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Prolongation of two phases in the model of fluid displacement through a capillary

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Abstract

The problem of a piston-like displacement of a fluid by another in a capillary is examined. It is suggested that each fluid is prolonged into the domain occupied by the other fluid. This enables the replacement of the two-phase flow problem by a transient single-phase flow problem, with discontinuity in velocity and pressure on a film interface. The problems related to the triple point are solved by introducing a limit fluid near the pore wall. The demonstration of the Washburn equation contributes to the physical justification of our model. ***To cite this article: Y. Lucas et al., C. R. Mecanique 334 (2006).***

Résumé

Prolongement de deux phases dans le modèle du déplacement d'un fluide dans un capillaire. Dans le cadre du déplacement d'un fluide par un autre dans un capillaire, lorsque le mouvement du ménisque est de type piston, nous proposons un prolongement de chaque fluide dans le domaine occupé par l'autre fluide, qui nous permet de remplacer le problème d'écoulement diphasique par un problème d'écoulement monophasique transitoire, avec une discontinuité de la vitesse et de la pression sur une interface de type film. Les problèmes liés au point triple sont résolus par l'introduction d'un fluide limite près de la paroi. La démonstration de l'équation de Washburn contribue à la justification physique de notre modèle. ***Pour citer cet article : Y. Lucas et al., C. R. Mecanique 334 (2006).***

Keywords: Computational fluid mechanics; Capillary; Meniscus; Surface conditions; Triple point; Two-phase flow

Mots-clés : Mécanique des fluides numérique ; Capillaire ; Conditions de surface ; Écoulement diphasique ; Ménisque ; Point triple

Version française abrégée

A l'échelle du capillaire, l'étude d'un type d'écoulement diphasique (déplacement d'un liquide non-mouillant par un liquide mouillant) est proposée, dans la configuration où l'interface séparant les deux fluides est un ménisque

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rigide, dont la courbure reste la même au cours de l'écoulement. L'existence d'un film fin de la phase mouillante en avant du ménisque [1,2] permet un mouvement de type piston [1].

Nous introduisons de nouvelles variables de vitesses $\tilde{\boldsymbol{u}}^\alpha$ et de pressions $\tilde{p}^\alpha$ pour la phase fluide α, définies sur l'ensemble de la phase fluide : ces variables définissent deux « pseudo-fluides » $\bar{\mathrm{I}}$ et $\overline{\mathrm{II}}$ étendus à l'ensemble de la phase liquide, coïncidant respectivement avec les fluides réels I et II sur les domaines occupés par chacun de ces fluides. Les conditions à l'interface sur le ménisque, continuité des vitesses et égalité des contraintes normales de part et d'autre de l'interface, se réduisent alors à des relations d'identité, et n'apparaissent plus comme des conditions contraignantes dans le nouveau système d'équations régissant l'écoulement.

Dans le cas d'un écoulement dans un capillaire aux parois parallèles, les nouvelles variables en pression sont définies dans (∗) et en vitesse dans (∗∗). Les problèmes liés au point triple sont résolus par l'introduction d'un fluide limite avant et après le ménisque. Les nouvelles vitesses et pressions vérifient les Éqs. (5)–(8). Dans l'espace des nouvelles variables, les conditions sur le ménisque ont disparu. En ajoutant aux conditions initiales de pression et de vitesse la position du ménisque, le nouveau système d'équations est fermé. L'étude de l'écoulement diphasique considéré se ramène ainsi à l'étude d'un écoulement monophasique avec une discontinuité de vitesse et de pression sur l'interface de type film. Moyennant certaines hypothèses, ce système permet de déduire l'équation de Washburn [3]. Le cas d'un capillaire de type tube courbé revient au cas précédent par un changement de variables d'espace.

1. Introduction

We study the immiscible displacement of a non-wetting fluid by a wetting fluid in a capillary, where the interface between fluids is a rigid meniscus, whose curvature remains invariable. We begin this study with the case where the capillary has parallel pore walls.

The introduction of a rigid meniscus leads to the well-known problem of the triple point in two dimensions, or the triple line in three dimensions (Fig. 1). Due to the no-slip condition at the pore walls, the point of the wall intersection with the meniscus must be immobile. From experimental studies [1], it was shown that a thin precursor wetting film exists, which enables a piston-like motion of the meniscus. The thin film thickness ε of this is of the order of $RCa^{2/3}$ [2], where, R is the mean meniscus curvature and Ca is the capillary number (ratio of the hydrodynamic force to the capillary force). In the case of important capillary effects, $Ca \ll 1$ and then the film thickness is negligible in front of the meniscus: $\varepsilon \ll R$. The film displacement is complex, requiring an analysis at molecular scale [1,4], but as its thickness is negligible, the following model can be used (Fig. 2). It describes the displacement in a capillary whose pore walls are parallel. This is the first case we dealt with.

The precursor film bounded by Γ_f and S is parallel to the pore walls and its thickness is ε. The curvature of Γ_f is $1/(r-\varepsilon)$. The flow of each phase is ruled by Navier–Stokes equations, with no-slip condition at the pore wall. At the interface, the continuity of velocity and normal stress tensor components are assumed. The piston-like motion of the meniscus means that its velocity is the same for each of its points, parallel to the pore wall.

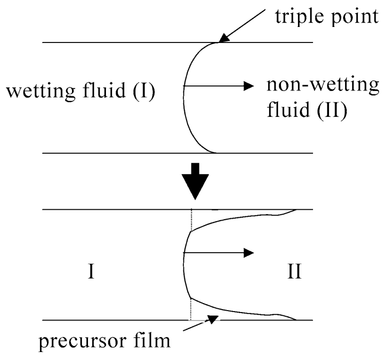

Fig. 1. The simple model above cannot explain the moving of the meniscus. Actually, a precursor film exists in front of the meniscus.

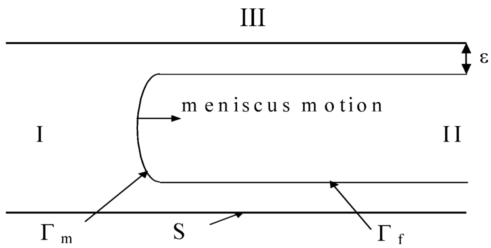

Fig. 2. Model of displacement in capillary: ε tends to zero.

2. Usual equations

Let two fluids in movement in the capillary $\Omega \subset \Re^3$ which consists of three non-intersecting sub-domains: Ω^{I} and Ω^{II} are occupied by the appropriate fluid and Ω^{III} is the solid. Each sub-domain Ω^{I} or Ω^{II} is connected and varies in time. Let Γ be the interface between fluids, composed of the meniscus interface Γ_m, and the film interface Γ_f, while S is the pore wall. The flow of each phase is described by the Navier–Stokes equations in the following dimensionless form (velocity $\boldsymbol{u}$ and pressure p), when inertial effects are negligible:

$$-\mathrm{Re}\,\frac{\partial \boldsymbol{u}^{\mathrm{I}}}{\partial \tau} - \mathbf{grad}\, p^{\mathrm{I}} + \bar{\mu}\Delta \boldsymbol{u}^{\mathrm{I}} = 0, \quad \operatorname{div} \boldsymbol{u}^{\mathrm{I}} = 0, \quad \boldsymbol{x} \in \Omega^{\mathrm{I}} \tag{1}$$

$$-\mathrm{Re}\,\frac{1}{\bar{\rho}}\frac{\partial \boldsymbol{u}^{\mathrm{II}}}{\partial \tau} - \mathbf{grad}\, p^{\mathrm{I}} + \Delta \boldsymbol{u}^{\mathrm{I}} = 0, \quad \operatorname{div} \boldsymbol{u}^{\mathrm{II}} = 0, \quad \boldsymbol{x} \in \Omega^{\mathrm{II}} \tag{2}$$

$\boldsymbol{u}^{\mathrm{I}} = 0$ $\boldsymbol{x} \in S$; $\boldsymbol{u}^{\mathrm{I}}$, $(\boldsymbol{x}, \tau = 0)$, $p^{\mathrm{I}}(\boldsymbol{x}, \tau = 0)$, $\boldsymbol{x} \in \Omega^{\mathrm{I}}$, $\boldsymbol{u}^{\mathrm{II}}(\boldsymbol{x}, \tau = 0)$, $p^{\mathrm{II}}(\boldsymbol{x}, \tau = 0)$, $\boldsymbol{x} \in \Omega^{\mathrm{II}}$, are known.

At the interface $x \in \Gamma$ velocity and the normal stress tensor components are continuous:

$$\boldsymbol{u}^{\mathrm{I}}\big|_{\Gamma} = \boldsymbol{u}^{\mathrm{II}}\big|_{\Gamma} \tag{3}$$

$$\left(\bar{\mu}\left(\frac{\partial u_i^{\mathrm{I}}}{\partial x_j} + \frac{\partial u_j^{\mathrm{I}}}{\partial x_i}\right)n_j - p^{\mathrm{I}} n_i\right)_{\Gamma} = \left(\left(\frac{\partial u_i^{\mathrm{II}}}{\partial x_j} + \frac{\partial u_j^{\mathrm{II}}}{\partial x_i}\right)n_j - p^{\mathrm{II}} n_i\right)_{\Gamma} + \sigma h(x) n_i, \quad \text{with } i = 1, 2, 3 \tag{4}$$

where U^0 is the characteristic velocity, like the injection velocity of the first phase at the entry of the capillary; d is the pore diameter; μ^{α} is the viscosity of fluid α; Σ is the surface tension between two liquids; $h(x) = 1/r_{\min}(x) + \theta/r_{\max}(x)$; $r_i = R_i(x)/\langle R_i\rangle$ $(i = \min, \max)$; $\langle R_{\min}\rangle$ and $\langle R_{\max}\rangle$ are two mean curvature radii of the interface Γ; $R_{\min}(x)$ and $R_{\max}(x)$ are the local curvature radii of the interface Γ; τ is the dimensionless time; $\boldsymbol{n}$ is the interface normal vector.

The dimensionless variables are obtained using d as the scale for space variables, U^0 as the scale for velocity, $\Delta P = \mu^{\mathrm{II}} U^0/d$ as the scale for pressure, and $t^0 = d/U^0$ as the scale for time. Five dimensionless parameters define the flow: the Reynolds number $Re = \rho^{\mathrm{I}} d U^0/\mu^{\mathrm{II}}$; the inverse capillary number which is the ratio of the capillary force to the hydrodynamic force: $\sigma = 2\Sigma/(\langle R_{\min}\rangle \Delta P)$; the viscosity ratio $\bar{\mu} = \mu^{\mathrm{I}}/\mu^{\mathrm{II}}$; the density ratio $\bar{\rho} = \rho^{\mathrm{I}}/\rho^{\mathrm{II}}$; the ratio between the interface curvature radii $\theta = \langle R_{\min}\rangle/\langle R_{\max}\rangle$.

3. Phase prolongation

For each fluid α, its prolongation is introduced into the sub-domain occupied by the other:

- *pressure prolongation*

$$\tilde{p}^{\mathrm{I}} = \begin{cases} p^{\mathrm{I}}(\boldsymbol{x}, \tau), & \boldsymbol{x} \in \Omega^{\mathrm{I}} \\ \bar{\mu} p^{\mathrm{II}}(\boldsymbol{x}, \tau) + c^{\mathrm{I}}(\tau), & \boldsymbol{x} \in \Omega^{\mathrm{II}} \end{cases} \qquad \tilde{p}^{\mathrm{II}} = \begin{cases} \frac{1}{\bar{\mu}} p^{\mathrm{I}}(\boldsymbol{x}, \tau) + c^{\mathrm{II}}(\tau), & \boldsymbol{x} \in \Omega^{\mathrm{I}} \\ p^{\mathrm{II}}(\boldsymbol{x}, \tau), & \boldsymbol{x} \in \Omega^{\mathrm{II}} \end{cases} \tag{$*$}$$

where

$$c^{\mathrm{I}}(\tau) = -\bar{\mu}\sigma h + (1 - \bar{\mu})\tilde{p}^{\mathrm{I}}(\boldsymbol{x}|_{x\in\Gamma_m}, \tau), \qquad c^{\mathrm{II}}(\tau) = \frac{\sigma h}{\bar{\mu}} + \left(1 - \frac{1}{\bar{\mu}}\right)\tilde{p}^{\mathrm{II}}(\boldsymbol{x}|_{x\in\Gamma_m}, \tau)$$

The motion of a meniscus is a piston-like motion, so the pressure of fluid I is the same on the meniscus, as for fluid II, otherwise a transversal flow would occur. Furthermore, we consider that curvature is the same on the whole meniscus. This means that the small variations in curvature are negligible. So $h = h(\tau)$. Thus c^{α} is only a function of time.

- *velocity prolongation*

$$\tilde{\boldsymbol{u}}^{\mathrm{I}} = \begin{cases} \boldsymbol{u}^{\mathrm{I}}(\boldsymbol{x}, \tau), & \boldsymbol{x} \in \Omega^{\mathrm{I}} \\ \frac{1}{\bar{\mu}}\boldsymbol{u}^{\mathrm{II}}(\boldsymbol{x}, \tau) + (1 - \frac{1}{\bar{\mu}})\boldsymbol{u}_m(\tau), & \boldsymbol{x} \in \Omega^{\mathrm{II}} \end{cases} \qquad \tilde{\boldsymbol{u}}^{\mathrm{II}} = \begin{cases} \bar{\mu}\boldsymbol{u}^{\mathrm{I}}(\boldsymbol{x}, \tau) + (1 - \bar{\mu})\boldsymbol{u}_m(\tau), & \boldsymbol{x} \in \Omega^{\mathrm{I}} \\ \boldsymbol{u}^{\mathrm{II}}(\boldsymbol{x}, \tau), & \boldsymbol{x} \in \Omega^{\mathrm{II}} \end{cases} \tag{$**$}$$

The new variables define two 'pseudo-fluids' $\bar{\mathrm{I}}$ and $\bar{\mathrm{II}}$. Each of them occupies a whole space: pseudo-fluid $\bar{\mathrm{I}}$ is equal to fluid I in Ω^{I} and pseudo-fluid $\bar{\mathrm{II}}$ is equal to fluid II in Ω^{II}. The fluid prolongation is determined in such a way that the conditions at the meniscus interface would be satisfied by identity. Indeed, for the pseudo-fluid $\bar{\mathrm{I}}$: $\boldsymbol{u}^{\mathrm{I}}|_{\Gamma_m^{\mathrm{I}}} = \boldsymbol{u}^{\mathrm{II}}|_{\Gamma_m^{\mathrm{II}}} = \boldsymbol{u}_m$ which leads to $\tilde{\boldsymbol{u}}^{\mathrm{I}}|_{\Gamma_m^{\mathrm{II}}} = \frac{1}{\bar{\mu}}\boldsymbol{u}_m + (1 - \frac{1}{\bar{\mu}})\boldsymbol{u}_m = \boldsymbol{u}_m = \tilde{\boldsymbol{u}}^{\mathrm{I}}|_{\Gamma_m^{\mathrm{I}}}$, which means identity

$$\bar{\mu}\left(\frac{\partial \tilde{u}_i^{\mathrm{I}}}{\partial x_j} + \frac{\partial \tilde{u}_j^{\mathrm{I}}}{\partial x_i}\right)n_j - \tilde{p}^{\mathrm{I}} n_i\Big|_{\Gamma_m^{\mathrm{I}}} = \bar{\mu}\left(\frac{\partial \tilde{u}_i^{\mathrm{I}}}{\partial x_j} + \frac{\partial \tilde{u}_j^{\mathrm{I}}}{\partial x_i}\right)n_j - \frac{\tilde{p}^{\mathrm{I}} + \bar{\mu}\sigma h + (\bar{\mu} - 1)\tilde{p}^{\mathrm{I}}}{\bar{\mu}} n_i\Big|_{\Gamma_m^{\mathrm{II}}} + \sigma h n_i, \quad i = 1, 2, 3$$

as $\boldsymbol{u}_m$ only depends on time, it follows:

$$\bar{\mu}\left(\frac{\partial \tilde{u}_i^{\mathrm{I}}}{\partial x_j} + \frac{\partial \tilde{u}_j^{\mathrm{I}}}{\partial x_i}\right)n_j - \tilde{p}^{\mathrm{I}} n_i\Big|_{\Gamma_m^{\mathrm{I}}} = \bar{\mu}\left(\frac{\partial \tilde{u}_i^{\mathrm{I}}}{\partial x_j} + \frac{\partial \tilde{u}_j^{\mathrm{I}}}{\partial x_i}\right)n_j - \tilde{p}^{\mathrm{I}} n_i\Big|_{\Gamma_m^{\mathrm{II}}}, \quad i = 1, 2, 3$$

which is identity.

Conditions at the meniscus for fluid $\bar{\mathrm{II}}$ are similar.

For the film interface, as the flow close to the film can be considered as parallel to it, i.e., $\boldsymbol{n} = (0, n_2)$ and $\boldsymbol{u}^{\mathrm{I}} = (u_1^{\mathrm{I}}, 0)$, so the stress tensor does not take velocity variations into account. With the new variables, surface conditions for fluid $\bar{\mathrm{I}}$ are:

$$\tilde{\boldsymbol{u}}^{\mathrm{I}}\big|_{\Gamma_f^{\mathrm{I}}} = \bar{\mu}\tilde{\boldsymbol{u}}^{\mathrm{I}}\big|_{\Gamma_f^{\mathrm{II}}} + (1 - \bar{\mu})\boldsymbol{u}_m$$
$$-\tilde{p}^{\mathrm{I}}\big|_{\Gamma_f^{\mathrm{I}}} = -\frac{1}{\bar{\mu}}\tilde{p}^{\mathrm{I}}\big|_{\Gamma_f^{\mathrm{II}}} + \left(\frac{1}{\bar{\mu}} - 1\right)\tilde{p}^{\mathrm{I}}\big|_{\Gamma_m} + \sigma\left(\frac{1}{r - \varepsilon} - h\right)$$

There are pressure and velocity discontinuities at the film interface. Conditions at the film interface for fluid $\bar{\mathrm{II}}$ give:

$$\tilde{\boldsymbol{u}}^{\mathrm{II}}\big|_{\Gamma_f^{\mathrm{II}}} = \frac{1}{\bar{\mu}}\tilde{\boldsymbol{u}}^{\mathrm{II}}\big|_{\Gamma_f^{\mathrm{I}}} + \left(1 - \frac{1}{\bar{\mu}}\right)\boldsymbol{u}_m$$
$$-\tilde{p}^{\mathrm{II}}\big|_{\Gamma_f^{\mathrm{I}}} = -\tilde{p}^{\mathrm{II}}\big|_{\Gamma_f^{\mathrm{II}}} - (1 - \bar{\mu})\tilde{p}^{\mathrm{II}}\big|_{\Gamma_m} + \sigma\left(\frac{1}{r - \varepsilon} - h\right)$$

Thus, each pseudo-fluid has continuous velocity and continuous normal stress tensor at the true interface location.

A contradiction appears in the model: indeed, there is a continuity of velocity and pressure through meniscus, a discontinuity through the film, what about the point linking film and meniscus? In fact, this problem comes from the fact that in reality the meniscus curvature is continuous till it becomes parallel to pore walls. To solve this contradiction, we consider that a film of a limit fluid exists on the whole length of capillary. This limit fluid occupies the sub-domain Ω^{L} of phase I, separating two fluids of the same phase (Fig. 3); there is a superficial tension between both phases, in such a way that the limit fluid is continuous.

This way can seem artificial a priori, but it enables us to solve the contradiction. We can notice some similarities between the limit fluid, which integrates continuity problems related to the phase interface, and some methods which introduce a thickness of the interface [5] or of the surface tension [6].

Furthermore, we can already notice that the velocity is continuous through this new interface, and that the existence of a superficial tension in this case, where the interface is almost parallel to flow, will slightly modify the pressure

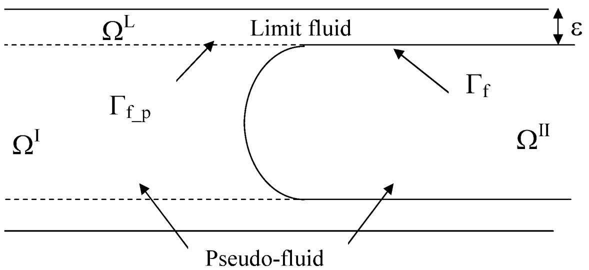

Fig. 3. Model of displacement in capillary with the limit fluid.

gradient of limit fluid. The deduction of the Washburn equation in the framework of this model (see Section 4) will contribute to its physical justification.

By injecting the new variables into Eqs. (1)–(4), and remembering that c^α and $\boldsymbol{u}_m$ are only functions of time, new equations for two pseudo-fluids (5)–(8) are obtained:

• *pseudo-fluid $\bar{I}$*

$$-\mathrm{Re}\,\frac{\partial\tilde{\boldsymbol{u}}^{\mathrm{I}}}{\partial\tau}-\mathbf{grad}\,\tilde{p}^{\mathrm{I}}+\bar{\mu}\Delta\tilde{\boldsymbol{u}}^{\mathrm{I}}=0,\quad \operatorname{div}\tilde{\boldsymbol{u}}^{\mathrm{I}}=0,\quad \boldsymbol{x}\in\Omega^{\mathrm{I}} \tag{5}$$

$$-\mathrm{Re}\,\frac{1}{\bar{\rho}}\left(\bar{\mu}\frac{\partial\tilde{\boldsymbol{u}}^{\mathrm{I}}}{\partial\tau}+(1-\bar{\mu})\frac{\partial\tilde{\boldsymbol{u}}_m}{\partial\tau}\right)-\frac{1}{\bar{\mu}}\mathbf{grad}\,\tilde{p}^{\mathrm{I}}+\bar{\mu}\Delta\tilde{\boldsymbol{u}}^{\mathrm{I}}=0,\quad \operatorname{div}\tilde{\boldsymbol{u}}^{\mathrm{I}}=0,\quad \boldsymbol{x}\in\Omega^{\mathrm{II}} \tag{6}$$

$\tilde{\boldsymbol{u}}^{\mathrm{I}}=0$, $\boldsymbol{x}\in S$; $\tilde{\boldsymbol{u}}^{\mathrm{I}}(\boldsymbol{x},0)$, $\tilde{p}^{\mathrm{I}}(\boldsymbol{x},0)$, $\boldsymbol{x}\in\Omega^{\mathrm{I}}\cup\Omega^{\mathrm{II}}$, are known;

• *pseudo-fluid $\bar{II}$*

$$-\mathrm{Re}\,\frac{\partial\tilde{\boldsymbol{u}}^{\mathrm{II}}}{\partial\tau}-\mathbf{grad}\,\tilde{p}^{\mathrm{II}}+\Delta\tilde{\boldsymbol{u}}^{\mathrm{II}}=0,\quad \operatorname{div}\tilde{\boldsymbol{u}}^{\mathrm{II}}=0,\quad \boldsymbol{x}\in\Omega^{\mathrm{II}} \tag{7}$$

$$-\mathrm{Re}\,\frac{1}{\bar{\rho}}\left(\frac{1}{\bar{\mu}}\frac{\partial\tilde{\boldsymbol{u}}^{\mathrm{II}}}{\partial\tau}+\left(1-\frac{1}{\bar{\mu}}\right)\frac{\partial\tilde{\boldsymbol{u}}_m}{\partial\tau}\right)-\bar{\mu}\,\mathbf{grad}\,\tilde{p}^{\mathrm{II}}+\Delta\tilde{\boldsymbol{u}}^{\mathrm{II}}=0,\quad \operatorname{div}\tilde{\boldsymbol{u}}^{\mathrm{II}}=0,\quad \boldsymbol{x}\in\Omega^{\mathrm{I}} \tag{8}$$

$\tilde{\boldsymbol{u}}^{\mathrm{II}}=0$, $\boldsymbol{x}\in S$; $\tilde{\boldsymbol{u}}^{\mathrm{II}}(\boldsymbol{x},0)$, $\tilde{p}^{\mathrm{II}}(\boldsymbol{x},0)$, $\boldsymbol{x}\in\Omega^{\mathrm{I}}\cup\Omega^{\mathrm{II}}$, are known;

• *limit fluid $\bar{I}$*

The limit fluid coincides with fluid I on Ω^{L}:

$$-\mathrm{Re}\,\frac{\partial\tilde{\boldsymbol{u}}^{\mathrm{L}}}{\partial\tau}-\mathbf{grad}\,\tilde{p}^{\mathrm{L}}+\bar{\mu}\Delta\tilde{\boldsymbol{u}}^{\mathrm{L}}=0,\quad \operatorname{div}\tilde{\boldsymbol{u}}^{\mathrm{L}}=0,\quad \boldsymbol{x}\in\Omega^{\mathrm{L}} \tag{9}$$

Initial and boundary conditions are:

$$\tilde{\boldsymbol{u}}^{\mathrm{L}}=0,\quad \boldsymbol{x}\in S;\quad \tilde{\boldsymbol{u}}^{\mathrm{L}}(\boldsymbol{x},0),\quad \tilde{p}^{\mathrm{L}}(\boldsymbol{x},0),\quad \boldsymbol{x}\in\Omega^{\mathrm{L}}$$

Equations of the prolonged fluids through the real film are, as the flow close to the film can be considered as parallel to it:

• *with pseudo-fluid $\bar{I}$*

$$\tilde{\boldsymbol{u}}^{\mathrm{L}}\big|_{\Gamma_f^{\mathrm{L}}}=\bar{\mu}\tilde{\boldsymbol{u}}^{\mathrm{I}}\big|_{\Gamma_f^{\mathrm{II}}}+(1-\bar{\mu})\boldsymbol{u}_m \tag{10}$$

$$-\tilde{p}^{\mathrm{L}}\big|_{\Gamma_f^{\mathrm{L}}}=-\frac{1}{\bar{\mu}}\tilde{p}^{\mathrm{I}}\big|_{\Gamma_f^{\mathrm{I}}}+\left(\frac{1}{\bar{\mu}}-1\right)\tilde{p}^{\mathrm{I}}\big|_{\Gamma_m^{\mathrm{I}}}+\sigma\left(\frac{1}{r-\varepsilon}-h\right) \tag{11}$$

• *with pseudo-fluid $\bar{II}$*

$$\tilde{\boldsymbol{u}}^{\mathrm{L}}\big|_{\Gamma_f^{\mathrm{L}}}=\tilde{\boldsymbol{u}}^{\mathrm{II}}\big|_{\Gamma_f^{\mathrm{II}}} \tag{12}$$

$$-\tilde{p}^{\mathrm{L}}\big|_{\Gamma_f^{\mathrm{L}}}=-\tilde{p}^{\mathrm{II}}\big|_{\Gamma_f^{\mathrm{II}}}+\sigma\left(\frac{1}{r-\varepsilon}\right) \tag{13}$$

Equations of the prolonged fluids through the prolonged film:

• *with pseudo-fluid $\bar{I}$*

$$\tilde{\boldsymbol{u}}^{\mathrm{L}}\big|_{\Gamma_{f\_p}^{\mathrm{L}}}=\tilde{\boldsymbol{u}}^{\mathrm{I}}\big|_{\Gamma_{f\_p}^{\mathrm{I}}} \tag{14}$$

$$-\tilde{p}^{\mathrm{L}}\big|_{\Gamma_{f\_p}^{\mathrm{L}}}=-\tilde{p}^{\mathrm{I}}\big|_{\Gamma_{f\_p}^{\mathrm{I}}}+\sigma'\left(\frac{1}{r-\varepsilon}\right) \tag{15}$$

In order to have continuity on the limit fluid, we must have:

$$\sigma'(x)=\frac{1}{\bar{\mu}}\sigma\big(1-h(r-\varepsilon_p)\big)-\left(1-\frac{1}{\bar{\mu}}\right)\left(\frac{\tilde{p}^{\mathrm{I}}(x)|_{\Gamma_{f\_p}}-\tilde{p}^{\mathrm{I}}|_{\Gamma_m}}{\bar{\mu}}+\tilde{p}^{\mathrm{L}}(x)\big|_{\Gamma_{f\_p}}\right)(r-\varepsilon) \tag{16}$$

And thus the normal stress conditions are:

$$-\frac{1}{\bar{\mu}}\tilde{p}^{\mathrm{L}}|_{\Gamma^{\mathrm{L}}_{f\_p}} = -\frac{1}{\bar{\mu}^2}\tilde{p}^{\mathrm{I}}|_{\Gamma^{\mathrm{I}}_{f\_p}} + \frac{1}{\bar{\mu}}\left(\frac{1}{\bar{\mu}}-1\right)\tilde{p}^{\mathrm{I}}|_{\Gamma_m} + \sigma\frac{1}{\bar{\mu}}\left(\frac{1}{r-\varepsilon}-h\right) \tag{17}$$

i.e., the same equation as on the real film.

- *with pseudo-fluid $\overline{II}$*

$$\tilde{\boldsymbol{u}}^{\mathrm{L}}|_{\Gamma^{\mathrm{L}}_{f\_p}} = \frac{1}{\bar{\mu}}\tilde{\boldsymbol{u}}^{\mathrm{II}}|_{\Gamma^{\mathrm{II}}_{f\_p}} + \left(1-\frac{1}{\bar{\mu}}\right)\tilde{\boldsymbol{u}}_m \tag{18}$$

$$-\tilde{p}^{\mathrm{L}}|_{\Gamma^{\mathrm{L}}_{f\_p}} = -\bar{\mu}\tilde{p}^{\mathrm{II}}|_{\Gamma^{\mathrm{II}}_{f\_p}} + (1-\bar{\mu})\tilde{p}^{\mathrm{II}}|_{\Gamma_m} + \sigma h + \sigma''\left(\frac{1}{r-\varepsilon}\right) \tag{19}$$

In order to have continuity on the limit fluid, we must have:

$$\sigma''(x) = \frac{1}{\bar{\mu}}\sigma\big(1-h\bar{\mu}(r-\varepsilon_p)\big) + (1-\bar{\mu})\left(\tilde{p}^{\mathrm{II}}|_{\Gamma_m} - (\bar{\mu}+1)\frac{\tilde{p}^{\mathrm{II}}(x)|_{\Gamma_{f\_p}}}{\bar{\mu}} + \frac{\tilde{p}^{\mathrm{L}}(x)|_{\Gamma_{f\_p}}}{\bar{\mu}}\right)(r-\varepsilon) \tag{20}$$

and thus the normal stress conditions are:

$$-\frac{1}{\bar{\mu}}\tilde{p}^{\mathrm{L}}|_{\Gamma^{\mathrm{L}}_{f\_p}} = -\frac{1}{\bar{\mu}}\tilde{p}^{\mathrm{II}}n_i|_{\Gamma^{\mathrm{II}}_{f\_p}} + \sigma\left(\frac{1}{r-\varepsilon}\right)n_i \tag{21}$$

i.e., the same equation as on the real film. We can notice that to ensure the continuity of the limit fluid in the case of pseudo-fluid $\overline{\mathrm{II}}$, we have to use an other surface tension.

Thus the meniscus interface Γ_m disappears in the explicit form. Compared to the classical system of two-phase flow equations, an additional term appears which is the meniscus velocity.

The systems (5), (6) and (7), (8) are two systems where two independent equations have two independent variables, $\tilde{\boldsymbol{u}}$ and $\tilde{p}$, and require the knowledge of the meniscus position at time τ. As the meniscus velocity is the fluids velocity of Γ_m, the initial meniscus position is a sufficient condition for the two systems being closed. Although for the sake of interest the prolongation of two fluids is performed, in the homogenization process for instance, the solution of the problem requires only one prolongation.

4. General equation of stationary flow in a tube capillary

Let us examine a slow flow with a meniscus in the middle of a long planned capillary (Fig. 4) such that it can be considered as stationary over a time period. A zone exists near the capillary where the flow is not rectilinear, but elsewhere, for reasons of symmetry, it is legitimate to assume it is rectilinear. The problem is solved only for pseudo-fluid $\bar{\mathrm{I}}$.

$$\text{on } \Omega^{\mathrm{I}} \quad -\frac{\partial\tilde{p}^{\mathrm{I}}}{\partial x_1} + \bar{\mu}\frac{\partial^2\tilde{\boldsymbol{u}}^{\mathrm{I}}_{x_1}}{\partial x_2^2} = 0,\ \boldsymbol{x}\in\Omega^{\mathrm{I}},\ x < x_m - a$$

$$\text{on } \Omega^{\mathrm{II}} \quad -\frac{1}{\bar{\mu}}\frac{\partial\tilde{p}^{\mathrm{I}}}{\partial x_1} + \bar{\mu}\frac{\partial^2\tilde{\boldsymbol{u}}^{\mathrm{I}}_{x_1}}{\partial x_2^2} = 0,\ \boldsymbol{x}\in\Omega^{\mathrm{II}},\ x > x_m + b$$

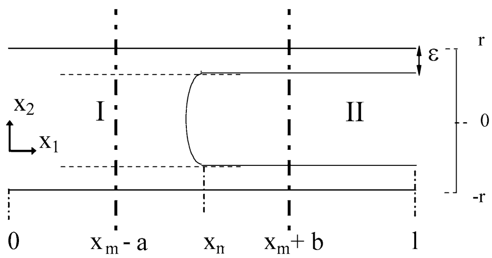

Fig. 4. Flow for $x_1 < x_m - a$ and $x_1 > x_m + b$ is rectilinear.

$$\text{on } \Omega^{\mathrm{L}} \quad -\frac{\partial \tilde{p}^{\mathrm{L}}}{\partial x_1} + \bar{\mu}\frac{\partial^2 \tilde{\boldsymbol{u}}^{\mathrm{L}}_{x_1}}{\partial x_2^2} = 0,\ \boldsymbol{x} \in \Omega^{\mathrm{I}},\ x < x_m - a,\ x > x_m + b$$

with the limit conditions:

$$\tilde{u}^{\mathrm{L}} = 0, \quad \boldsymbol{x} \in S$$

By integrating these relations on their corresponding domains, it comes:

$$\frac{(A^{\mathrm{II}}x_2 + B^{\mathrm{II}} - x_2^2)}{2\bar{\mu}}\frac{\tilde{p}^{\mathrm{I}}_{x_m-a} - \tilde{p}^{\mathrm{I}}_0}{(x_m - a)} = \tilde{u}^{\mathrm{I}}_{x_1}, \quad x \in \Omega^{\mathrm{I}},\ x_1 < x_m - a$$

$$\frac{(A^{\mathrm{II}}x_2 + B^{\mathrm{II}} - x_2^2)}{2\bar{\mu}^2}\frac{\tilde{p}^{\mathrm{I}}_l - \tilde{p}^{\mathrm{I}}_{x_m+b}}{l - (x_m + b)} = \tilde{u}^{\mathrm{I}}_{x_1}, \quad x \in \Omega^{\mathrm{II}},\ x_1 > x_m + b$$

where A^{I}, B^{I}, A^{II} and B^{II} are unknown. It comes by symmetry that $A^{\mathrm{I}} = 0$ and $A^{\mathrm{II}} = 0$.

$$\frac{(r^2 - x_2^2)}{2\bar{\mu}^2}\frac{\tilde{p}^{\mathrm{L}}_{x_m-a} - \tilde{p}^{\mathrm{L}}_0}{x_m - a} = \tilde{u}^{\mathrm{L}}_{x_1}, \quad x \in \Omega^{\mathrm{L}},\ x_1 < x_m - a$$

$$\frac{(r^2 - x_2^2)}{2\bar{\mu}^2}\frac{\tilde{p}^{\mathrm{L}}_l - \tilde{p}^{\mathrm{L}}_{x_m+b}}{l - (x_m + b)} = \tilde{u}^{\mathrm{L}}_{x_1}, \quad x \in \Omega^{\mathrm{L}},\ x_1 > x_m + b$$

Let us examine the conditions on the real and the prolonged film:

(i) on the real film:

$$\text{from (10)} \quad \frac{(r^2 - (r-\varepsilon)^2)}{2\bar{\mu}}\frac{\tilde{p}^{\mathrm{I}}_l - \tilde{p}^{\mathrm{I}}_{x_m+b}}{l - (x_m + b)} = \frac{(B^{\mathrm{II}} - (r-\varepsilon)^2)}{2\bar{\mu}}\frac{\tilde{p}^{\mathrm{I}}_l - \tilde{p}^{\mathrm{I}}_{x_m+b}}{l - (x_m + b)} + (1 - \bar{\mu})\tilde{u}_m$$

$$\text{from (11)} \quad \tilde{p}^{\mathrm{L}}_l - \tilde{p}^{\mathrm{L}}_{x_m+b} = \frac{1}{\bar{\mu}}\big(\tilde{p}^{\mathrm{I}}_l - \tilde{p}_{x_m+b}\big)$$

$$\text{thus} \quad B^{\mathrm{II}} = (r-\varepsilon)^2\left(1 - \frac{1}{\bar{\mu}}\right) + \frac{r^2}{\bar{\mu}} + (\bar{\mu} - 1)2\bar{\mu}u_m \Big/ \frac{\tilde{p}^{\mathrm{I}}_l - \tilde{p}^{\mathrm{I}}_{x_m+b}}{l - (x_m + b)}$$

(ii) on the prolonged film:

$$\text{from (14)} \quad \frac{(r^2 - (r-\varepsilon)^2)}{2\bar{\mu}}\frac{\tilde{p}^{\mathrm{L}}_{x_m-a} - \tilde{p}^{\mathrm{L}}_0}{x_m - a} = \frac{(B^{\mathrm{II}} - (r-\varepsilon)^2)}{2\bar{\mu}}\frac{\tilde{p}^{\mathrm{I}}_{x_m-a} - \tilde{p}^{\mathrm{I}}_0}{x_m - a}$$

$$\text{from (17)} \quad \tilde{p}^{\mathrm{L}}_{x_m-a} - \tilde{p}^{\mathrm{L}}_0 = \frac{1}{\bar{\mu}}\big(\tilde{p}^{\mathrm{I}}_{x_m-a} - \tilde{p}^{\mathrm{I}}_0\big)$$

$$\text{thus} \quad B^{\mathrm{I}} = (r-\varepsilon)^2\left(1 - \frac{1}{\bar{\mu}}\right) + \frac{r^2}{\bar{\mu}} = B^{\mathrm{II}} - (\bar{\mu} - 1)2\bar{\mu}u_m \Big/ \frac{\tilde{p}^{\mathrm{I}}_l - \tilde{p}^{\mathrm{I}}_{x_m+b}}{l - (x_m + b)}$$

As fluids are incompressible, the average velocity on a section of fluid $\bar{\mathrm{I}}$ is the same as meniscus velocity when ε tends to 0:

$$\text{on } \Omega^{\mathrm{I}}: \quad u_{\mathrm{I}} = \frac{1}{2(r-\varepsilon)}\int_{-r+\varepsilon}^{r-\varepsilon} \tilde{u}^{\mathrm{I}}_{x_1}\,\mathrm{d}x_2 = -\frac{\tilde{p}^{\mathrm{I}}_{x_m-a} - \tilde{p}^{\mathrm{I}}_0}{(x_m - a)}\frac{1}{2\bar{\mu}}\left(B^{\mathrm{I}} - \frac{(r-\varepsilon)^2}{3}\right)$$

as $\lim u_{\mathrm{I}\varepsilon\to 0} = u_m$, it comes: $u_m = -\frac{\tilde{p}^{\mathrm{I}}_{x_m-a} - \tilde{p}^{\mathrm{I}}_0}{x_m - a}\frac{r^2}{3\bar{\mu}}$;

$$\text{on } \Omega^{\mathrm{II}}: \quad u_{\mathrm{II}} = \frac{1}{2(r-\varepsilon)}\int_{-r+\varepsilon}^{r-\varepsilon} \tilde{u}^{\mathrm{I}}_{x_1}\,\mathrm{d}x_2 = \frac{\tilde{p}^{\mathrm{I}}_l - \tilde{p}^{\mathrm{I}}_{x_m+b}}{l - (x_m + b)}\frac{1}{2\bar{\mu}^2}\left[\left(B^{\mathrm{I}} - \frac{(r-\varepsilon)^2}{3}\right) + 2(\bar{\mu} - 1)\bar{\mu}u_m\right]$$

as $\lim u_{\mathrm{II}\varepsilon\to 0} = u_m$, it comes $u_m = -\frac{\tilde{p}_l^{\mathrm{I}} - \tilde{p}_{x_m+b}^{\mathrm{I}}}{l-(x_m+b)}\frac{r^2}{3\bar{\mu}}$, and using (∗) to come back to the real pressure, we obtain the following expression for meniscus velocity

$$u_m = -\frac{-p_0^{\mathrm{I}} + p_{x_m-a}^{\mathrm{I}} - p_{x_m+b}^{\mathrm{II}} + p_l^{\mathrm{II}}}{\bar{\mu}(x_m - a) + (l - (x_m + b))}$$

Thus, we see that if $a = b = 0$,

$$u = -\frac{p_l^{\mathrm{II}} - p_0^{\mathrm{I}} - \sigma h}{\bar{\mu}x_m + (l - x_m)}$$

which is the Washburn equation. Physically, it means that this equation ignores rotational flow near the meniscus. For capillaries whose length is of the same order as the pore diameter, this hypothesis is no longer acceptable.

5. Curved tube capillary

The stress produced by superficial tension at the triple point depends on the contact angle between the interface and the solid. So the velocity of the part of the meniscus in contact with the solid depends on this angle. By changing space variables, this case can be studied by the previous method.

A curved capillary is examined. It was assumed that the meniscus should be rigid, which means that the capillary pore walls have a constant curvature (Fig. 5). Furthermore, it is assumed that the curvature remains identical along the pore walls. Following two consecutive transformations ψ and ϕ, the study of the displacement in this tube becomes identical to that in a parallel capillary.

The transformation ψ makes the capillary plane, preserving lengths and areas:

$$x_1' = \sqrt{x_1^2 + x_2^2} \times \arccos\left(\frac{x_2}{\sqrt{x_1^2 + x_2^2}}\right); \qquad x_2' = \sqrt{x_1^2 + x_2^2} - r_2$$

The meniscus C of radius r is deformed. We introduce another transformation ϕ in such a way that the meniscus becomes an arc of circle with curvature radius r. Its centre (a'', b'') is on the capillary median. We require that ϕ lets $\psi(C1)$ unchanged, $C1$ being known as an initial condition:

$$(x_{1C1}'' - a'')^2 + \left(x_{2C1}'' - \frac{r_1 - r_2}{2}\right)^2 = r^2$$

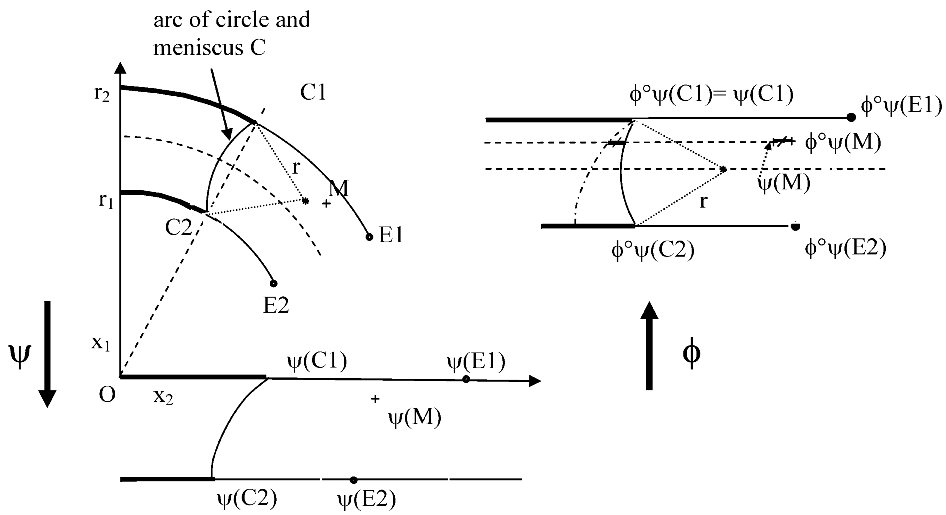

Fig. 5. Transformation of a curved tube capillary into a straight one. This transformation depends on the capillary curvature, so that for a capillary having several curvatures, a new transformation has to be defined for each of them.

thus,

$$a'' = \sqrt{x_{1C1}^2 + x_{2C1}^2} \times \operatorname{arcos}\left(\frac{x_{2C1}}{\sqrt{x_{1C1}^2 + x_{2C1}^2}}\right) + \left(r^2 - \left(\frac{r_1 + r_2}{2} - \sqrt{x_{1C1}^2 + x_{2C1}^2}\right)^2\right)^{1/2}$$

If the point M belongs to the meniscus C, then $\phi^\circ\psi(M)$ belongs to the new meniscus and arc of circle $\phi^\circ\psi(C)$:

$$(x_1'' - a'')^2 + \left(x_2'' - \frac{r_1 - r_2}{2}\right)^2 = r^2, \quad \text{thus } x_1'' = -\left(r^2 - \left(\frac{r_1 + r_2}{2} - \sqrt{x_{1C1}^2 + x_{2C1}^2}\right)^2\right)^{1/2} + a''$$

Let M be a point of the capillary Ω, $\psi(M)$ the transformed point in $\psi(\Omega)$. We transform $\psi(M)$ coordinates in such a way that: (i) the meniscus becomes an arc of circle as said previously; (ii) the areas are preserved. Knowing the ordinate of $\psi(M)$, we know the ordinate of $\phi^\circ\psi(M)$ $x_2'' = x_2'$. And the intersection, in $\psi(\Omega)$, of the horizontal line passing by $\psi(M)$ with $\psi(C)$ gives the abscissa $\tilde{x}_1'(x_2')$, and so $\tilde{x}_1''(x_2')$ by the previous method, and finally enables us to define:

$$x_1'' = x_1' + \tilde{x}_1''(x_2') - \tilde{x}_1'(x_2')$$

This operation is a translation, so preserving areas.

Thus, we find a capillary with parallel pore walls, with a meniscus of the same curvature. The geometry of the entry and the exit are changed, but known, as well as the boundary conditions.

This global transformation $\phi^\circ\psi$ is bijective. The problem can be studied in the space variable $x'' = \phi^\circ\psi(x)$, which leads to the previous case of parallel pore walls. The real behavior of fluids is found by the inverse transformation $x = (\phi^\circ\psi)^{-1}(x'')$.

6. Conclusions

By introducing new variables of velocity and pressure, it is possible to study an immiscible displacement of a non-wetting fluid by a wetting fluid in a parallel tube capillary as a transient single-phase flow problem. It has enabled us to point out some contradictions related to the triple point and to solve it by introducing a limit fluid. A conform transformation enables the case of a curved tube capillary such as the one above to be dealt with.

The developed procedure of fluid prolongation may be used in various aspects: (i) the resolution of the modified single-phase Navier–Stokes equations (5), (6) is obviously simpler than the use of a two-phase classic formulation; and (ii) the homogenization procedure of two-phase meniscus flow requires a fluid prolongation procedure to apply.

References

[1] G.C. Tzimas, T. Matsuura, D.G. Avraam, W. Van Der Brugghen, G.N. Constantinides, A.C. Patyatakes, The combined effect of the viscosity ratio and the wettability during forced imbibition through nonplanar porous media, Journal of Colloid and Interface Science 189 (1997) 27–36.

[2] P.I. Gutenev, A.M. Pyatnitskii, N.V. Klimova, Liquid entrainment from the meniscus of a liquid wedge by a moving horizontal plate, Colloid Journal 65 (2003) 301–304.

[3] E.W. Washburn, The dynamic of capillary flow, Phys. Rev. 17 (1921) 273–283.

[4] V.C. Weiss, J.O. Indekeu, Contact angle at the first-order transition in sequential wetting, Physica A 301 (2001) 37–51.

[5] D. Jamet, O. Lebaigue, N. Courtris, J.M. Delhaye, The second gradient method for direct numerical simulation of liquid–vapor flows with phase change, Journal of Computational Physics 169 (2001) 624–651.

[6] S.-J. Lee, K.-S. Chang, S.-J. Kim, Surface tension effect in the two-fluids equation system, International Journal of Heat and Mass Transfer 41 (1998) 2821–2826.