_S := Q^{\mathrm{T}}J_S Q, \quad \partial S := \partial S(0)$$

In order to keep compatible notations for both the two an three dimensional case, we assume for $d = 2$ that $\omega := (0, 0, \boldsymbol{\omega})$ and similarly $y\times[Tn] = (0, 0, -y_2(Tn)_1 + y_1(Tn)_2)$. For $d = 2$, Eq. $(8)_2$ reduces to a scalar equation.

In the body frame $\mathcal{R}(t)$ the direction of the gravitational force G depends on the time t and becomes therefore an unknown to be resolved. The third additional equation of (8) provides the needed equation describing its variation. Its derivation relies on simple calculus related to the transformation (6). For more details regarding the overall derivation of these equations we refer to Galdi [1,14,15].

2.2. *Formulation of the stationary free fall problem*

The solid body $\mathcal{S}$ is said to undergo a *free steady fall* if the translational and angular velocity V_C and ω are constant and if the motion of the liquid $\mathcal{L}$ is stationary in the frame $\mathcal{R}(t)$. The study of such a configuration is of great interest since it corresponds to so called *terminate state* motions of sedimenting particle for which many questions still remain open: e.g. the number of possible terminal states for a given body geometry, the orientation of the solid body, the stability of the corresponding solution (see [1] and references therein). The free steady fall is thus obtained

Table 1
Considered configurations for the free steady fall problem; d is the dimension of the flow region, while ω is the angular velocity. The body/fluid setup is said to be 'general' if Eq. (13) has to be included in the model

d	ω	Body/fluid setup	Formulation	Number of	
				unknowns	scalar equations
3	$\neq 0$	general	Problem 1	10	10
2	$\neq 0$	general	not possible due to (14)	–	–
3	$= 0$	general	overdetermined	9	10
2	$= 0$	general	Problem 2	6	6
3	$= 0$	symmetric	Problem 3	5	5
2	$= 0$	symmetric	Problem 3	4	4

by requiring that v, p, V_C, ω and G are time independent. Comparing with (7)–(8), this leads to the following system of equations:

$$\left.\begin{aligned}&\rho\big\{\big((v-V)\cdot\nabla\big)v+\omega\times v\big\}=\nabla\cdot T(v,p)+\rho G\\&\nabla\cdot v=0\end{aligned}\right\}\quad\text{for } y\in\big[\mathbb{R}^d\backslash S\big] \tag{9}$$

$$\lim_{|y|\to\infty} v(y)=0 \tag{10}$$

$$v(y)=V(y):=V_C+\omega\times y\quad\text{for } y\in\partial S \tag{11}$$

$$m_S(\omega\times V_C)=m_s G-\int_{\partial S} T(v,p)n\,\mathrm{d}\sigma \tag{12}$$

$$\omega\times(I_S\omega)=-\int_{\partial S} y\times\big[T(v,p)n\big]\,\mathrm{d}\sigma \tag{13}$$

$$G\times\omega=0 \tag{14}$$

The system of Eqs. (9)–(14) describes different class of free fall regimes and configurations which are outlined in Table 1. They lead to different problem formulations. For the most general setup, we assume $\omega\neq 0$. Due to Eq. (14), this configuration can be attained only for $d=3$. Furthermore, it imposes G parallel to ω. The free steady fall problem can then be stated as

Problem 1. Assume $d=3$. Given ρ, $T=T(v,p)$, $|G|=|g|$, I_S and m_S, find v, p, V_C, ω, G whereas $G=|g||\omega|^{-1}\omega$ if $\omega\neq 0$ (see Table 1), such that (9)–(13) holds.

An important subclass of free steady fall problems is given by the case $\omega=0$, i.e., for the solid $\mathcal{S}$ falling with a purely *translational* velocity (see [16]). The problem formulation for this case is subtle since it depends not only on the dimension d of the problem but also on the geometrical properties of the solid.

At first, we assume that Eq. (13) has to be enforced and can not be eliminated by means of any special geometrical properties of the solid $\mathcal{S}$ or on the flow configuration. For $d=3$ such a translational problem is overdetermined and will therefore not be further considered (see Table 1). For $d=2$ however this problem is well formulated in the sense that it involves six unknowns associated to six scalar equations. It can be stated as:

Problem 2. Assume $d=2$. Given ρ, $T=T(v,p)$, $|G|=|g|$, I_S, m_S and $\omega:=0$, find v, p, V_C and the direction $\widehat{G}$ of $G:=|g|\widehat{G}$ such that (9)–(13) holds.

From the physical point of view, the reason of the overdetermination of the translational free steady fall for $d=3$ can be interpreted through the fact that additional geometric properties of the solid body $\mathcal{S}$ have to prevent him from rotating (see [16]). Following Galdi [1], we consider now translational free steady fall problems for solid body with symmetric properties. Let $\{e_1,e_2,e_3\}$ be the canonical basis associated to $\mathbb{R}^3$. Assume that the solid body is homogeneous and symmetric around the axis e_2. Furthermore, the velocity field v and the pressure p describing the *terminal state* of the fluid $\mathcal{L}$ are assumed to be symmetric around the axis e_2. One can show (see [16]) that every sufficiently smooth v, p satisfies the following properties:

$$\int_{\partial S} T(v,p)n = \eta\, e_2, \quad \eta \in \mathbb{R} \tag{15}$$

$$\int_{\partial S} y \times \big[T(v,p)n\big] = 0 \tag{16}$$

$$V = \alpha_V e_2, \quad \alpha_V \in \mathbb{R} \tag{17}$$

Therefore for the symmetric case, Eqs. (12), (13) reduce to the following scalar equation

$$-\bigg\{\int_{\partial S} T(v,p)n\,\mathrm{d}s\bigg\}_2 - m_S|g| = 0 \tag{18}$$

since comparing Eqs. (13) with (16) leads to $G = \pm|g|e_2$. We choose the orientation $G = -|g|e_2$ for the force of gravity. Under these symmetry assumptions, the steady free fall problem can be formulated as

Problem 3. Given ρ, $T = T(v,p)$, $G = -|g|e_2$, I_S, m_S and $\omega := 0$ find v, p, and the scalar quantity α_V defining $V := \alpha_V e_2$ such that (9)–(11) and (18) hold.

Remark 1. Problem 3 is well formulated for both three of two dimensional problems.

3. Galerkin finite element discretization

For a domain $\Omega \subset \mathbb{R}^d$, let $L^2(\Omega)$ denote the Lebesgue space of square-integrable functions on Ω equipped with the inner product and norm

$$(f,g)_\Omega := \int_\Omega fg\,\mathrm{d}x, \qquad \|f\|_\Omega := \bigg(\int_\Omega |f|^2\,\mathrm{d}x\bigg)^{1/2}$$

Analogously, $L^2(\partial\Omega)$ denotes the space of square integrable functions defined on the boundary $\partial\Omega$. The L^2 functions with generalized (in the sense of distributions) first-order derivatives in $L^2(\Omega)$ form the Sobolev space $H^1(\Omega)$, while $H_0^1(\Omega) = \{v \in H^1(\Omega),\ v|_{\partial\Omega} = 0\}$.

3.1. Variational formulation

The Galerkin finite element method starts from a variational formulation of the equations to be solved. We first consider the most general setup of Problem 1, i.e., $\omega \neq 0$ and the related equations (9)–(13). The key ingredient for the derivation of a weak form of the equations (9)–(13) is an adequate choice of the velocity space allowing to eliminate the explicit formulation of the hydrodynamic force and torque on the solid body needed for the kinematic equations (12) and (13). This can be obtained by including the no-slip Dirichlet condition (11) in the velocity space:

$$\mathcal{H}_1(D) := \big\{(v,V,\omega):\ v \in \big[H^1_{\mathrm{loc}}(D)\big]^d,\ V \in \mathbb{R}^d,\ \omega \in \mathbb{R}^d,\ v = V + \omega \times y \text{ on } \partial S\big\} \tag{19}$$

where $D := \mathbb{R}^d \backslash S$. The pressure p is assumed to lie in the space

$$L_0^2(D) := \bigg\{q \in L^2(D):\ \int_{D'} q = 0\bigg\} \tag{20}$$

which defines it uniquely assuming $D' \subset D$ bounded. For $u := \{(v, V_C, \omega), p\} \in \mathcal{H}_1(D) \times L_0^2(D)$ and $\phi := \{(\varphi, \phi_1, \phi_2), q\} \in \mathcal{H}_1(D) \times L_0^2(D)$ we define the semi-linear form

$$\mathcal{A}_1(u;\phi) := \rho\big(\big(\big(v-(V_C+\omega\times y)\big)\cdot\nabla\big)v,\varphi\big)_D + (\omega\times v,\varphi)_D - (p,\nabla\cdot\varphi)_D + 2\mu\int_D D(v):D(\varphi)$$
$$-\big(\rho|g||\omega|^{-1}\omega,\varphi\big)_D - \phi_1\cdot\big[m_S\big(|g||\omega|^{-1}\omega-\omega\times V_C\big)\big] + \phi_2\cdot\big[\omega\times(I_S\omega)\big] - (\nabla\cdot v,q)_D \tag{21}$$

which is obtained by testing Eqs. (9) and (12), (13) by $\phi\in\mathcal{H}_1(D)\times L^2_0(D)$ and by partial integration of the diffusive terms and the pressure gradient in $(9)_1$. Above, $D(v)$ denotes the deformation tensor i.e. $D(v):=\frac{1}{2}(\nabla v+(\nabla v)^{\mathrm{T}})$. A weak form of Problem 1 is given therefore by

Problem V1. Find $u:=\{(v,V_C,\omega),p\}\in\mathcal{H}_1(D)\times L^2_0(D)$ such that

$$\mathcal{A}_1(u;\phi)=0,\quad \forall\phi\in\mathcal{H}_1(D)\times L^2_0(D) \tag{22}$$

The equation modeling the balance of the linear (resp. angular) momentum (12) (resp. (13)) can obviously be recovered by testing in (22) with the functions $\{(0,\phi_1,0),0\}$ (resp. $\{(0,0,\phi_2),0\}$).

Remark 2. The advantages of the formulation (22) rely on the fact that the force and torque on the solid body do not need to be computed explicitly. Numerical instabilities arising for the computation of these lower dimensional integrals can therefore be avoided (see [17,18]).

For the weak formulation of Problems 2 and 3, the formulation (22) simplifies greatly since the free steady fall is then assumed to be translational. For the velocity field we define

$$\mathcal{H}_2(D) := \big\{(v,V)\colon\ v\in\big[H^1_{\mathrm{loc}}(D)\big]^d,\ V\in\mathbb{R}^d,\ v=V \text{ on } \partial S\big\} \tag{23}$$

For $u:=\{(v,V_C),p,\theta\}\in\mathcal{H}_2(D)\times L^2_0(D)\times\mathbb{R}$ and $\phi:=\{(\varphi,\phi_1),q,\phi_2\}\in\mathcal{H}_2(D)\times L^2_0(D)\times\mathbb{R}$, we define the semi-linear form

$$\mathcal{A}_2(u;\phi) := \rho\big(\big((v-V_C)\cdot\nabla\big)v,\varphi\big)_D - (p,\nabla\cdot\varphi)_D + 2\mu\int_D D(v):D(\varphi)$$
$$-(\nabla\cdot v,q)_D - \rho(G,\varphi) - m_S G\cdot\phi_1 + \int_{\partial S}\big[-y_2\big\{T(v,p)n\big\}_1 + y_1\big\{T(v,p)n\big\}_2\big]\phi_2\,\mathrm{d}\sigma \tag{24}$$

where G is assumed to be $G:=|g|\binom{\cos\theta}{\sin\theta}$. A weak formulation of Problem 2 reads then as follows

Problem V2. Find $u:=\{(v,V_C),p,\theta\}\in\mathcal{H}_2(D)\times L^2_0(D)\times\mathbb{R}$ such that

$$\mathcal{A}_2(u;\phi)=0\quad \forall\phi\in\mathcal{H}_2(D)\times L^2_0(D)\times\mathbb{R} \tag{25}$$

For Problem 3 the direction of the gravitation force G is not a variable anymore. Furthermore, due to Eq. (17) the direction of V_C is known to be colinear to e_2. For this configuration we therefore define the following space

$$\mathcal{H}_3(D) := \big\{(v,\alpha_V)\colon\ v\in\big[H^1_{\mathrm{loc}}(D)\big]^d,\ \alpha_V\in\mathbb{R},\ v=\alpha_V e_2 \text{ on } \partial S\big\} \tag{26}$$

for the velocity field. For $u:=\{(v,\alpha_V),p\}\in\mathcal{H}_3(D)\times L^2_0(D)$ and $\phi:=\{(\varphi,\phi_1),q\}\in\mathcal{H}_3(D)\times L^2_0(D)$, we define the semi-linear form

$$\mathcal{A}_3(u;\phi) := \rho\big(\big((v-\alpha_V e_2)\cdot\nabla\big)v,\varphi\big)_D - (p,\nabla\cdot\varphi)_D + 2\mu\int_D D(v):D(\varphi) - (\nabla\cdot v,q)_D \tag{27}$$

and the functional

$$\mathcal{F}_3(\phi) := (\rho G,\varphi)_D + m_S\phi_1 e_2\cdot G \tag{28}$$

A weak formulation of Problem 3 reads then as follows

Problem V3. Find $u:=\{(v,\alpha_V),p\}\in\mathcal{H}_3(D)\times L^2_0(D)$ such that

$$\mathcal{A}_3(u;\phi)=\mathcal{F}_3(\phi),\quad \forall\phi\in\mathcal{H}_3(D)\times L^2_0(D) \tag{29}$$

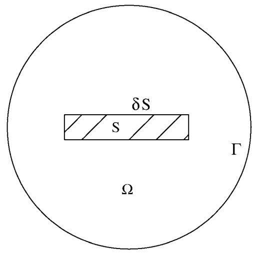

Fig. 1. Bounded domain considered for the finite element discretization and related notations.

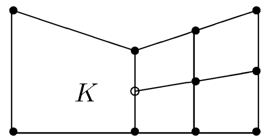

Fig. 2. Quadrilateral mesh patch with a 'hanging node'.

3.2. Finite element discretization

We first consider the general setting of Eq. (22) for the solution of Problem V1. The unbounded domain $D := \mathbb{R}^d \backslash S$ filled by the liquid $\mathcal{L}$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^d \backslash S$ which is chosen to be large enough in order that the liquid may be assumed to be at rest on Γ which denotes the boundary of Ω without ∂S, i.e., $\Gamma = \partial\Omega \backslash \partial S$ (see Fig. 1). In the remainder of this article, Ω is chosen such that the impact of this simplification for the quantities of interest is smaller than the discretization error. We refer to [19–21] for a detailed discussion on this issue.

The discretization uses a conforming finite element space $W_1^h \subset \mathcal{H}_1(\Omega) \times L_0^2(\Omega)$ defined from a quasi-uniform 'triangulation' $\mathcal{T}_h = \{K\}$ consisting of quadrilateral or hexahedral cells K covering the domain $\overline{\Omega}$. For the trial and test spaces $W_1^h \subset \mathcal{H}_1(\Omega) \times L_0^2(\Omega)$ we consider the standard Hood–Taylor finite element [22] i.e.

$$W_1^h := \bigl\{\bigl((v, V, \omega), p\bigr) \in \bigl\{\bigl[C(\overline{\Omega})\bigr]^d \times \mathbb{R}^d \times \mathbb{R}^d\bigr\} \times C(\overline{\Omega}), \quad v|_K \in [Q_2]^d,\ p|_K \in Q_1,\ v|_{\partial S} = V + \omega \times y\bigr\}$$

where Q_r describes the space of isoparametric tensor-product polynomials of degree r (for a detailed description of this standard construction process see e.g. [23]). This choice for the trial and test functions has the advantage that it guarantees a stable approximation of the pressure since the uniform *Babuska–Brezzi* inf–sup stability condition is satisfied uniformly (see [24,25] and references therein). Compared to equal order functions spaces for the pressure and the velocity, no additional stabilization terms are needed.Moreover, in order to facilitate local mesh refinement and coarsening, we allow the cells in the refinement zone to have nodes which lie on faces of neighboring cells (see Fig. 2). The degrees of freedom corresponding to such hanging nodes are eliminated by interpolation enforcing global conformity for the finite element functions. The discrete counterpart of Problem V1 reads as follows:

Problem V1′. Find $u_h := W_1^h$ such that

$$\mathcal{A}_1(u_h; \phi_h) = 0, \quad \forall \phi_h \in W_1^h \tag{30}$$

Analogously, we define for Problems V2 and V3, respectively, the following finite dimensional spaces

$$W_2^h := \bigl\{\bigl((v, V), p, \theta\bigr) \in \bigl\{\bigl[C(\overline{\Omega})\bigr]^d \times \mathbb{R}^d\bigr\} \times C(\overline{\Omega}) \times \mathbb{R},\ v|_K \in [Q_2]^d,\ p|_K \in Q_1,\ v|_{\partial S} = V\bigr\}$$

$$W_3^h := \bigl\{\bigl((v, \alpha_V), p\bigr) \in \bigl\{\bigl[C(\overline{\Omega})\bigr]^d \times \mathbb{R}\bigr\} \times C(\overline{\Omega}),\ v|_K \in [Q_2]^d,\ p|_K \in Q_1,\ v|_{\partial S} = \alpha_V e_2\bigr\}$$

The discrete counterpart of Problem V2 reads as follows

Problem V2′. Find $u_h := W_2^h$ such that

$$\mathcal{A}_2(u_h; \phi_h) = 0, \quad \forall \phi_h \in W_2^h \tag{31}$$

Analogously, the discrete counterpart of Problem V3 reads as follows:

Problem V3′. Find $u_h := W_3^h$ such that

$$\mathcal{A}_3(u_h; \phi_h) = \mathcal{F}_3(\phi_h), \quad \forall \phi_h \in W_3^h \tag{32}$$

4. A posteriori error estimation

4.1. A generic weighted a posteriori estimator

In this section, we outline the concepts related to dual-based error estimation following the general paradigm introduced in Eriksson et al. [26] and Becker and Rannacher [27]. We refer to Machiels et al. [28], Oden and Prudhomme [29], and Giles et al. [30,31] for related approaches to goal-oriented error estimation.

Let $A(\cdot;\cdot)$ be a differentiable semi-linear form and $F(\cdot)$ a linear functional defined over some functional space V. For $u \in V$ the directional derivatives of $A(u;\cdot)$ are denoted by $A'(u;\cdot,\cdot)$, i.e.,

$$A'(u;v)(\varphi) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\bigl\{A(u+\epsilon v;\varphi) - A(u;\varphi)\bigr\}$$

The second derivative is denoted by $A''(\cdot;\cdot)(\cdot,\cdot)$. We seek a solution $u \in V$ to the variational equation

$$A(u;\varphi) = F(\varphi), \quad \forall \varphi \in V \tag{33}$$

This problem is approximated by a *Galerkin method* using a sequence of finite dimensional subspaces $V_h \subset V$ parametrized by h. The corresponding discrete problem seeks $u_h \in V_h$ satisfying

$$A(u_h;\varphi_h) = F(\varphi_h), \quad \forall \varphi_h \in V_h \tag{34}$$

We assume that Eqs. (33) and (34) have unique solutions. A key feature of the discrete problem (34) is the *Galerkin orthogonality* property which reads as follows in the general nonlinear case

$$A(u;\varphi_h) - A(u_h;\varphi_h) = 0, \quad \forall \varphi_h \in V_h \tag{35}$$

Suppose that the quantity $J(u)$ has to be computed, where $J(\cdot)$ is a differentiable functional defined on V. To control the error with respect to the functional J we introduce the following dual problem

$$A'(\overline{uu_h};\varphi)(\hat{z}) = J'(\overline{uu_h})(\varphi), \quad \forall \varphi \in V \tag{36}$$

where

$$A'(\overline{uu_h};\varphi)(\psi) = \int_0^1 A'\bigl(su + (1-s)u_h;\varphi,\psi\bigr)\,\mathrm{d}s$$

$$J'(\overline{uu_h})(\varphi) = \int_0^1 J'\bigl(su + (1-s)u_h;\varphi\bigr)\,\mathrm{d}s$$

We assume that Eq. (36) possesses a solution. Based on the dual solution $\hat{z}$ and due to the Galerkin orthogonality property (35), we obtain the following error representation

$$\begin{aligned} J(u) - J(u_h) &= A'(\overline{uu_h};e,\hat{z}) = A(u;\hat{z}) - A(u_h,\hat{z}) = A(u;\hat{z}-\hat{z}_h) - A(u_h;\hat{z}-\hat{z}_h) \\ &= F(\hat{z}-\hat{z}_h) - A(u_h;\hat{z}-\hat{z}_h) = \rho(u_h,\hat{z}-\hat{z}_h) \end{aligned}$$

for any $\hat{z}_h \in V_h$ and where $\rho(u_h,\cdot) = F(\cdot) - A(u_h;\cdot)$ describes the *primal* residual and $e := u - u_h$. In practice, the previously derived error representation cannot be used directly since the adjoint problem (36) involves the unknown solution u. One alternative is to replace the exact solution u by its approximation u_h in the adjoint problem (36). The resulting adjoint problem reads

$$A'(u_h;\varphi)(z) = J'(u_h;\varphi) \quad \forall \varphi \in V \tag{37}$$

One can show (see [27]) that the following modified error representation holds

$$J(u) - J(u_h) = \rho(u_h, z - z_h) + R \tag{38}$$

for any $z_h \in V_h$, where the remainder term R depends on the second order derivatives of $A(\cdot;\cdot)$ and $J(\cdot)$ and is given by

$$R = \int_0^1 \left\{ A''(u_h + se; z)(e, e) - J''(u_h + se)(e, e) \right\} s \, \mathrm{d}s \tag{39}$$

The remainder term vanishes if $A(\cdot;\cdot)$ and $J(\cdot)$ are linear.

From now on, we consider procedures based on the error representation (38) for the a posteriori error control with respect to the functional J. The remainder term is neglected since, in our context, it involves higher order terms with respect to the discretization parameter h which can be neglected for h small enough.

4.2. Error control of the free fall velocity and body orientation

Our goal in this section is to derive an a posteriori error estimator to control the accuracy of the velocity of the falling solid body. At first, in order to avoid an overload of technicalities for the derivation, we consider the setup of the simplest Problem 3. For $u := \{(v, \alpha_V), p\} \in \mathcal{H}_3(D) \times L_0^2(D)$, the target functional for the control of the fall velocity of the solid body $\mathcal{S}$ is assumed to be

$$J_3(u) := \alpha_V, \quad \forall u \in \mathcal{H}_3(D) \times L_0^2(D) \tag{40}$$

The associated dual problem (resp. its discrete counterpart) is defined as

$$\mathcal{A}_3'(u; z)(\phi) = J_3'(u)(\phi), \quad \forall \phi \in \mathcal{H}_3(D) \times L_0^2(D) \tag{41}$$

$$\mathcal{A}_3'(u_h; z_h)(\phi_h) = J_3'(u_h)(\phi_h), \quad \forall \phi_h \in W_3^h \tag{42}$$

To the approximate solution $u_h \in W_3^h$ of the discrete Problem V3′ we associate the residual

$$\rho_3(u_h; \cdot) := \mathcal{F}_3(\cdot) - \mathcal{A}_3(u_h; \cdot) \tag{43}$$

Proposition 4.1. *Let $u := \{(v, \alpha_V), p\} \in \mathcal{H}_3(D) \times L_0^2(D)$ and $z := \{(z^v, z^\alpha), z^p\} \in \mathcal{H}_3(D) \times L_0^2(D)$ be the solutions of respectively* (29) *and* (41). *Let u_h and z_h be their discrete counterparts, i.e., the solutions of* (32) *and* (42), *respectively. We denote $e := u - u_h$, $e^v := v - v_h$ and $e^\alpha := \alpha_V - \alpha_V^h$. We then have*

$$\alpha_V - \alpha_V^h = \rho_3(u_h; z - z_h) + R_3 \tag{44}$$

where

$$R_3 := \rho\big((e^v \cdot \nabla)e^v, z^v\big)_D - \rho e^\alpha \big((e_2 \cdot \nabla)e^v, z^v\big)_D \tag{45}$$

Proof. The error representation (44) is a direct consequence of Eqs. (38), (39). To identify the remainder R_3, we note that

$$\mathcal{A}_3''(u_h + se; z)(e, e) = 2\rho\Big(\big((e^v - e^\alpha e_2) \cdot \nabla\big)e^v, z^v\Big)_D$$

$$J_3''(u_h + se)(e, e) = 0$$

This completes the proof. □

For the more complex setup of Problems V1 and V2, one can derive an error representation similar to (44). In that context, due to the existence of additional nonlinear terms for the description of the gravitation force $G := |g||\omega|^{-1}\omega$, the residual term becomes however much more complicated. In the context of Problem V2, particularly for the stability analysis of the terminal state, the error control of the orientation of the solid body may be of great interest, i.e.,

$$J_2(u) := \theta, \quad \forall u := \big\{(v, V_C), p, \theta\big\} \in \mathcal{H}_2(D) \times L_0^2(D) \times \mathbb{R} \tag{46}$$

The associated dual problem is defined as

$$\mathcal{A}_2'(u; z)(\phi) = J_2'(u)(\phi), \quad \forall \phi \in \mathcal{H}_2(D) \times L_0^2(D) \times \mathbb{R} \tag{47}$$

as well as its discrete counterpart

$$\mathcal{A}_2'(u_h; z_h)(\phi_h) = J_2'(u_h)(\phi_h), \quad \forall \phi_h \in W_2^h \tag{48}$$

To the approximate solution $u_h \in W_2^h$ of the discrete Problem V2′ we associate the residual

$$\rho_2(u_h; \cdot) := -\mathcal{A}_2(u_h; \cdot) \tag{49}$$

The discretization error on the orientation of the solid body $\mathcal{S}$ can be estimate by means of the following:

Proposition 4.2. *Let $u := \{(v, V_C), p, \theta\} \in \mathcal{H}_2(D) \times L_0^2(D) \times \mathbb{R}$ and $z := \{(z^v, z^{V_C}), z^p, z^\theta\} \in \mathcal{H}_2(D) \times L_0^2(D) \times \mathbb{R}$ be the solutions of* (25) *and* (47)*, respectively. Let u_h and z_h be their discrete counterparts, i.e., the solutions of* (31) *and* (48)*, respectively. We denote $e := u - u_h$, $e^v := v - v_h$, $e^{V_C} := V_C - V_C^h$ and $e^\theta := \theta - \theta_h$. We then have*

$$\theta - \theta_h = \rho_2(u_h; z - z_h) + R_2 \tag{50}$$

where

$$R_2 := \rho\big(\big(\big(e^v - e^{V_C}\big)\cdot\nabla\big)e^v, z^v\big)_D + \frac{1}{2}|g|\left[\rho\left(\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, z^v\right)_D + m_S\left(\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\cdot z^{V_C}\right)\right]\big|e^\theta\big|^2 \tag{51}$$

Proof. The error representation (50) is a direct consequence of Eqs. (38), (39). To identify the remainder R_2, we note that

$$\mathcal{A}_2''(u_h + se; z)(e, e) = 2\rho\big(\big(\big(e^v - e^{V_C}\big)\cdot\nabla\big)e^v, z^v\big)_D + \rho|g|\left(\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, z^v\right)_D\big|e^\theta\big|^2 + \rho m_S\left(\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\cdot z^{V_C}\right)\big|e^\theta\big|^2$$

$$J_2''(u_h + se)(e, e) = 0$$

This completes the proof. □

4.3. Error control of the hydrodynamical force and torque

The implicit treatment of the hydrodynamical force and torque acting on the solid body $\mathcal{S}$ by way of the natural boundary conditions (see Section 3.1), allows one to derive a specific a posteriori error control strategy. The proposed approach, inspired by the work of Giles et al. [17], takes advantage of the special structure of the free steady fall problem and of the considered weak formulation leading to a remarkable natural derivation of error bounds for the hydrodynamical force and torque.

We consider the most general setup of Problem 1 and define for $u := \{(v, V_C, \omega), p\} \in \mathcal{H}_1(D) \times L_0^2(D)$ the following weighted functional

$$J_\psi(u) := \int_{\partial S} \big[T(v, p)n\big]\cdot\psi \,\mathrm{d}\sigma \tag{52}$$

where $\psi := \psi_1 + \psi_2 \times y \in \mathbb{R}^3$ with $\psi_1, \psi_2 \in \mathbb{R}^3$. For $\psi = \psi_1$ (resp. $\psi = \psi_2 \times y$), the functional $J_\psi(u)$ corresponds obviously to the weighted hydrodynamical force (resp. hydrodynamical torque) since

$$J_{\psi_1}(u) = \psi_1 \cdot \int_{\partial S} \big[T(v, p)n\big]\,\mathrm{d}\sigma \tag{53}$$

$$J_{\psi_2\times y}(u) = \psi_2 \cdot \int_{\partial S} y \times \big[T(v, p)n\big]\,\mathrm{d}\sigma \tag{54}$$

Now, we define the following semi-linear form

$$\begin{aligned}\mathcal{A}(u; \phi) := {}& \rho\big(\big(\big(v - (V_C + \omega\times y)\big)\cdot\nabla\big)v, \varphi\big)_D + (\omega \times v, \varphi)_D \\ & - (p, \nabla\cdot\varphi)_D + 2\mu\int_D D(v) : D(\varphi) - \big(\rho|g||\omega|^{-1}\omega, \varphi\big)_D - (\nabla\cdot v, q)_D\end{aligned} \tag{55}$$

which, once the boundary terms ϕ_1 and ϕ_2 have been deleted, corresponds to the semi-linear form $\mathcal{A}_1(u;\phi)$. Next, we define the following velocity space

$$\mathcal{H}_1^{\psi}(D) := \mathcal{H}_1(D) \cap \big\{(v, V, \omega)\colon\ \nabla\cdot v = 0 \text{ in } \Omega,\ V = \psi_1,\ \omega = \psi_2\big\} \tag{56}$$

Then following lemma holds:

Lemma 4.3. *Under sufficient regularity assumptions for the solution u of Problem* V1*, we have*

$$J_\psi(u) = \mathcal{A}(u; w), \quad \forall w \in \mathcal{H}_1^{\psi}(D) \times L_0^2(D) \tag{57}$$

Proof. Eq. (57) is obtained by replacing the stress force in (52) by its components given for the solution u by mean of Eq. $(9)_1$. Applying the standard Green's identity leads to the equality (57). This completes the proof. □

The discrete counterpart of $\mathcal{H}_1^{\psi}(D) \times L_0^2(D)$ is defined as

$$W_1^{\psi,h} := W_1^h \cap \big\{\big((v, V, \omega), p\big)\colon\ \nabla\cdot v = 0 \text{ in } \Omega,\ V = \psi_1,\ \omega = \psi_2\big\}$$

Let $u_h \in W_1^h$ be the solution of the discrete Problem V1′. One can easily shows that the functional

$$\tilde{J}_\psi(u_h) := \mathcal{A}(u_h; w) \quad \forall w \in W_1^{\psi,h} \tag{58}$$

is well defined since $\mathcal{A}(u_h; w)$ depends uniquely on the boundary value ψ of w. It is of importance to notice that in general

$$\tilde{J}_\psi(u_h) \neq J_\psi(u_h)$$

As shown in [17], the functional $\tilde{J}_\psi(u_h)$, rather than $J_\psi(u_h)$ is the appropriate approximation of $J_\psi(u)$. From now on, our purpose is then to derive error bounds for $J_\psi(u_h) - \tilde{J}_\psi(u_h)$. In order to derive an error representation for the error $J_\psi(u_h) - \tilde{J}_\psi(u_h)$, we define the following *linearized dual problem*:

Problem 10. Find $z := \{(z^v, z^{V_C}, z^\omega), z^p\} \in \mathcal{H}_1^{\psi}(D) \times L_0^2(D)$ such that

$$L(u, u_h; z, \phi) = 0, \quad \forall \phi \in \mathcal{H}_1^{\psi=0}(D) \times L_0^2(D) \tag{59}$$

Here, $L(u, u_h; z, \phi)$ is assumed to be a bilinear form in z and ϕ chosen such that the following equality holds

$$L(u, u_h; z, u - u_h) = \mathcal{A}(u; z) - \mathcal{A}(u_h, z), \quad \forall z \in \mathcal{H}_1(D) \times L_0^2(D) \tag{60}$$

where u (resp. u_h) describes the solution of Problem V1 (resp. V1′).

Due to the special nature of the nonlinear terms in $\mathcal{A}(\cdot;\cdot)$, $L(u, u_h; \cdot, \cdot)$ can be defined explicitly. Considering $u := \{(v, V_C, \omega), p\} \in \mathcal{H}_1(D) \times L_0^2(D)$ (resp. $u_h := \{(v_h, V_C^h, \omega_h), p_h\} \in W_1^h$) solution of the Problem V1 (resp. V1′) as well as $z := \{(z^v, z^{V_C}, z^\omega), z^p\} \in \mathcal{H}_1(D) \times L_0^2(D)$ and $\phi := \{(\varphi, \phi_1, \phi_2), q\} \in \mathcal{H}_1(D) \times L_0^2(D)$, the bilinear form $L(u, u_h; \cdot, \cdot)$ can be formulated as

$$L(u, u_h; z, \phi) := a(z, \phi) + b(z, \phi) + b(\phi, z) + a_1(u, u_h; z, \phi) + a_2(u, u_h; z, \phi) + a_3(u, u_h; z, \phi) \tag{61}$$

where

$$a(z, \phi) := 2\mu \int_\Omega D(z^v) : D(\varphi)$$

$$b(z, \phi) := -\int_\Omega q(\nabla\cdot z^v)$$

$$a_1(u, u_h; z, \phi) := -\rho\Big(\big(\big(v - (V + \omega\times y)\big)\cdot\nabla\big)z^v, \varphi\Big)_\Omega + \rho\Big(\big(\big(\varphi - (\phi_1 + \phi_2\times y)\big)\cdot\nabla\big)v_h, z^v\Big)_\Omega$$

$$a_2(u, u_h; z, \phi) := \frac{1}{2}\big(\phi_2 \times (v + v_h), z^v\big)_\Omega + \frac{1}{2}\big((\omega + \omega_h)\times\varphi, z^v\big)_\Omega$$

$$a_3(u, u_h; z, \phi) := -\rho g\left[1 + \frac{\omega\cdot\omega_h}{|\omega||\omega_h|}\right]^{-1}\left\{\left(\frac{\omega\cdot\phi_2}{|\omega|^2|\omega_h|}\omega + \frac{\omega_h\cdot\phi_2}{|\omega||\omega_h|^2}\omega_h, z^v\right)_\Omega - \left(\left(\frac{1}{|\omega|} + \frac{1}{\omega_h}\right)\phi_2, z^v\right)_\Omega\right\}$$

Using these functional we are now able to derive the needed error representation of $J_\psi(u_h) - \tilde{J}_\psi(u_h)$.

Proposition 4.4. *Let z be the solution of Problem* 10. *Further, let $\Pi : \mathcal{H}_1^\psi(D) \times L_0^2(D) \to W_1^{\psi,h}$ be some interpolation operator. We then have*

$$J_\psi(u_h) - \tilde{J}_\psi(u_h) = \mathcal{A}(u_h, z - \Pi z) \tag{62}$$

Proof. First notice that

$$\begin{aligned}\mathcal{A}(u, \Pi z) - \mathcal{A}(u_h, \Pi z) &= \mathcal{A}(u, z) - \mathcal{A}(u_h, z) - \big(\mathcal{A}(u, z - \Pi z) - \mathcal{A}(u_h, z - \Pi z)\big)\\ &= L(u, u_h, z, u - u_h) - \big(\mathcal{A}(u, z - \Pi z) - \mathcal{A}(u_h, z - \Pi z)\big) \quad \text{due to (60)}\\ &= -\big(\mathcal{A}(u, z - \Pi z) - \mathcal{A}(u_h, z - \Pi z)\big) \quad \text{due to (59)}\end{aligned}$$

However from the definitions of $J_\psi(u_h)$ and $\tilde{J}_\psi(u_h)$ we have

$$\begin{aligned}J_\psi(u_h) - \tilde{J}_\psi(u_h) &= \mathcal{A}(u, \Pi z) - \mathcal{A}(u_h, \Pi z)\\ &= -\big(\mathcal{A}(u, z - \Pi z) - \mathcal{A}(u_h, z - \Pi z)\big)\\ &= \mathcal{A}(u_h, z - \Pi z)\end{aligned}$$

The last equality relies on the fact that for the test function $z - \Pi z$, which verifies the homogeneous Dirichlet boundary conditions, the semi-linear forms $\mathcal{A}(u, z - \Pi z)$ and $\mathcal{A}_1(u, z - \Pi z)$ are identical. Since u is solution of Problem V1, this implies

$$\mathcal{A}(u, z - \Pi z) = \mathcal{A}_1(u, z - \Pi z) = 0$$

This completes the proof. □

Remark 3. The error representation (62) allows not only to control separately the hydrodynamical force and torque but also a weighted combination of both quantities. This can be done by an adequate definition of the weights ψ_1 and ψ_2 of the trace $\psi = \psi_1 + \psi_2 \times y$ in (57) and (58) respectively. The dual solution z depends on ψ exclusively through the enforcement of the Dirichlet boundary condition $z^v|_{\partial S} = \psi$.

5. Numerical experiments

We consider the free fall of a rectangular body $[-0.5, 0.5] \times [-0.1, 0.1]$ with density $\rho_S = 10$ in a viscous fluid. The shear viscosity (resp. the density) is assumed to be $\mu = 0.1$ (resp. $\rho = 1$). Our numerical simulations lead to both horizontal and vertical position as terminal state. The vertical fall is however an instable terminal state (see e.g. [13]) and will not be further considered in the following. The terminal Reynolds number which is based on the length of the rectangle as characteristic length is equal to $Re = 17$.

Table 2 clearly shows that despite a careful treatment of boundary conditions on the outmost part of the computational domain Ω (see [20]) one needs to consider vessels which size are several order of magnitude larger than the

Table 2
Convergence of the relative error on the drag acting on the body assuming a computational domain with diameter D_Ω in the range $[40, 400]$ (the body width is assumed to be $l = 1$). The depicted relative error corresponds to the best attainable accuracy for the drag assuming for the outmost part of Ω the second order accurate artificial boundary conditions described in [20]. The third and fourth columns depict the needed number of unknowns of the finite element discretization assuming respectively a global refinement and a local refinement based on the error estimator (62). Note that in order to control the drag we impose $\psi = (1, 0)^{\mathrm{T}}$ for the solution of the dual problem (59)

Diameter of Ω	Relative error for the drag	# Unknowns	
		Global refinement	Local refinement
$D_\Omega = 40$	7.2×10^{-2}	153 456	22 654
$D_\Omega = 60$	1.3×10^{-2}	281 432	28 882
$D_\Omega = 100$	4.5×10^{-3}	723 524	34 432
$D_\Omega = 400$	2.3×10^{-4}	1 140 124	54 868

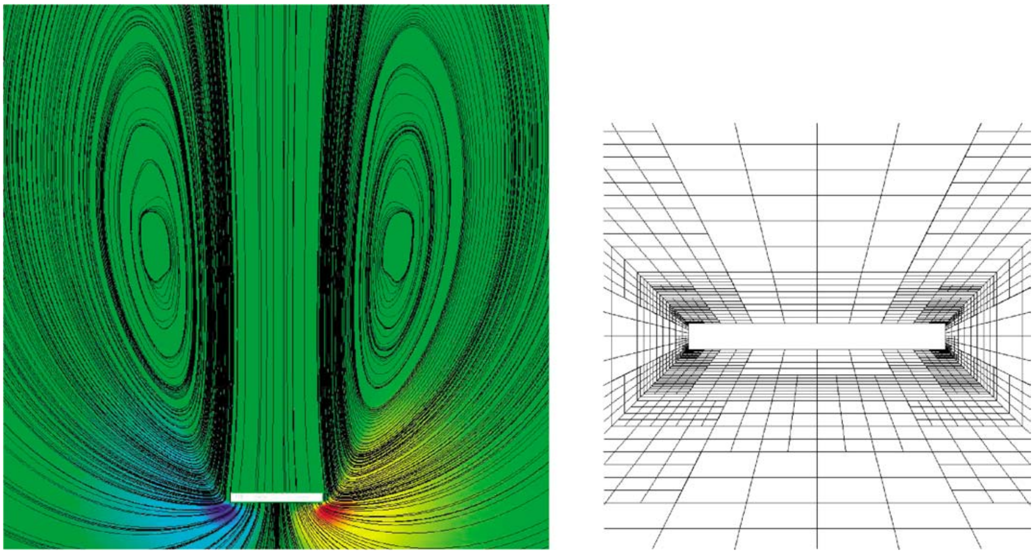

Fig. 3. (Left) Streamlines around the falling body for $\mu = 0.1$; (Right) Zoom on the local refined mesh obtained by means of the error estimator (62) toward the drag computation on a computational domain of diameter $D = 100$ (to be compared with the width of the body $l = 1$).

considered body in order to obtain accurate results. Similar results have already been experimentally observed (see e.g. [1]). The large size needed for the computational domain imposes a careful mesh design. The derived error estimators (44), (50) and (62) rely on the solution of an additional dual problem. This additional problem which is linear is solved numerically by means of the method described in Section 3.2. We refer to [30,27,32] for the derivation of techniques leading to local refinement strategies on the basis of such error estimators. The fourth column of Table 2 clearly show that the proposed approach allows us to solve such fluid/structure interaction problem in a very efficient way. A prototypical mesh adapted toward the drag computation is depicted in Fig. 3.

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