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A harmonic balance method for the non-linear vibration of viscoelastic shells

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Abstract

In this Note, we deal with the non-linear vibration of viscoelastic shell structures. Coupling a harmonic balance method with a one mode Galerkin's procedure, one obtains an amplitude equation depending on two complex coefficients. These are determined by solving a classical eigenvalue problem and two linear problems. This permits us to characterize the evolution of the loss factor with the vibration amplitude. To validate our approach, the amplitude-frequency and the amplitude-loss factor relationships are illustrated in the case of a circular ring. ***To cite this article: E.H. Boutyour et al., C. R. Mecanique 334 (2006).***

Résumé

Vibrations non linéaires des structures viscoélastiques par une méthode de la balance harmonique. Dans cette Note, on s'intéresse aux vibrations non linéaires des structures courbes viscoélastiques. Couplant une méthode de la balance harmonique linéarisée et la technique de Galerkin à un mode, nous obtenons une équation d'amplitude dépendant de deux coefficients complexes. Ces derniers sont déterminés en résolvant un problème aux valeurs propres classique et deux autres problèmes linéaires. Cela permet de caractériser l'évolution du facteur de perte avec l'amplitude des vibrations. Pour valider notre approche, les relations de la fréquence non linéaire modale et du facteur de perte non linéaire modale en fonction de l'amplitude des vibrations sont illustrées dans le cas d'un anneau circulaire. ***Pour citer cet article : E.H. Boutyour et al., C. R. Mecanique 334 (2006).***

Keywords: Computational solid mechanics; Solids and structures

Mots-clés : Mécanique des solides numérique ; Solides et structures

1. Introduction

In structural mechanics, viscoelastic materials are widely used to reduce vibration and noise in many domains (e.g., the aerospace industry). Indeed, these materials can induce an effective damping especially when they are sandwiched

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between two elastic hard layers. In the linear range, the damping properties are characterized by two modal parameters which are the frequency and the loss factor. Several analytical and numerical works have been developed to determine these quantities in the linear vibration analysis of viscoelastic shells [1]. In the case of non-linear viscoelastic structures, only a few investigations have been devoted to take into account the non-linear geometrical effects. These studies concern viscoelastic structures with simple geometry as beams or plates [2–4]. It is well known that the non-linear geometrical effect induce some dependence of the frequencies and the loss factor with respect to the amplitude [4].

The goal of this Note is to establish a simple general methodology to describe non-linear vibration properties of viscoelastic shell structures. The method is limited to the periodic or damped responses and it couples an approximated harmonic balance method with a one mode Galerkin's procedure. This coupling leads to an amplitude-frequency equation whose resolution permits to get amplitude-frequency and amplitude-loss factor relationships. An application to a viscoelastic circular ring will be given. This will extend the approach of [3] that can only be applied to straight beams or plates.

2. Formulation

Using the principal of virtual work, the equations describing the free non-linear vibrations of a 3D viscoelastic structure can be written in the general following form:

$$\begin{cases} L(U) + Q(U,U) + M(\ddot{U}) = 0 \\ S = D \otimes \dot{\gamma}, \quad \gamma = \varepsilon(u) + \gamma_{nl}(u,u) \end{cases} \tag{1}$$

where $U = (u, S)$ is a mixed vector; its components are the generalized displacement 'u' and the second Piola Kirchhoff stress tensor 'S', and the dots denote the derivative with respect to time. $D(t)$ is the relaxation function and $\otimes$ denotes the classical convolution product. γ is the Green–Lagrange strain which can be decomposed into a linear part ε and a quadratic one $\gamma_{nl}(u,u)$.

$L(\cdot)$ is a linear operator with respect to the mixed unknown $U = (u, S)$, $Q(\cdot,\cdot)$ a bilinear and symmetric one and $M(\cdot)$ is the inertial operator. All these operators are defined by, ρ being the mass density and Ω being the reference configuration of the structure:

$$\begin{cases} \langle L(U), \delta U\rangle = \int_\Omega S : \varepsilon(\delta u)\,\mathrm{d}\Omega, \quad \langle M\ddot{U}, \delta U\rangle = \int_\Omega \rho \ddot{u}_i \delta u_i\,\mathrm{d}\Omega \\ \langle Q(U,U), \delta U\rangle = \int_\Omega \{\delta S : \gamma_{nl}(u,u) + 2S : \gamma_{nl}(u,\delta u)\}\,\mathrm{d}\Omega \end{cases} \tag{2}$$

3. An approximated harmonic balance method

The aim of this section is to get approximate solutions of the non-linear problem (1). As a first approximation, the solution is assumed to be harmonic in time and almost parallel to a single mode in space with an arbitrary complex amplitude. This approximation assumes that the frequency is near the frequency of an associated linear elastic structure. As in non-linear elastodynamics, the harmonic response has to be corrected to balance the quadratic terms in (1), (2). Thus, a non-linear complex frequency-amplitude relationship is obtained by using the one mode Galerkin's procedure.

3.1. First order modal approximation

Let consider a first approximated solution U_h of the problem (1), (2), that is supposed harmonic and proportional to the linear mode:

$$U_h = \frac{1}{2} U_n \left(a\, \mathrm{e}^{\mathrm{i}\omega t} + \mathrm{CC}\right) \tag{3}$$

where 'CC' denotes the conjugate complex of the preceding term, 'a' is an unknown complex amplitude, ω the complex frequency. U_n is the n-th linear vibration mode of the associated elastic system and it is defined by a classical real eigenvalue problem:

$$\begin{cases} L(U_n) - \omega_n^2 M(U_n) = 0 \\ S_n = D(0)\varepsilon(u_n) \end{cases} \tag{4}$$

3.2. How to get the correction term?

Let us consider a second order approximated solution of (1), (2) by adding a corrective term U_c to the linear response (3):

$$U = U_h + U_c \tag{5}$$

The corrective term is assumed to be small with respect to the main term. That is why the equations defining the correction are linearised with respect to U_c. This U_c balances the quadratic terms in (1), (2):

$$\begin{cases} L(U_c) + M\ddot{U}_c = -Q(U_h, U_h) & \text{(6a)} \\ S_c = D \otimes \left[\dot{\varepsilon}(u_c) + \dot{\gamma}_{nl}(u_h, u_h)\right] & \text{(6b)} \end{cases}$$

The correction term U_c combines a time independent term and a harmonic term with a double frequency:

$$U_c = |a|^2 U_0 + \frac{1}{2}\left(a^2 U_2\, \mathrm{e}^{2\mathrm{i}\omega t} + \mathrm{CC}\right) \tag{7}$$

When restricted to the elastic case, the approximations (3)–(6) correspond to the two first terms of a Poincaré–Lindsted expansion [6], that yields a parabolic approximation of the backbone curve. It holds for moderately large amplitude: the first harmonic term (3) is small ($\mathrm{O}(a)$) and the correction term is smaller than the first one ($\mathrm{O}(a)^2$). This is way the coupling term $Q(U_h, U_c)$ can be neglected in (6) ($\mathrm{O}(a^3)$), as well as the quadratic term $Q(U_c, U_c)$ (($\mathrm{O}(a^4)$).

The substitution of (7) into (6) leads to two linear time independent problems satisfied by the amplitudes U_0 and U_2, the corresponding stresses S_0 and S_2 being deduced from the constitutive law (6b).

$$\begin{cases} L(U_0) = -\dfrac{1}{2} Q(U_n, U_n) \\ S_0 = D(0)\left[\gamma_l(u_0) + \dfrac{1}{2}\gamma_{nl}(u_n, u_n)\right] \end{cases} \tag{8a}$$

$$\begin{cases} L(U_2) - 4\omega_n^2 M(U_2) = -\dfrac{1}{2} Q(U_n, U_n) \\ S_2 = D(2\mathrm{i}\omega)\left[\gamma_l(u_2) + \dfrac{1}{2}\gamma_{nl}(u_n, u_n)\right] \end{cases} \tag{8b}$$

where $D(0)$ is the tensor of the delayed elasticity of the viscoelastic material and $D(2\mathrm{i}\omega)$ is the viscoelastic tensor at frequency 2ω. Thus, the general solution of (1) induces a principal harmonic and two secondary ones.

$$U = \frac{1}{2} U_m\left(a\, \mathrm{e}^{\mathrm{i}\omega t} + \mathrm{CC}\right) + |a|^2 U_0 + \frac{1}{2}\left(a^2 U_2\, \mathrm{e}^{2\mathrm{i}\omega t} + \mathrm{CC}\right) \tag{9}$$

As previously said, the approximation (3) assumes that the structure oscillates with a frequency ω near the linear one ω_n. So, the tensor $D(2\mathrm{i}\omega)$ in (8b) will be replaced by $D(2\mathrm{i}\omega_n)$.

3.3. Amplitude equation

To get the non-linear frequency-amplitude relationship, one applies the one-mode Galerkin's procedure, which consists to project Eq. (1) on $U_n\, \mathrm{e}^{-\mathrm{i}\omega t}$, the displacement being given by (9).

$$\int_0^{\frac{2\pi}{\omega}} \left\langle L(U) + Q(U, U) + M(\ddot{U}), U_n\, \mathrm{e}^{-\mathrm{i}\omega t}\right\rangle \mathrm{d}t = 0 \tag{10}$$

Eq. (9) leads to an equation for the complex amplitude in the following form:

$$a\big(k_l - \omega^2 m\big) + a|a|^2 k_{nl} = 0 \tag{11a}$$

where k_l and k_{nl} are complex constants, which correspond, respectively, to the linear and non-linear modal stiffness and m is the modal mass.

$$k_l = \big\langle L(U_n), U_n\big\rangle, \quad k_{nl} = 2\big\langle Q(U_n, U_0\big\rangle + Q\langle U_n, U_2\rangle, \quad m = \big\langle M(U_n), U_n\big\rangle \tag{11b}$$

The amplitude equation can be considered as a generic bifurcation equation, that holds for any form of the non-linearity. It has been first derived in [3], but with a procedure that can only be applied in specific cases, as straight beams or flat plates. When it restricted to an elastic material, the amplitude equation (11) coincides with the parabolic approximation of the backbone curve, that can be deduced for instance by the Poincaré–Lindstedt asymptotic procedure. The linearised form of (11) permits to recover the results of the modal strain energy method [3], that is a classical approach in the analysis of viscoelastic linear structures. The ratio k_l/m permits to define the damped linear frequency Ω_l and the linear loss factor η_l.

$$\frac{k_l}{m} = \Omega_l^2(1 + \mathrm{i}\eta_l), \quad \Omega_l^2 = \frac{k_l^{\mathrm{R}}}{m}, \quad \eta_l = \frac{k_l^{\mathrm{I}}}{k_l^{\mathrm{R}}} \tag{12}$$

where $(k_l^{\mathrm{R}}, k_l^{\mathrm{I}})$ are, respectively, the real and imaginary parts of k_l. Eq. (11a) establishes that the non-linear complex frequency is a function of the amplitude $|a|$.

$$\omega^2 = \frac{k_l}{m} + |a|^2\frac{k_{nl}}{m} \tag{13}$$

A non-linear modal frequency Ω_{nl}^2 and a non-linear modal loss factor η_{nl} are deduced from the complex frequency in the same way as in the linear case.

$$\Omega_{nl}^2 = \omega_n^2\big(1 + C^{\mathrm{R}}|a|^2\big), \qquad \eta_{nl} = \eta_l\frac{1 + C^{\mathrm{I}}|a|^2}{1 + C^{\mathrm{R}}|a|^2} \tag{14}$$

where $C^{\mathrm{R}} = k_{nl}^{\mathrm{R}}/k_l^{\mathrm{R}}$ and $C^{\mathrm{I}} = k_{nl}^{\mathrm{I}}/k_l^{\mathrm{I}}$.

So, the present procedure defines a non linear frequency and a non linear loss factor as in [3]. Nevertheless, the method of reference [3] defines the main term U_h as a bending mode and the correction term U_c as a membrane one and this restricts the application to straight beams or flat plates. In this paragraph, the latter technique has been extended to any viscoelastic structure. We refer also to [3] for comparison with direct numerical studies.

4. An application

In this section, the presented approach is applied to study the in-plane free non-linear vibrations of a viscoelastic circular ring. For this, the rotations are assumed to be moderate, the shear deformation and the rotary inertia terms of the kinetic energy are neglected. The geometrical data are: radius $R = 100$, thickness $h = 1$ and width $b = 1$. The motion equations describing the non-linear free vibrations and the constitutive law are given by:

$$\begin{cases} -RN' - M' + R(\beta N) + \rho h S R\ddot{v} = 0 \\ RN - M'' + R(N\beta)' + \rho h S R\ddot{w} = 0 \\ N = SD \otimes \dot{\gamma}, \quad M = \dfrac{I}{R} D \otimes \dot{\beta}' \\ \gamma(u) = \varepsilon(u) + \dfrac{1}{2}\beta^2, \quad \varepsilon(u) = \dfrac{h}{R}(v' + w), \quad \beta = \dfrac{h}{R}(v - w') \end{cases} \tag{15}$$

where N is the normal force, M is the bending moment, I is the moment of inertia, S the cross-sectional area, v and w denote respectively the radial and tangential non-dimensional displacements and β the rotation of the cross-section.

The classical linear mode $U_n = (v_n, w_n)$ and the eigenfrequency ω_n of the conservative associated system are obtained by solving (15) neglecting the non-linear terms and using a real Young modulus [5].

$$\omega_n^2 = \frac{k_1}{2}\left(1 \pm \sqrt{1 - 4\frac{k_2}{k_1^2}}\right), \quad v_n = V\cos(n\theta), \quad w_n = W\sin(n\theta) \tag{16}$$

where n is the circumferential wave number, V and W are constants, the ratio V/W being deduced from the linearised form of (15). The constants k_1 and k_2 are defined by:

$$k_1 = \frac{1+n^2}{\rho h R^2}\left(\frac{Dn^2}{R^2}+k\right), \quad k_2 = \frac{kDn^2(1-n)^2}{(\rho h)^2 R^6}, \quad k = Eh, \quad D = \frac{Eh^3}{12}$$

The correction term is obtained in the same way as in the general case. In the case of the ring, the linear problems (8a) and (8b) give the following differential equations:

$$\begin{cases} RN_0' + M_0' = \frac{1}{2}RN_n\beta_n, \quad RN_0 - M_0'' = -\frac{1}{2}R(N_n\beta_n)' \\ N_0 = SD(0)\gamma(u_0), \quad M_0 = \frac{I}{R}D(0)\beta_0' \end{cases} \tag{17a}$$

$$\begin{cases} (RN_2' + M_2') + 4\omega_n^2\rho hSRv_2 = \frac{1}{2}RN_n\beta_n, \quad RN_2 - M_2'' - 4\omega_n^2\rho hSRw_2 = -\frac{1}{2}R(N_n\beta_n)' \\ N_2 = SD(2\mathrm{i}\omega_n)\gamma(u_2), \quad M_2 = \frac{I}{R}D(2\mathrm{i}\omega_n)\beta_2' \end{cases} \tag{17b}$$

with

$$N_n = SD(0)\varepsilon(u_n), \quad \gamma(u_\alpha) = \frac{h}{R}(v_\alpha' + w_\alpha) + \frac{1}{4}\beta_m^2 \quad \text{for } \alpha = 0, 2 \tag{18}$$

Injecting (18) in (17) and solving of the obtained problems, one gets $u_0 = (v_0, w_0)$ and $u_2 = (v_2, w_2)$ in the following form:

$$\begin{Bmatrix} v_0 \\ w_0 \end{Bmatrix} = \begin{Bmatrix} d_{01}\sin(2n\theta) \\ d_{00} + d_{02}\cos(2n\theta) \end{Bmatrix}, \qquad \begin{Bmatrix} v_2 \\ w_2 \end{Bmatrix} = \begin{Bmatrix} d_{21}\sin(2n\theta) \\ d_{20} + d_{22}\cos(2n\theta) \end{Bmatrix} \tag{19}$$

where d_{0j} are real constants and d_{2j} complex ones depending on V and W. We do not report here the values of the latter coefficients, that can be obtained in a straightforward manner. Inserting (19) in the constitutive laws (17) and using (11b), one gets the constants of the amplitude equation (11a) as follows:

$$k_l = D(\mathrm{i}\omega)\int_0^{2\pi}\left\{N_n\varepsilon(u_n) + \mu M_n\beta_n'\right\}\mathrm{d}\theta, \qquad m = \int_0^{2\pi}\rho h^2\left\{v_n^2 + w_n^2\right\}\mathrm{d}\theta$$

$$k_{nl} = \int_0^{2\pi}\left\{D(0)N_0\beta_n^2 + D(\mathrm{i}\omega)N_n\beta_0\beta_n + \frac{D(2\mathrm{i}\omega)}{2}N_2\beta_n^2 + \frac{\bar{D}(\mathrm{i}\omega)}{2}N_n\beta_2\beta_n\right\}\mathrm{d}\theta$$

where $\mu = I/SR^2$.

A numerical application is presented, where, for simplicity, the complex Young modulus is assumed constant, i.e. $D(\alpha\mathrm{i}\omega) = D(0)(1+\mathrm{i}\eta_l)$, η_l being the material loss factor and $D(0)$ the delayed elasticity modulus. Table 1 presents both constants C^{R} and C^{I} for different vibration modes. From these results, one notes that C^{R} is a negative number and that C^{I} is greater than C^{R}. In Fig. 1, one presents backbone curves corresponding to modal non-linear frequencies and modal loss factors with respect of the radial displacement $w_{\max} = |a|W$. The modal non-linear frequencies decrease with the displacement (non-linearities of soft type), the modal loss factors increase with the displacement. The increasing and the decreasing of frequencies and loss factors, respectively, are more important for higher vibration modes.

Table 1
Modal coefficients C^{R} and C^{I}, as functions of the circumferential waves number m

m	2	6	10
C^{R}	-6.55×10^{-9}	-9.24×10^{-8}	-2.60×10^{-7}
C^{I}	-1.55×10^{-9}	-4.55×10^{-9}	-4.81×10^{-9}

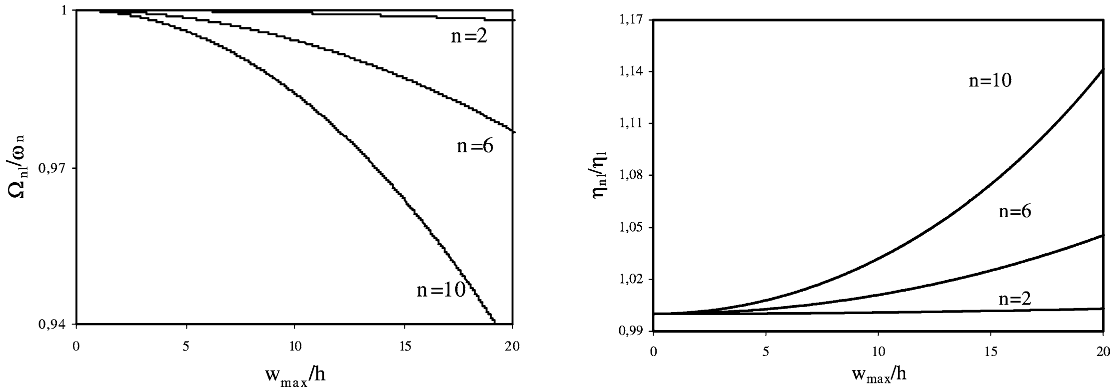

Fig. 1. Modal non-linear frequency and modal non-linear loss factor ratio versus the radial displacement near the linear frequencies.

5. Conclusion

In this study, an amplitude equation has been presented for the free non-linear vibrations analysis of curved viscoelastic structures. As in a classical bifurcation analysis, this amplitude equation is obtained by coupling an approximated harmonic balance method with a one-mode Galerkin's procedure. This leads to two modal parameters C^{R} and C^{I}, which account for the non-linear effects. These constants are determined by solving three classical problems. The first is a real eigenvalue problem. The two others are linear problems. In this way, one obtains a frequency-amplitude relationship, which is similar to classical backbone curves. This approach yields also a definition of the loss factor that depends on the amplitude.

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References

[1] B. Kovâcs, Vibration analysis of a damped arch using an iterative laminate model, J. Sound Vib. 254 (2) (2002) 367–378.
[2] H.-H. Lee, Non-linear vibration of a multilayer sandwich beam with viscoelastic layers, J. Sound Vib. 216 (4) (1998) 601–621.
[3] E.M. Daya , L. Azrar, M. Potier-Ferry, An amplitude equation for the non-linear vibration of viscoelastically damped sandwich beams, J. Sound Vib. 271 (3–5) (2004) 789–813.
[4] B.P. Patel, M. Ganapathi, D.P. Makhecha, P. Shah, Large amplitude free flexural vibration of rings using finite element approach, Int. J. Non Linear Mech. 38 (6) (2003) 911–921.
[5] W. Soedel, Vibrations of Shells and Plates, second ed., Marcel Dekker, New York, 1993.
[6] A.H. Nayfeh, D.T. Mook, Non-Linear Oscillations, Wiley, New York, 1979.