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Uniformly valid approximation for channel flow

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Abstract

The flow at high Reynolds number in a two-dimensional channel whose walls are slightly deformed is considered. This Note addresses the problem of constructing a uniformly valid approximation leading to a better understanding of two-dimensional steady laminar incompressible separated flow. It is proposed to use a new asymptotic approach: the Successive Complementary Expansions Method (SCEM). The starting point is an assumed form of the approximation. The matching principle is a by-product of the method not at all necessary to construct the uniformly valid approximation. ***To cite this article: J. Mauss et al., C. R. Mecanique 334 (2006).***

Résumé

Approximation valable uniformément pour l'écoulement dans un canal. On considère l'écoulement à grand nombre de Reynolds dans un canal bidimensionnel dont les parois sont légèrement déformées. Cette étude est liée à la construction d'une approximation uniformément valable de la solution conduisant à une meilleure compréhension de la séparation pour des écoulements laminaires de fluides visqueux incompressibles. On propose d'utiliser une nouvelle approche asymptotique appelée « Méthode des approximations successives complémentaires » dont l'acronyme est MASC. Le point de départ est une forme supposée de l'approximation conduisant à l'utilisation d'un développement asymptotique généralisé. La méthode des développements asymptotiques raccordés devient une conséquence de la MASC et le principe du raccordement n'est plus nécessaire dans cette méthode. ***Pour citer cet article : J. Mauss et al., C. R. Mecanique 334 (2006).***

Keywords: Fluid mechanics; Boundary layer; Asymptotic analysis; Singular perturbation; Channel flow

Mots-clés : Mécanique des fluides ; Couche limite ; Analyse asymptotique ; Perturbation singulière ; Écoulement en canal

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1. Introduction

We consider a laminar steady two-dimensional flow of an incompressible Newtonian fluid in a channel at high Reynolds number. When small perturbations, e.g. indentations, are placed on the walls, adverse pressure gradients are generated and separation can occur. The analysis of the flow structure has been done essentially by Smith [1]. Later, a systematic asymptotic analysis has been performed by Mauss and Cousteix [2] and Saintlos and Mauss [3]. An extensive analysis of the triple deck structure can be found in Sobey [4]. With the SCEM, we assume a uniformly valid approximation (UVA) based on generalised expansions. This method developed by Cousteix and Mauss [5] has been used by Dechaume et al. [6].

2. Formulation

If the characteristic length, velocity and pressure are chosen respectively as h, U_0 and ρU_0^2, the dimensionless equations can be written

$$\vec{V} \cdot \overline{\text{grad}}\, \vec{V} = -\overrightarrow{\text{grad}}\, \Pi + \frac{1}{Re} \Delta \vec{V}, \qquad \text{div}\, \vec{V} = 0 \tag{1}$$

where $\vec{V}$ is the velocity, Π the pressure and Re the Reynolds number.

If $(v_{(x)}, v_{(y)})$ and (x, y) are respectively the longitudinal and transverse velocity components and coordinates. The basic plane Poiseuille flow can be written,

$$v_{(x)} = u_0 = y - y^2, \quad v_{(y)} = 0, \quad \Pi = \Pi_0 = -\frac{2x}{Re} + p_0$$

where, p_0 being a constant pressure, the characteristic velocity U_0 is linked to the basic pressure gradient or flow rate by,

$$U_0 = -\frac{h^2}{2\eta_0} \frac{\partial \Pi_0^*}{\partial x_*} = 6 \frac{Q^*}{Lh}$$

The flow is perturbed, for instance, by indentations such as,

$$y = \varepsilon F(x, \varepsilon) \quad \text{and} \quad y = 1 - \varepsilon G(x, \varepsilon) \tag{2}$$

where ε is a small parameter. If we seek a solution in the form,

$$v_{(x)} = u_0(y) + \varepsilon^a u(x, y, \varepsilon), \quad v_{(y)} = \varepsilon^a v(x, y, \varepsilon), \quad \Pi - p_0 = -\frac{2x}{Re} + \varepsilon^b p(x, y, \varepsilon) \tag{3}$$

where a and b are yet undetermined, we obtain the equations,

$$u_x + v_y = 0 \tag{4a}$$

$$L_\varepsilon u = \varepsilon^a (u u_x + v u_y) + u_0 u_x + u_0' v + \varepsilon^{b-a} p_x - \frac{1}{Re} \Delta u = 0 \tag{4b}$$

$$L_\varepsilon v = \varepsilon^a (u v_x + v v_y) + u_0 v_x + \varepsilon^{b-a} p_y - \frac{1}{Re} \Delta v = 0 \tag{4c}$$

The operators $L_\varepsilon u$ and $L_\varepsilon v$ denote respectively the x- and y-momentum equations.

It is clear that, for high Reynolds numbers, the reduced equations are of first order leading to a singular perturbation. In the core flow, we are looking for approximations coming from asymptotic generalised expansions such that,

$$u = u_1(x, y, \varepsilon) + \cdots, \quad v = v_1(x, y, \varepsilon) + \cdots, \quad p = p_1(x, y, \varepsilon) + \cdots \tag{5}$$

With the SCEM, no generality is lost by taking $a = b$. Formally neglecting terms of the order $O(\varepsilon^a, 1/Re)$, we obtain the equations,

$$u_{1x} + v_{1y} = 0 \tag{6a}$$

$$u_0 u_{1x} + u_0' v_1 = -p_{1x}, \qquad u_0 v_{1x} = -p_{1y} \tag{6b}$$

It is very interesting to observe the singular behaviour of the solution of (6) as we approach the boundaries. For instance, when $y \to 0$, we obtain,

$$\begin{aligned} u_1 &= -2p_{10} \ln y + c_{10} + \cdots \\ v_1 &= -p_{10x} + 2p_{10x} y \ln y - y(2p_{10x} + c_{10x}) + \cdots \\ p_1 &= p_{10} + \frac{y^2}{2} p_{10xx} + \cdots \end{aligned} \tag{7}$$

Similar results are obtained when $(1-y) \to 0$. In (7), p_{10} and c_{10} are functions of x and ε.

3. The uniformly valid approximation

In order to fulfil the no-slip condition at the walls, boundary layer variables are required,

$$Y = \frac{y}{\varepsilon} \quad \text{and} \quad Y^* = \frac{1-y}{\varepsilon} \tag{8}$$

As in the boundary layers, $u_0 = \mathrm{O}(\varepsilon)$, we have to choose $a = 1$ in order to be able to describe separated flows. Then, following the SCEM, the approximation already obtained in the core flow is complemented as follows,

$$\begin{aligned} u &= U_1(x, Y, \varepsilon) + U_1^*(x, Y^*, \varepsilon) + u_1(x, y, \varepsilon) \\ v &= \varepsilon V_1(x, Y, \varepsilon) - \varepsilon V_1^*(x, Y^*, \varepsilon) + v_1(x, y, \varepsilon) \\ p &= \Delta(\varepsilon) P_1(x, Y, \varepsilon) + \Delta(\varepsilon) P_1^*(x, Y^*, \varepsilon) + p_1(x, y, \varepsilon) \end{aligned} \tag{9}$$

Here, the triplet (u, v, p) is no longer the exact solution of the problem but only an approximation. For instance, if all boundary conditions are fulfilled, from (4b) and (4c), $L_\varepsilon u$ and $L_\varepsilon v$ are not zero but must be small, in a sense.

The gauge function $\Delta(\varepsilon)$ is not yet known. Finally, the v-approximation comes from the continuity equations,

$$U_{1x} + V_{1Y} = 0, \qquad U_{1x}^* + V_{1Y^*}^* = 0 \tag{10}$$

It is clear that we can write,

$$\begin{aligned} &(U_1, V_1, P_1) \to 0 \quad \text{when } Y \to \infty \\ &(U_1^*, V_1^*, P_1^*) \to 0 \quad \text{when } Y^* \to \infty \end{aligned}$$

Moreover, boundary conditions must be written, on the lower and upper boundaries given by $Y = F(x, \varepsilon)$ and $Y^* = G(x, \varepsilon)$,

$$u_0 + \varepsilon u = 0, \qquad v = 0 \tag{11}$$

For instance, with our approximation, we can write on the lower boundary,

$$u_0 + \varepsilon U_1 + \varepsilon u_1 = 0, \qquad \varepsilon V_1 + v_1 = 0$$

It must be kept in mind that, when y tends to 0 or 1, each term u_1 and v_1 or their derivatives are singular, which is not the case for u and v. Thus, this channel flow shows clearly the interest of the SCEM.

Assuming that $1/Re = \mathrm{o}(\varepsilon)$, it can be interesting to refine the approximation of the core equations. The continuity equation keeping the same form, we have then,

$$\begin{aligned} u_0 u_{1x} + u_0' v_1 + \varepsilon(u_1 u_{1x} + v_1 u_{1y}) &= -p_{1x} \\ u_0 v_{1x} + \varepsilon(u_1 v_{1x} + v_1 v_{1y}) &= -p_{1y} \end{aligned} \tag{12}$$

4. The lower interactive boundary layer model

To obtain the interactive boundary layer for the lower boundary and the core flow, we set

$$\begin{aligned} u &= U_1(x, Y, \varepsilon) + u_1(x, y, \varepsilon) \\ v &= \varepsilon V_1(x, Y, \varepsilon) + v_1(x, y, \varepsilon) \\ p &= \Delta(\varepsilon) P_1(x, Y, \varepsilon) + p_1(x, y, \varepsilon) \end{aligned}$$

where, for the sake of simplicity, the same notation (u, v, p) as for the preceding UVA has been used. The triplet (u_1, v_1, p_1) comes from the core approximation. In order to have inertia terms of the same order of magnitude as the viscous terms, we take $Re = \frac{1}{k}\varepsilon^{-3}$, where k is, as we shall see, a useful normalisation factor. From (4b) and (4c) and (12), we obtain the equations,

$$(U_1 + u_1)U_{1x} + \left(V_1 + \frac{v_1}{\varepsilon}\right)U_{1Y} + U_1 u_{1x} + \frac{u_0}{\varepsilon}U_{1x} + u_0' V_1 + \varepsilon V_1 u_{1y}$$
$$= -\frac{\Delta}{\varepsilon}P_{1x} + k(U_{1YY} + \varepsilon^2 u_{1yy}) + \mathrm{O}(k\varepsilon^2)$$
$$(U_1 + u_1)V_{1x} + \left(V_1 + \frac{v_1}{\varepsilon}\right)V_{1Y} + U_1\frac{v_{1x}}{\varepsilon} + V_1 v_{1y} + \frac{u_0}{\varepsilon}V_{1x} = -\frac{\Delta}{\varepsilon^3}P_{1Y} + k(V_{1YY} + \varepsilon v_{1yy}) + \mathrm{O}(k\varepsilon)$$

From the second equation, we are led to take $\Delta(\varepsilon) = \varepsilon^3$. Then, neglecting terms $\mathrm{O}(\varepsilon^2)$, the first equation becomes

$$(U_1 + u_1)U_{1x} + \left(V_1 + \frac{v_1}{\varepsilon}\right)U_{1Y} + U_1 u_{1x} + \frac{u_0}{\varepsilon}U_{1x} + u_0' V_1 + \varepsilon V_1 u_{1y} = k(U_{1YY} + \varepsilon^2 u_{1yy})$$

The second equation enables us to calculate the transverse pressure gradient P_Y as soon as the velocity field is known,

$$(U_1 + u_1)V_{1x} + \left(V_1 + \frac{v_1}{\varepsilon}\right)V_{1Y} + \frac{v_{1x}}{\varepsilon}U_1 + V_1 v_{1y} + \frac{u_0}{\varepsilon}V_{1x} = -P_{1Y} + k(V_{1YY} + \varepsilon v_{1yy})$$

Now, as the behaviour of the core flow is singular when $y \to 0$, with the preceding definitions of (u, v, p), it is better to write the momentum equations in the form

$$\varepsilon(uu_x + vu_y) + u_0 u_x + u_0' v = -p_{1x} + k\varepsilon^3 u_{yy} \tag{13}$$
$$\varepsilon(uv_x + vv_y) + u_0 v_x = -p_y + k\varepsilon^3 v_{yy} \tag{14}$$

5. The global interactive boundary layer model

The generalised asymptotic expansions for the velocity are given by,

$$v_{(x)} = u_0(y) + \varepsilon u(x, y, \varepsilon) + \cdots, \qquad v_{(y)} = \varepsilon v(x, y, \varepsilon) + \cdots \tag{15}$$

The problem we have to solve is the continuity equation $u_x + v_y = 0$ together with Eq. (13). However, now, we have to solve simultaneously the continuity equation (6a) and the core equations (12) or (6b) depending on the accuracy desired.

The same form as Prandtl's equations is recovered if we let

$$U = u_0 + \varepsilon u, \quad V = \varepsilon v, \quad \Pi_x = -2k\varepsilon^3 + \varepsilon p_{1x}$$

leading to,

$$U_x + V_y = 0 \tag{16a}$$
$$UU_x + VU_y = -\Pi_x + k\varepsilon^3 U_{yy} \tag{16b}$$

The boundary conditions are now, $U = V = 0$ on the walls.

As we have four conditions, it is clear that the pressure gradient must be adjusted in order to ensure the global mass flow conservation in the channel.

In addition, in a first approximation, the pressure must satisfy the equation

$$\Delta p_1 - 2\frac{u_0'}{u_0}p_{1y} = 0 \tag{17}$$

To ensure the link between pressure and velocity in the core flow it is necessary that, from (6b), in this region,

$$u_0 v_{1x} = -p_{1y} \tag{18}$$

For this problem,

$$L_\varepsilon u = (p_x - p_{1x}) - k\varepsilon^3 U_{xx} \quad \text{and} \quad L_\varepsilon v = -k\varepsilon^3 V_{xx}$$

It must be kept in mind that $(p_x - p_{1x})$ is a boundary layer term, small in the core flow.

It is why we call it 'global interactive boundary layer model' (GIBL).

6. Regular asymptotic analysis of GIBL

For regular expansions, we have to write the hump equations more precisely,

$$y = \varepsilon F(x\varepsilon^\alpha) \quad \text{and,} \quad y = 1 - \varepsilon G(x\varepsilon^\alpha)$$

where α is the searched longitudinal perturbation giving possibly separation. As k may depend on α our aim is to find these two unknowns in such a way that we get a significant wall deformation to ensure the possibility of separation.

We introduce the variable

$$X = x\varepsilon^\alpha \tag{19}$$

With

$$u = u^*(X, y, \varepsilon), \quad v = \varepsilon^\alpha v^*(X, y, \varepsilon), \quad p_1 = p^*(X, y, \varepsilon)$$

the continuity equation is $u^*_X + v^*_y = 0$, and the longitudinal equation becomes,

$$\varepsilon(u^* u^*_X + v^* u^*_y) + u_0 u^*_X + u'_0 v^* = -p^*_X + k\varepsilon^{3-\alpha} u^*_{yy} \tag{20}$$

The pressure equation becomes

$$\varepsilon^{2\alpha} p^*_{XX} + p^*_{yy} - 2\frac{u'_0}{u_0} p^*_y = 0$$

showing that, using regular expansions, $\alpha \geqslant 0$. We have also, in the core flow

$$u_0 v^*_X = -\varepsilon^{-2\alpha} p^*_y$$

For $\alpha > 0$, in studying both the core flow and the boundary layer, we easily find,

$$p^* = \varepsilon \overline{P}(X) + \frac{\varepsilon^{\beta+2\alpha} A''(X)}{30}(6y^5 - 15y^4 + 10y^3) + \cdots \tag{21}$$

$$u^* = \varepsilon^\beta A(X) u'_0 + \cdots \quad \text{and} \quad v^* = -\varepsilon^\beta A'(X) u_0 + \cdots \tag{22}$$

where β is not known and $A(X)$ is the displacement function. It is noted that (22) is a first order solution to Eq. (20) valid in the core flow.

In the lower boundary, to obtain a significant longitudinal equation, the first order regular approximation must be written,

$$u^* = \overline{U}(X, Y) + \cdots, \quad v^* = \varepsilon \overline{V}(X, Y) + \cdots, \quad p^* = \varepsilon \overline{P}(X) + \cdots \tag{23}$$

giving the equations,

$$\begin{aligned} &\overline{U}_X + \overline{V}_Y = 0 \\ &Y\overline{U}_X + \overline{V} + \overline{U}\,\overline{U}_X + \overline{V}\,\overline{U}_Y = -\overline{P}_X + \overline{U}_{YY} \end{aligned} \tag{24}$$

With $k = \varepsilon^\alpha$. The Reynolds number is written $Re = \varepsilon^{-3-\alpha}$.

We thus obtain three cases for $\alpha > 0$ which can be also classified with the slope of the indentation $\delta = \varepsilon^{1+\alpha}$. This can be written,

$$\varepsilon = Re^{-1/(3+\alpha)} \quad \text{or} \quad \delta = Re^{-(1+\alpha)/(3+\alpha)}$$

1) $\alpha > \frac{1}{2}$ or $\delta \ll Re^{-3/7}$.

In that case, $\beta = 0$, $p^* = \varepsilon \overline{P}_1(X) + \cdots$ and $\lim_{Y\to\infty} \overline{U}_1 = A(X)$.

2) $\alpha = \frac{1}{2}$ or $\delta = Re^{-3/7}$.

This is the classical triple deck case. We have always $\beta = 0$ and $\lim_{Y\to\infty} \overline{U}_1 = A(X)$, but the pressure is given by

$$p^* = \varepsilon\left(\overline{P}_1(X) + \frac{A''(X)}{30}(6y^5 - 15y^4 + 10y^3)\right) + \cdots$$

3) $0 < \alpha < \frac{1}{2}$ or $Re^{-3/7} \ll \delta \ll Re^{-1/3}$.

In that case, $\beta = 1 - 2\alpha$, $\lim_{Y\to\infty} \overline{U}_1 = 0$ and the pressure is given as in the case 2. The height of the perturbation is smaller but the slope is larger. In fact, at this order, there is no displacement of the boundary layer.

For $\alpha = 0$, which is the limiting case of our model, the only change is in the core flow, but from (7) the regular expansion are singular at the boundaries. It is very clear that the use of SCEM is necessary. If we use the SCEM, the symmetric and asymmetric cases are treated in the same way.

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