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Optimal control theory and Newton–Euler formalism for Cosserat beam theory ☆

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Abstract

A Newton–Euler formalism is derived for Cosserat beam theory in a *purely deductive manner*, thanks to an analogy with optimal control theory. The method relies upon joint use of Gauss least constraint principle, Appell's equations and optimal control theory, that was used successfully in a previous work for the classical case of discrete Newton–Euler backward and forward recursions of multibody systems. ***To cite this article: G. Le Vey, C. R. Mecanique 334 (2006).***

Résumé

Commande optimale et formalisme de Newton–Euler pour les poutres de Cosserat. Un formalisme de Newton–Euler pour les théories de poutres de Cosserat est obtenu de manière purement déductive, grâce à une analogie avec la théorie de la commande optimale. La méthode repose sur l'utilisation conjointe du principe de la moindre contrainte de Gauss, des équations d'Appell et de la théorie de la commande optimale, de façon analogue à un travail précédent sur le formalisme de Newton–Euler bien connu pour les systèmes multicorps. ***Pour citer cet article : G. Le Vey, C. R. Mecanique 334 (2006).***

Keywords: Dynamics of rigid or flexible systems; Newton–Euler; Gauss principle; Appell's equations; Optimal control

Mots-clés : Dynamique des systèmes rigides ou flexibles ; Newton–Euler ; principe de Gauss ; équations d'Appell ; Commande optimale

Version française abrégée

Le formalisme de Newton–Euler est bien connu et utilisé depuis longtemps en mécanique des systèmes multicorps. Les récurrences correspondantes sont obtenues par application des lois fondamentales de la mécanique. Ce travail présente une approche purement déductive pour l'obtention d'un formalisme analogue des modèles de poutres continues déformables. L'approche repose sur l'utilisation conjointe du principe de la moindre contrainte de Gauss, des équations d'Appell et de la théorie de la commande optimale. Cette approche a déjà été utilisée avec succès dans le cas des systèmes multicorps, ce qui a permis d'obtenir les récurrences bien connues de la même manière déductive. Le principe général est très simple et consiste à résoudre un problème d'optimisation contrainte, où le critère, fourni

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par le principe de Gauss, est une fonction quadratique des accélérations (« énergie d'accélération ») et la contrainte provient de la géométrie du système mécanique étudié. Dans le formalisme de Newton–Euler classique, il subsiste des inconnues à ce problème : soit les accélérations articulaires, si les couples articulaires sont donnés (récurrence « directe ») soit l'opposé (récurrence « inverse »). En posant alors ce problème comme un problème de commande optimale où les entrées de commande sont l'une ou l'autre de ces inconnues, le formalisme de Newton–Euler s'en déduit simplement et l'extension à des modèles continus est immédiate.

1. Introduction

The Newton–Euler approach to obtain dynamical equations for multibody systems has been for a long time widely used for modelling in multibody systems and robotics. Actually, this approach does not make explicit the dynamical equations but, instead, relies upon considering the relative positions, velocities and accelerations of each body with respect to its neighbours, in a recursive manner, leading to low complexity algorithms. Two questions are usually addressed in the framework of Newton–Euler formalism for multibody systems: either the joint accelerations are known and one searches for the corresponding joint torques (*backward algorithm*, for control objectives) or the converse (*forward algorithm*, for simulation purposes). These two questions are addressed here in the case of Cosserat continuous beam theory, on the illustrative example of a Kirchoff, inextensible model. On another hand, analogy of recursive Newton–Euler equations for multibody systems with optimal filtering [1] and discrete-time optimal control [2] has been evidenced. This analogy was not really used as a principle for deriving equations but simply pointed out while using the trick of separating linear from nonlinear effects in [2]. A significant improvement was given in [3] for the classical case of (discrete) Newton–Euler formalism, in the sense that the Newton–Euler recursions were derived in a purely deductive way, thanks to a joint use of Gauss least constraint principle, Appell's equations and multistage optimal control theory. In the present work, this approach is extended in order to obtain a Newton–Euler formalism for continuous beam theory, where recursions are replaced by ordinary differential equations, as expected. The concrete motivation for deriving a Newton–Euler formalism in the continuous case comes from the recent derivation of a continuous 3D model of a swimming eel-robot [4], where the analogy with (backward) Newton–Euler has been pointed out but derivation of a forward analogous is far from being evident. It is worth mentioning that all the ingredients on which the present work rely upon have been known for a long time but that their conjunction leads to derivations from first principles that are new and allow for interesting geometrical considerations as well as for low complexity algorithms. Also the analogy with optimal control theory opens new perspectives for simulation, control and identification of flexible mechanical structures.

2. Model of a continuous Kirchoff beam

Notations: in the sequel, dots over some quantity will indicate differentiation with respect to time and primes differentiation with respect to the space variable. Dependence on independent variables is implicitly understood. For a vector $y \in R^3$, $\hat{y}$ is the antisymmetric matrix such that $\forall z \in R^3$, $\hat{y}z = y \times z$ and $\times$ is the usual vector product. I_3 is the identity matrix 3×3.

The three dimensional model of the considered beam is quickly recalled (see [4], and Fig. 1 and [4] for more details). The beam, with length normed to one, is considered as a Cosserat beam, i.e., a set of stacked microsolids,

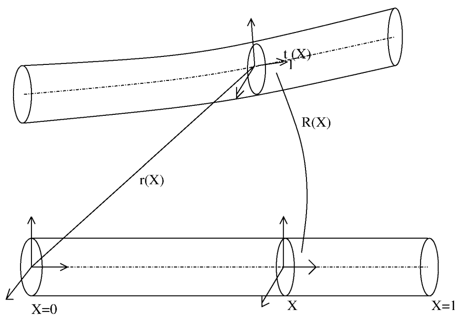

Fig. 1. The geometry of a Cosserat beam.

named hereafter 'sections', labelled by the material abscissa, X, along some abstract neutral line. It is assumed, for the sake of exposition, that there is neither shearing (Kirchoff hypothesis) nor extensibility of the body. But this does not prevent generality of the method and the more general Reissner beam theory, e.g., could be used as well. The set of equations is simply listed below for fixing notations. The configuration space of the beam is the principal bundle $R^3 \times SO(3)$, a cross section, labelled X, being described by $r(X) \in R^3$, the position of its mass center in a reference frame, and $R(X) \in SO(3)$, its attitude. Its kinematics is described by a twist-curvature tensor field and the constraint imposed by the 'spherical kinematics':

$$\forall X \in [0,1], \quad \hat{K}_d(X,t) = R^{\mathrm{T}}(\partial R/\partial X) = R^{\mathrm{T}} R', \quad \partial r/\partial X = r' = t_1 \tag{1}$$

where t_1 is the unit tangent vector to the neutral line of the beam, R is the rotation matrix mapping the X mobile basis before deformation onto that after. The question of boundary conditions is not discussed here but can easily be taken into account in the present framework (see [3], e.g., in the discrete case). Two differentiations of the previous set of equations with respect to time give a kinematic model of the accelerations, that is best written in matrix–vector form for the ease of subsequent analogy with optimal control:

$$\begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix}' = \begin{pmatrix} 0 & -\hat{t}_1 \\ 0 & \hat{k} \end{pmatrix} \begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix} + \begin{pmatrix} 0 \\ \ddot{k} \end{pmatrix} + \begin{pmatrix} \omega \times (\omega \times t_1) \\ \dot{k} \times \omega \end{pmatrix} \tag{2}$$

where ω is the axial vector corresponding to the matrix $\hat{\omega} = \dot{R} R^{\mathrm{T}}$, and k, the axial vector corresponding to the matrix $\hat{k} = R \hat{K}_d R^{\mathrm{T}}$. Notice that in the above equation, $\ddot{r}$, $\dot{\omega}$ appear as elements of a 'state' vector and $\ddot{k}$ as a 'control input', of a linear 'state equation', when using the language of control systems theory. Also, the last term is an inhomogeneity, not depending on this so defined state, as it can be computed for each X.

3. Gauss least constraint principle and Gibbs–Appell equations

Appell's approach [5] (also known as 'Gibbs–Appell') to deriving motion equations of a mechanical system is based on the consideration of an 'acceleration energy' instead of the kinetic energy that is used for deriving Lagrange equations. For the sake of easy reference, Appell notations [5] are in order for a while. S is the acceleration energy of the mechanical system under consideration: let $\gamma(P,q)$ be the acceleration of particle with mass $\mathrm{d}m_P$ located in P. Then: $S = \int \frac{1}{2}|\gamma(P,q)|^2 \,\mathrm{d}m_P$ where the integral extends to the whole system. q is the configuration parameter (generalized coordinate), Q the vector of applied efforts and R the *analytical expression of the constraint* in the sense of Gauss principle, defined as $R = S - Q^{\mathrm{T}}.q''$. Then Appell has shown that the motion equations take the form: $\frac{\partial S}{\partial q''} = Q$ and observed that his equations have a tight connection with Gauss least constraint principle [6], in the sense that they are also those obtained when searching for the minimum of R, a quadratic function of q''.

Returning now to the notations of Section 2, consider a Cosserat type beam model, with mass density ρ, section area A and inertia matrix per section ρI at section X (Fig. 1). The acceleration energy of one section at X, can be shown to be (see [7]):

$$S(X) = \left[\frac{1}{2} \left(\ddot{r}^{\mathrm{T}} \dot{\omega}^{\mathrm{T}} \right) \begin{pmatrix} \rho A I_3 & 0 \\ 0 & \rho I \end{pmatrix} \begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix} + \begin{pmatrix} 0 \\ (\omega \times (\rho I \omega)) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix} \right] \tag{3}$$

As for the applied efforts, it is necessary to consider the nonvanishing work of active efforts, which are summarized in the vector: $(f_{\mathrm{ext}}^{\mathrm{T}} | c_{\mathrm{ext}}^{\mathrm{T}})^{\mathrm{T}}$. Consider now the torque at X, denoted Γ, and the corresponding generalized coordinate, denoted k, the curvature density at X. In the next section, either $\ddot{k}$ or Γ will be considered as unknown (control inputs) hence introducing external energy to the system. Thus, the contribution of k to the constraint is $\Gamma^{\mathrm{T}} \ddot{k}$. Eventually, gathering results, the expression of the constraint at section X is:

$$R(X) = \left[\tfrac{1}{2} \left(\ddot{r}^{\mathrm{T}} \dot{\omega}^{\mathrm{T}} \right) \begin{pmatrix} \rho A I_3 & 0 \\ 0 & \rho I \end{pmatrix} \begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix} + \begin{pmatrix} -f_{\mathrm{ext}} \\ -c_{\mathrm{ext}} + (\omega \times (\rho I \omega)) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix} - \Gamma^{\mathrm{T}} \ddot{k} \right] \tag{4}$$

Define the 'analytical expression of the constraint' on the whole body as the sum: $R = \int_0^1 R(X)\,\mathrm{d}X$. Gauss least constraint principle stipulates that one has to search for the minimum of this quadratic form (in $\ddot{r}$ and $\dot{\omega}$), subject to the 'dynamical' equation (2), while considering either $\ddot{k}$ or Γ as unknown.

4. A non-homogeneous, singular, variable coefficients, optimal control problem

Results of the previous paragraphs suggest considering the following optimal control, variable coefficients problem (x is some independent variable):

$$\min_{\upsilon} J(\xi,\upsilon)=\int_{x_0}^{x_f}\left(\frac{1}{2}\xi^{\mathrm T}\Sigma\xi+b^{\mathrm T}\xi+c^{\mathrm T}\upsilon\right)\mathrm{d}x\quad \textbf{s.t. } \xi'=F\xi+G\upsilon+h \tag{5}$$

It is apparent that, because the control appears linearly in the functional, the optimization problem to be solved is a singular one [8, Chapter 8]. As it is a rather routine task, only the main steps are given for computing the optimal control (see [8,7] for details). Introducing a Lagrange multiplier (costate) λ, define the following Hamiltonian:

$$H=\frac{1}{2}\xi^{\mathrm T}\Sigma\xi+b^{\mathrm T}\xi+c^{\mathrm T}\upsilon+\lambda^{\mathrm T}(F\xi+G\upsilon+h) \tag{6}$$

First order necessary conditions for optimality [8] write:

$$\xi'=\frac{\partial H}{\partial\lambda}=F\xi+G\upsilon+h,\qquad \lambda'=-\frac{\partial H}{\partial\xi}^{\mathrm T}=-F^{\mathrm T}\lambda-\Sigma\xi-b,\qquad 0=\frac{\partial H}{\partial\upsilon}^{\mathrm T}=c+G^{\mathrm T}\lambda \tag{7}$$

A popular method of solution is the *sweep method*. It uses the fact that the costate λ can be written as an affine function of the state: $\lambda=\zeta\xi+\kappa$, thence amounts to finding an affine feedback law. As the last condition ($\frac{\partial H}{\partial\upsilon}=0$) does not give an expression for the optimal control, making the second derivative $\frac{\partial^2 H}{\partial\upsilon^2}$ vanish, one can say that the above first order necessary conditions together with the affine form for λ constitute a differential system of equations in (ξ,λ) that is not *formally integrable* because zero order equations are present. The search for formal integrability leads, after straightforward computations, to the expression of the optimal control υ^*:

$$\upsilon^*=\left(G^{\mathrm T}\Sigma G\right)^{-1}(K_1\lambda+K_2\xi+K_3) \tag{8}$$

where K_1,K_2,K_3 are intermediate quantities that are readily computed, and with $\lambda=\zeta\xi+\kappa$, computed thanks to the following two matrix–vector differential equations (compatibility conditions), for ζ (*Riccati*) and κ:

$$\begin{cases}\zeta'+\zeta G(G^{\mathrm T}\Sigma G)^{-1}K_1\zeta+\zeta(F+G(G^{\mathrm T}\Sigma G)^{-1}K_2)+F^{\mathrm T}\zeta+\Sigma=0\\ \kappa'+(F^{\mathrm T}+\zeta G(G^{\mathrm T}\Sigma G)^{-1}K_1)\kappa+\zeta h+\zeta G(G^{\mathrm T}\Sigma G)^{-1}K_3+b=0\end{cases} \tag{9}$$

Summarizing, to solve the optimal control problem (5), proceed along the following steps: given the data Σ,b,c,F,G,h, (i) solve Eq. (9) for ζ and κ; (ii) compute the optimal control with Eq. (8) while simultaneously solving the state equation in (5) for ξ.

5. The Newton–Euler continuous formalism

It is an easy task now, by identifying the data in Section 3 and those in Section 4, to write down the backward and forward equations for a continuous Newton–Euler formalism. Comparing problem (5) to Eqs. (4) and (2), let:

$$\begin{aligned}&x=X;\quad \xi=\begin{pmatrix}\ddot r\\ \dot\omega\end{pmatrix};\quad \lambda=\begin{pmatrix}n\\ M\end{pmatrix};\quad \upsilon=\ddot k;\quad F=\begin{pmatrix}0&-\hat t_1\\ 0&\hat k\end{pmatrix};\quad G=\begin{pmatrix}0\\ I_3\end{pmatrix};\quad h=\begin{pmatrix}\omega\times(\omega\times t_1)\\ \dot k\times\omega\end{pmatrix}\\ &\Sigma=\begin{pmatrix}\sigma_1 I_3&0\\ 0&\sigma_2\end{pmatrix};\quad b=\begin{pmatrix}-f_{\mathrm{ext}}\\ -c_{\mathrm{ext}}+(\omega\times(\rho I\omega))\end{pmatrix};\quad c=-\Gamma;\quad \sigma_1=\rho A;\quad \sigma_2=\rho I\end{aligned} \tag{10}$$

5.1. Backward algorithm

In the language of Newton–Euler formalism, the backward algorithm takes the accelerations as inputs and aims at giving the necessary joint torques (i.e., λ) as outputs. Firstly, writing down the optimality condition of the Hamiltonian wrt the control (third equation of (7)) gives: $-\Gamma+M=0$, as expected. Getting the searched for efforts thus simply amounts to solving the adjoint equation of the state equation in problem (5), i.e., using the second optimality equation in (7):

$$\begin{pmatrix} n \\ M \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ -\hat{t}_1 & \hat{k} \end{pmatrix}\begin{pmatrix} n \\ M \end{pmatrix} - \begin{pmatrix} \rho A I_3 & 0 \\ 0 & \rho I \end{pmatrix}\begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix} - \begin{pmatrix} -f_{\mathrm{ext}} \\ -c_{\mathrm{ext}} + (\omega \times (\rho I \omega)) \end{pmatrix} \tag{11}$$

5.2. *Forward algorithm*

The forward algorithm is known, in the usual discrete case, to be significantly more difficult to obtain than the backward one and to necessitate one recursion more. It aims at finding joint accelerations when joint torques are known. In the continuous case, it would be a really huge task to obtain an analogous algorithm when proceeding along the same path. It is in that respect that the analogy with optimal control shows real powerfulness, making these derivations systematic. Although tedious, they are nevertheless straightforward with the followed approach. The forward algorithm is thus, with given initial/boundary data and suitable parameters:

(1) Solve the following equations (see Eq. (9)) for ζ and κ, using the data identification set in (10):

$$\begin{cases} \zeta' + \zeta\begin{pmatrix} 0 & 0 \\ -\sigma_2^{-1}(\hat{t}_1' + \hat{k}\hat{t}_1) & \sigma_2^{-1}(\hat{k}' + \hat{k}\hat{k}) \end{pmatrix}\zeta + \zeta\begin{pmatrix} 0 & -\hat{t}_1 \\ \sigma_1\sigma_2^{-1}\hat{t}_1 & -\sigma_2'\sigma_2^{-1} - \hat{k} \end{pmatrix} \\ \quad + \begin{pmatrix} 0 & 0 \\ \hat{t}_1 & -\hat{k} \end{pmatrix}\zeta + \begin{pmatrix} \sigma_1 I_3 & 0 \\ 0 & \sigma_2 \end{pmatrix} = 0 \\ \kappa' + \left(\begin{pmatrix} 0 & 0 \\ \hat{t}_1 & -\hat{k} \end{pmatrix} + \zeta\begin{pmatrix} 0 & 0 \\ -\sigma_2^{-1}(\hat{t}_1' + \hat{k}\hat{t}_1) & \sigma_2^{-1}(\hat{k}' + \hat{k}\hat{k}) \end{pmatrix}\right)\kappa \\ \quad + \sigma_2^{-1}\zeta\begin{pmatrix} 0 \\ -\Gamma'' + (c_{\mathrm{ext}} - \omega \times (\rho I \omega))' - \hat{t}_1 f_{\mathrm{ext}} + \hat{k}(c_{\mathrm{ext}} - \omega \times (\rho I \omega)) \end{pmatrix} \\ \quad + \begin{pmatrix} -f_{\mathrm{ext}} \\ -c_{\mathrm{ext}} + (\omega \times (\rho I \omega)) \end{pmatrix} = 0 \end{cases} \tag{12}$$

(2) solve the state equation (Eq. (2)) for $\ddot{r}, \dot{\omega}$:

$$\begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix}' = \begin{pmatrix} 0 & -\hat{t}_1 \\ 0 & \hat{k} \end{pmatrix}\begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix} + \begin{pmatrix} 0 \\ \ddot{k} \end{pmatrix} + \begin{pmatrix} \omega \times (\omega \times t_1) \\ \dot{k} \times \omega \end{pmatrix} \tag{13}$$

still with initial/boundary data depending on the specific problem addressed while simultaneously computing the accelerations (i.e., the optimal control, Eq. (8)), which are given explicitly here as:

$$\begin{aligned} \sigma_1\ddot{k} = {} & \left[-(\hat{t}_1' + \hat{k}\hat{t}_1)\zeta_{11} + (\hat{k}' + \hat{k}\hat{k})\zeta_{21} + \sigma_1\hat{t}_1\right]\ddot{r} + \left[-(\hat{t}_1' + \hat{k}\hat{t}_1)\zeta_{12} + (\hat{k}' + \hat{k}\hat{k})\zeta_{22} - \sigma_2' I - 2\sigma_2\hat{k}\right]\dot{\omega} \\ & - (\hat{t}_1' + \hat{k}\hat{t}_1)\kappa_1 + (\hat{k}' + \hat{k}\hat{k})\kappa_2 - \Gamma'' + \left(c_{\mathrm{ext}} - \omega \times (\rho I \omega)\right)' - \hat{t}_1 f_{\mathrm{ext}} + \hat{k}\left(c_{\mathrm{ext}} - \omega \times (\rho I \omega)\right) \\ & - \sigma_2\dot{k} \times \omega \end{aligned} \tag{14}$$

where a block decomposition of matrix ζ, corresponding to $\ddot{r}, \dot{\omega}$, is used. As mentioned in Section 4, notice that the set of Eqs. (13) and (14) is actually non formally integrable (or an index two differential algebraic system [9]), due to the zero-order equation (in the space dimension) for $\ddot{k}$. Nevertheless, as $\ddot{k}$ is given explicitly, substitution into the state equation makes (13) an ordinary differential equation, which is obviously linear and inhomogeneous with variable coefficients. Thus, the forward algorithm is obtained straightforwardly. Among interesting applications, the one that motivated the present work concerns locomotion of an eel-robot, which can be modelled as a inextensible flexible beam with torque density as control. Obviously, the present approach is relevant to the study of other types of continua.

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