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On the impact of anisotropy on dispersion spectra of acoustic waves in plates

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Abstract

A brief overview is given of specific features that can (or cannot) appear in the dispersion spectra of traction-free elastic homogeneous plates due to anisotropy. Its effect on the overall spectral configuration and on the short and long wave trends is illuminated with a link to anisotropic traits of bulk and surface waves. Relevant classical and recent results are put together, and new points are established. ***To cite this article: A.L. Shuvalov, C. R. Mecanique 334 (2006).***

Résumé

Impact de l'anisotropie sur les spectres de dispersion des ondes élastiques dans des plaques. Une courte vue d'ensemble des traits spécifiquement liés à l'anisotropie, qui peuvent (ou ne peuvent pas) apparaître dans le spectre de dispersion des plaques élastiques homogènes libres est proposée. L'effet de l'anisotropie sur la configuration spectrale dans son ensemble, ainsi que sur ses limites petites et grandes longueurs d'onde, est mis en lumière à travers un lien avec les propriétés anisotropes des ondes de volume et des ondes de surface. Les résultats classiques les plus marquants, ainsi que les résultats les plus récents, sont rassemblés dans ce papier, et des points nouveaux sont établis. ***Pour citer cet article : A.L. Shuvalov, C. R. Mecanique 334 (2006).***

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Mots-clés : Acoustique ; Ondes ; Plaque élastique ; Anisotropie ; Spectre de dispersion

1. Introduction

Acoustic-wave propagation in anisotropic plates receives much attention from theoretical and practical points of view, see, e.g., [1–4]. It is well recognized, since the fundamental studies by Mindlin and coauthors [5–8], Solie and Auld [9], Li and Thompson [10] and others, that plate dispersion spectra can be essentially affected and complexified by anisotropy. This does not imply a plate material of necessarily low class of elastic symmetry. For any non-isotropic material, taking a non-symmetric orientation of plate faces and of propagation direction can unfold, in principle, all the variety of spectral ramifications. Anisotropy brings in new 'degrees of freedom' in the configuration and trends of

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dispersion curves, in the layout of their low- and high-frequency limits, etc. Are there any limitations on the existence of extreme points and non-monotonicity of the velocity branches? How does the shape of bulk-wave slowness surface affect the plate spectrum? Can all three low-frequency plate waves be slower than the bulk modes? Is it always the case that the Rayleigh velocity is a high-frequency limit for two fundamental branches, and what does happen when there is no or two Rayleigh waves? Knowing answers to these and other questions alike enables a qualitative prediction of the spectrum, which is useful for its computation and for thoughtful interpretation of the numerical results. Besides direct practical implication, it is both important and interesting to establish frontiers of spectral trends 'unlocked' by anisotropy.

The objective of the present short survey, which is certainly far from being all-inclusive, is to outline certain principal particularities that can (or cannot) occur in the dispersion spectrum of a free plate due to anisotropy. In this context, some of the classical results are put together with recent findings and some new observations are made. A detailed statement of the problem and rigorous discussion of analytical and numerical treatment of the dispersion equation are available in the ample literature, which may be traced via the bibliography provided. The motivation of this article is, however, to avoid equations and to adhere insomuch as possible to the pure reasoning grounds. Such manner is appropriate and helpful for a survey; moreover, as we shall see, it can sometimes yield a new insight at a remarkably 'low cost', saving lengthy derivations. Another methodological aspect of the exposition is highlighting and exploiting a link between the plate-wave perspective and the background of the surface-wave theory. To this end, it is pertinent to refer in the first lines to the recent reviews on surface and bulk acoustic waves [11–13] and on elastostatics [14] for generally anisotropic media. The present precis is hoped to contribute, to some extent, to the same purpose of elaborating the vast landscape of anisotropic elasticity.

For future use, introduce a nomenclature of geometry settings. They are identified in terms of anisotropy of three reference planes: the boundary plane $\mathbf{R_n}$, the sagittal plane $\mathbf{R_t}$, and the plane $\mathbf{R_m}$ orthogonal to them ($\mathbf{n}$ is the normal to the boundary, $\mathbf{m}$ is the wave normal, $\mathbf{t} = \mathbf{m} \times \mathbf{n}$). Either all these planes, or any one, or none of them may be symmetry planes. These options decide about factorization of the free-plate dispersion equation. Having symmetry planes along all three reference planes uncouples the shear-horizontal (*SH*) from the inplane branches, and splits the latter ones into two families of the symmetric and antisymmetric branches (S and A). Symmetry of $\mathbf{R_t}$ alone keeps the inplane/*SH* uncoupling. A single symmetry plane along either $\mathbf{R_n}$ or $\mathbf{R_m}$ ensures the uncoupled A and S families of wave branches, whose 3D displacements and tractions are appropriately symmetric/antisymmetric about midplane of the plate.[1] A generic geometry with non-symmetric reference planes lifts any uncoupling and corresponding intersections of dispersion branches. Obviously the strength of coupling induced by a certain dissymmetry is subject to the quantitative measure of this dissymmetry.

2. Background

Consider a traction-free infinite plate of a uniform thickness d, which consists of a linearly elastic non-dissipative material with constant density ρ and elastic coefficients c_{ijkl}. Plane acoustic waves, composed of bulk (homogeneous) and/or inhomogeneous partial modes, propagate along the plate with a real trace velocity v depending on a real horizontal wavenumber k or frequency $\omega = kv$. As a starting point, recall the configuration of the dispersion spectrum in the form $v(k)$ or $v(\omega)$ for inplane (Lamb) waves in isotropic plates. The two fundamental branches originate at $k, \omega = 0$: the flexural branch goes up (from zero velocity) and the extensional branch goes down (from the beam velocity) towards the Rayleigh wave velocity v_{R}, approaching it at high k, ω exponentially ($\propto \mathrm{e}^{-kd}$) from the opposite sides. The upper branches start at the cutoff frequencies of thickness resonances and uniformly decrease in k (monotonicity in ω is subject to the sign of group velocity $\mathrm{d}\omega/\mathrm{d}k$). They form a terrace-like structure above the longitudinal wave velocity v_l, then drop down and ultimately tend to the transverse wave velocity v_t. The short wave rate $\propto k^{-2}$ of the upper branches flattening near v_l and of their asymptotical tendency to v_t has been described by Mindlin [16,17] (see also [18,19]) by way of the grids of bounds $v_{l,t}[1 + \frac{1}{2}(\pi n_{l,t}/kd)^2]$, $n_l, n_t = 1, 2, \ldots$. Juxtapos-

[1] The case of a single symmetry plane along $\mathbf{R_m}$, studied in [15] and [5–8] but rather scarcely mentioned since then, differs from the well-elaborated treatment of the case of symmetric $\mathbf{R_n}$ by mere interchanging the vector components along $\mathbf{m}$ and $\mathbf{n}$. This implies a difference in partitioning into symmetric/antisymmetric families. For instance, the flexural branch belongs to a different symmetry type than the two other fundamental branches when $\mathbf{R_n}$ is a single symmetry plane, but it falls into the same type as the quasi-SH_0 branch when $\mathbf{R_m}$ is a single symmetry plane (note a misprint for the latter case in [21]).

ing this picture with the *SH* branches, unfastening shear horizontal and vertical wave velocities ($v_t \to v_{SH} \neq v_{SV}$), and switching on the inplane/*SH* coupling produces a basic pattern of the spectrum for anisotropic plates. Its various aspects have been investigated through Mindlin's method of bounds and an explicit analysis of the uncoupled symmetric/antisymmetric dispersion equations for orthorhombic and monoclinic plates in the pioneer papers [5–10] and in many subsequent studies; the Stroh formalism has been applied for arbitrary anisotropic plates in [20–23]. Some of the principal spectral attributes remain typical or invariant in the presence of anisotropy: e.g., there always exist three fundamental branches, normally two of them tend exponentially to v_{R}; the terracing patterns of the upper branches and their short wave limit (incorporating also the third fundamental branch) at the bulk-wave threshold usually maintain the k^{-2}-trend. At the same time, this common framework leaves plenty of room for novel properties due to anisotropy.

3. Effect of anisotropy of bulk-wave slowness surface

3.1. Preamble: the transonic states (TS)

The problems involving planar interfaces essentially depend on the local geometry of the bulk-wave slowness surface $\mathbf{S}$ at the so-called transonic states (TS) [24–26]. For a given orientation of $\mathbf{m}$ and $\mathbf{n}$, the occurrence of a TS at velocity $v = v^{(t.s.)}$ implies that the slowness curve(s) in the cut by $\mathbf{R_t}$ touches the straight line parallel to $\mathbf{n}$, and hence that (usually) a pair of partial modes merge into a grazing mode (a bulk mode with energy flux lying in the boundary plane $\mathbf{R_n}$). It propagates along the slowness vector directed to the point of tangency and has the trace velocity $v^{(t.s.)}$, which is the inverse of this slowness vector projection on $\mathbf{m}$. The transonic state with the least $v^{(t.s.)}$ ($= v_L$, see Section 4) is specified as the first TS,[2] the rest are termed the subsequent TS. For a fixed $\mathbf{m}$ and $\mathbf{n}$, there are may be at least 3 and at most 15 TS [24]. If $\mathbf{R_n}$, $\mathbf{R_m}$, $\mathbf{R_t}$ are symmetric, then all three bulk modes travelling along $\mathbf{m}$ are assuredly grazing and so their velocities $v = v^{(\mathbf{m})}$ indicate the TS: $v^{(\mathbf{m})} = v^{(t.s.)}$; if $\mathbf{R_n}$ or $\mathbf{R_m}$ is a single symmetry plane, then at least one of these bulk modes is grazing. Other TS, arising due to anisotropic distortion of $\mathbf{S}$ (concavities and/or misorientation about $\mathbf{m}$), are oblique relatively to $\mathbf{m}$ in the sense that they are related to obliquely propagating grazing modes. There may be up to three grazing modes with a given $v^{(t.s.)}$—e.g., these occur in pairs symmetric with respect to $\mathbf{m}$ when $\mathbf{R_n}$ and/or $\mathbf{R_m}$ are symmetry planes.

3.2. Fundamental branches

An oblique TS gives rise to partial inhomogeneous modes with nonzero real parts of their complex vertical wavenumbers. The resulting phase interference permits the exponentially flattening Rayleigh plateau of the pair of fundamental branches to acquire a non-uniform, rippling shape. The effect is especially pronounced in the case of $\mathbf{R_n}$ and/or $\mathbf{R_m}$ being symmetry planes and the first TS being oblique. Then for $v \sim v_{\mathrm{R}}$ there are two pairs of interfering inhomogeneous modes, which are equally prominent at the plate faces; moreover, the symmetric/antisymmetric uncoupling allows intersection of the fundamental branches and hence their weaving pattern with multiple crossings. It has been demonstrated in [9,10] for orthorhombic and monoclinic plates with symmetric $\mathbf{R_n}$. Remarkably, when both $\mathbf{R_n}$, $\mathbf{R_m}$ and hence $\mathbf{R_t}$ are symmetry planes, the weaving pattern of the inplane fundamental A_0 and S_0 branches, if it occurs, has all the crossing points at exactly the Rayleigh velocity v_{R} [10]. Fig. 1 replots two typical examples of this case from [9,10], with v_{R} lying below or above the SH_0 non-dispersive branch (see Section 4). Among other things,[3] the weaving structure of fundamental branches is interesting for it necessitates an occurrence of local extrema of the curves $v(k)$ or $v(\omega)$, which is strictly ruled out if the sagittal plane $\mathbf{R_t}$ is isotropic. Note to this end that, in the case of anisotropy, the short wave e^{-kd}-asymptote applies rather to the envelopes of the fundamental branches above and below the Rayleigh plateau, whereas their possible modulation within the overall exponential trend follows from a more detailed calculation.

[2] In the context of Section 3, if $\mathbf{R_t}$ is a symmetry plane then the first TS is tacitly understood as the first *inplane* one. In Sections 4, 5, the first TS at v_L implies the overall bulk-wave threshold, related in the case of symmetric $\mathbf{R_t}$ to either the inplane or the *SH* modes.

[3] Note that all crossing points of symmetric and antisymmetric branches for a free plate remain the (real-valued) solutions when this plate is loaded from one side by an arbitrary non-viscous fluid [21].

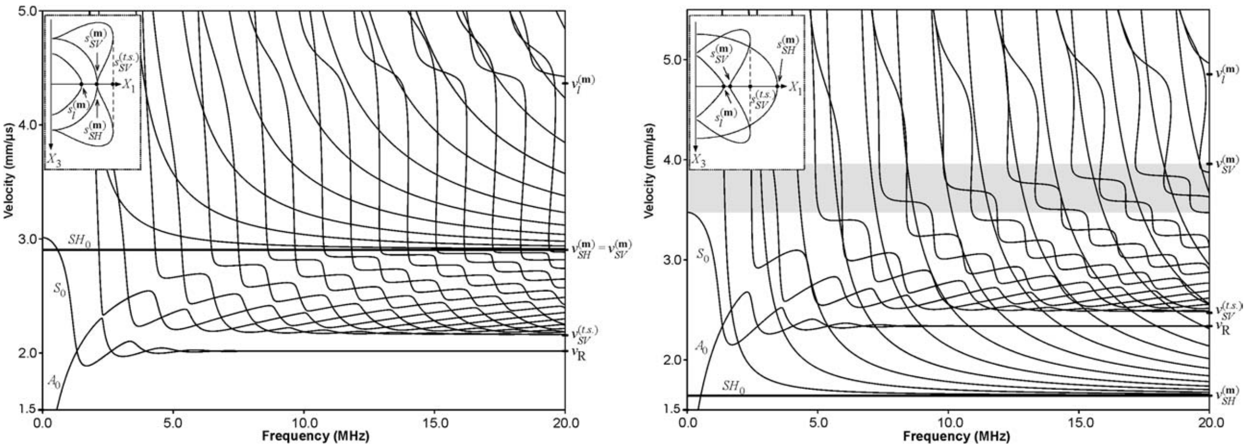

Fig. 1. The dispersion spectra $v(\omega)$ (a) for a cubic copper plate, after [9], and (b) for an orthorhombic SB copper plate, after [10], both with the orientation $\mathbf{m} \parallel X_1, \mathbf{n} \parallel X_3$ so that the boundary and sagittal planes are symmetry planes. The spectra are plotted by means of the DISPERSE package [27]. The long wave onset of the flexural branch is left out; the frequency is re-scaled to $f = \omega/2\pi$. The insets in the upper left corner show the bulk-wave slowness curves in the sagittal plane ($s = v^{-1}$ is the horizontal slowness). The elasticity coefficients of the SB copper [10] ensure that the concave TS at $v_{SV}^{(\mathbf{m})}$ lies above the long wave origin point of the S_0-branch; however, we have further increased c_{55} (140 instead of 109.4 GPa in [10]) in order to blow up the shaded area of the spectrum (b), which is discussed in Sections 3 and 5. The curvatures q of the TS at $v_l^{(\mathbf{m})}$, $v_{SV}^{(\mathbf{m})}$ and $v_{SV}^{(t.s.)}$ are: $q \approx 12.6$, -9.5 and 5.1 mm/μs for (a); 27.5, -25.0 and 6.8 mm/μs for (b).

3.3. Short wave asymptotes for the upper branches

Anisotropy of the slowness surface **S** greatly diversifies the short wave behaviour of the upper branches at their limit towards the first TS and at their intermediate broken plateaux related to the subsequent TS.

For one, anisotropy brings into play the curvature of TS, i.e., of the slowness curve(s) at the corresponding tangency point. The role of curvature at the first (convex) TS has been brought out in [23]. The predominate trend of the dispersion branches $v(k)$, tending from above to the velocity v_L of the first TS with the curvature q, has been found to be $v(k) - v_L \propto \frac{1}{2}q(\pi n/kd)^2$ (cf. Section 2, where $v_L = v_t = q$ for the isotropic case). Extending this consideration to the branch-terracing near the velocity $v^{(t.s.)}$ of other TS yields a similar formula, now with q standing for the subsequent TS curvature (and with $v^{(t.s.)}$ instead of v_L). Thus it is readily seen that if a given TS is concave ($q < 0$) and so the bulk modes tend to grazing propagation when v increases towards $v^{(t.s.)}$, then the corresponding terracing structure of plate branches approaches $v^{(t.s.)}$ from below—unlike the case of isotropy. It also follows that, generally speaking, the flattening and extent of spectral plateaux near a subsequent TS is scaled up or down by the smaller or greater curvature of this TS. For the first or subsequent TS with a zero curvature $q = 0$ [25,26], the power rate $k^{-\sigma}$ of the short wave asymptote has an exponent $\sigma > 2$, where σ is the order of lowest non-zero angular derivative of slowness curve at the point of its tangency with **n**.

In the similar sense as mentioned above with regard to the fundamental branches, it is carefully noted that the k^{-2}-asymptote, determined by the TS curvature, estimates only an overall collective trend of the short wave limit and of the terracing structure near, respectively, the first and subsequent TS. The shape of individual branches within the collective trend is subject to the particularities of the given TS (whether it is oblique,[4] involves one grazing mode or more, etc.), its proximity to neighbouring TS, and in fact other causes. In the case of symmetric $\mathbf{R_n}$ and/or $\mathbf{R_m}$, a detailed arrangement of the symmetric/antisymmetric dispersion branches near TS may be elaborated through Mindlin's method of bounds. It explains [9,10] the weaving patterns, which ascend towards the concave TS and descend towards the convex TS (like in Fig. 1). What is significant is that, given an increasing in k trend of the terracing pattern tending as a whole to a concave TS, the individual branches $v(k)$ inside those spectral clusters may

[4] The uncoupled *SH* branches and their derivatives remain monotonic even if the *SH* transonic state is oblique ($v_{SH}^{(t.s.)} \neq v_{SH}^{(\mathbf{m})}$, which is when $\mathbf{R_t}$ is a single symmetry plane).

be decreasing—see the high-frequency range of the shaded area in Fig. 1(b). The reason for such behaviour is not directly related to the TS properties and will be addressed in Section 5.

4. Link to the surface-wave theory

4.1. Leaky waves and supersonic surface waves (SSW)

Another 'channel of influence' of anisotropy on the plate dispersion spectra is via the behaviour of surface waves, which are responsible for the short wave limit of, normally, the fundamental branches. This perspective is linked inevitably to the theory of surface waves developed by Lothe, Barnett, Chadwick, Alshits, Ting and others (see bibliography in [28] and [11,12]). There, the trace velocity at the first TS is termed the limiting velocity v_L and is said to separate subsonic ($v < v_L$) and supersonic ($v > v_L$) ranges. The surface wave usually occurs as a unique subsonic Rayleigh wave but certain orientations in an anisotropic halfspace admit a two-partial supersonic surface wave (SSW), which, upon generic perturbation of the geometry, couples with the bulk mode and gives rise to the branch of pseudo-Rayleigh, or leaky, waves with complex velocity [29]. A major insight into the admissible options is rendered by the Barnett–Lothe theorem [30]. It states uniqueness of the subsonic Rayleigh wave and restricts its possible non-existence to particular occasions, for which the bulk-wave threshold v_L admits a single grazing bulk mode leaving the surface free of traction and termed exceptional (see [11] for the precise formulation and its track record).

A particular clarity pertains to the case when the sagittal plane $\mathbf{R_t}$ is a symmetry plane and so the *SH* and inplane motions are uncoupled. Then the inplane (two-partial) Rayleigh wave always exists [31]. On varying elastic coefficients, its velocity can freely pass through the first TS once it is associated with the *SH*, certainly exceptional, grazing mode: $v_L = v_{SH}^{(t.s.)}$. Thus the unique inplane Rayleigh wave is either subsonic, $v_R < v_L$, or supersonic (or transonic), $v_R \geqslant v_L = v_{SH}^{(t.s.)}$. The latter is often specified as the symmetrical SSW—to emphasize that $\mathbf{R_t}$ is a symmetry plane. Analyzing its implication for the plate dispersion spectrum has been another significant stride made by Solie and Auld [9]; further important qualitative and analytical considerations to this end are due to [10] and [23]. The occurrence of a symmetrical SSW $v_R > v_L$ means that, in contrast to the typical spectral configuration, the short wave extent of two inplane fundamental branches tending to the Rayleigh velocity v_R is supersonic. Both branches are either steady or form a weaving pattern, subject to the shape of the inplane slowness curves as discussed in Section 3. Perturbing the orientation of $\mathbf{R_t}$ so that it is no longer a symmetry plane modifies the symmetrical SSW into the leaky wave and brings back a 'slightly subsonic' (quasi-bulk [32,33]) Rayleigh wave with velocity $v_R \lesssim v_L$, splitting from the quasi-*SH* first TS. This velocity is now the short wave limit for the quasi-SH_0 branch and for the flexural branch, whose shape transforms appropriately to restore the subsonic extent. Notably, if $\mathbf{R_n}$ is symmetric, those two fundamental branches can cross each other and then it is the flexural branch which tends to the subsonic Rayleigh velocity v_R from above—rather than from below as usual. Simultaneously, the supersonic extents of the 'former' fundamental inplane branches, which have been exponentially approaching the SSW velocity $v_R > v_L$, now break due to coupling with the (quasi) *SH* family and transform into prolonged segments of successive pairs of the upper branches. They form a pairwise terracing pattern (of a somewhat different 'microstructure' than that near TS), which is asymptotically close at high frequency to the real part of the leaky-wave velocity in the measure of its imaginary part. This pattern is more distinct given a steady supersonic extent of the fundamental branches for the symmetric orientation of $\mathbf{R_t}$; otherwise their unfolding may produce a fairly intricate picture, probably complicated by the other trends towards neighbouring TS. Instructive examples and observations are provided in [9,10,23].

The state of affairs, predicted by the Barnett–Lothe theorem, is more diverse in the general case of a non-symmetric sagittal plane $\mathbf{R_t}$. Now the options for surface wave do not reduce to the subsonic/supersonic alternative. Firstly, there exists a possibility, exemplified by Chadwick [34], that the exceptional wave at v_L is not accompanied by any surface wave at all. In this case, two dispersive fundamental branches at high frequency tend to the non-dispersive one $v(k) = v_L$ from above with the k^{-2}-rate alongside the upper branches. Another special occasion, relatively more common, is concerned with those SSW, termed secluded or non-symmetrical, which come about for certain lines of $\mathbf{m}$ and $\mathbf{n}$ orientations in 3D space Ψ^3 of surface-wave geometry parameters [35,36]. Such orientations do not generally admit an exceptional wave at v_L and in that ensure the subsonic Rayleigh wave—thus two surface waves, above and below v_L, coexist. The subsonic surface-wave velocity $v_R < v_L$ is, as usual, the short wave limit of two fundamental plate branches. At first glance, the non-symmetrical SSW might also be thought to set the limit for a pair of plate branches. It is noted, however, that, in contrast to the case of inplane/*SH* uncoupling, the packet of plate modes with the velocity $v_R > v_L$ of a non-symmetrical SSW must include the bulk modes propagating through reflections from the plate faces. The reflection is uncoupled from the SSW within the boundary problem for a halfspace (single free surface) [37], but not for a plate once $\mathbf{R_t}$ is not a symmetry plane. That is why the velocity $v_R > v_L$ of a non-symmetrical SSW cannot be a limit of continuous plate branches, as is the case for a symmetrical SSW, but it leads instead to the pairwise terracing of successive branches, which is basically similar to their formation in the case of leaky wave evolving from a symmetrical SSW. An analytical account for this behaviour has been given in [23]. A small perturbation of $\mathbf{m}$ and $\mathbf{n}$ away from the line of their orientations, admitting the non-symmetrical SSW, also transforms the latter into a leaky wave; however, this has rather a slight quantitative effect on the terracing pattern (see [23]). From the inverse viewpoint, it may thus be said that the vanishing of the imaginary part of leaky-wave velocity does not imply a topological transformation of the plate spectrum, unless it vanishes due to the confluence of $\mathbf{R_t}$ with a symmetry plane, in which case SH-uncoupling 'heals up' the broken terracing into the continuous curves of two inplane fundamental branches tending to the symmetrical SSW velocity $v_R > v_L$.

4.2. Degenerate modes

Surface-wave solutions in anisotropic media can have certain particularities due to degeneracy of complex vertical wavenumbers, see [38]. Their double degeneracy, occurring on the lines of $\mathbf{m}, \mathbf{n}$ orientations in the space Ψ^3 of surface-wave geometry [39], renders the Rayleigh wave with a linear, on top of exponential, dependence on the vertical coordinate. When so, then the short wave tendency of the fundamental branches towards v_R becomes of the order of $\sqrt{kd}\,\mathrm{e}^{-kd}$, which is slower than the usual rate e^{-kd} (it may be related to a weaker localization at the plate faces). Of methodological interest for plates is a one-partial, necessarily supersonic, surface wave stipulated by degeneracy yet retaining usual, pure exponential form [40,41]. It is classified in [38] as W1A. Its occurrence entails a non-dispersive, hence fundamental, plate branch consisting of two inhomogeneous plane modes. This is, however, a sheer theoretical possibility, because W1A can come about only for appropriately fixed model values of material constants.

4.3. Exceptional waves

It is clear from the definition of an exceptional wave, possible among one-partial bulk grazing modes at various TS (not necessarily at the first TS as in the context of the Barnett–Lothe theorem[5]), that it always corresponds to a non-dispersive fundamental branch in the plate dispersion spectrum. (The inverse is true to within barring a theoretical possibility of the W1A-branch.) Chadwick has shown [24] that, for a fixed orientation of $\mathbf{m}$ and $\mathbf{n}$, there may be up to two exceptional waves with different trace velocity. Following [24], their plausible existence can be exemplified in a simple way by taking a symmetric sagittal plane $\mathbf{R_t}$, so that there is an exceptional SH mode yielding the non-dispersive SH_0 branch $v_{SH}^{(t.s.)}$ for any $\mathbf{m}$ in $\mathbf{R_t}$, and then finding the angular orientation φ of such propagation direction $\mathbf{m}$ in $\mathbf{R_t}$, for which the longitudinal bulk mode with velocity $v_l^{(\mathbf{m})}$ is also exceptional and hence also produces a non-dispersive branch. The equation to solve is $c_{iijj}(\varphi) = 0$, where $X_i \parallel \mathbf{m}$, $X_j \parallel \mathbf{n}$, and it yields an appropriate root for $\sin^2 2\varphi$ subject to a rather stringent but quite tenable inequality on material constants (see details in [24, pp. 219, 220]). The Rayleigh surface wave for the case in hand always exists and can always be arranged as subsonic relatively to the SH_0 branch: $v_R < v_L = v_{SH}^{(t.s.)}$. Thus, here is an example ensuring the plate spectrum with subsonic v_R and two non-dispersive fundamental branches (here, $v_{SH}^{(t.s.)}$ and $v_l^{(\mathbf{m})}$). The Rayleigh velocity $v_R < v_L$ must render a limit for two dispersive velocity branches, so one is the 'remaining' fundamental, flexural branch, whereas the other, contrary to the typical spectral layout, is not a fundamental branch but extends from a thickness-resonance frequency. To underline the usefulness of the complementary surface-wave and plate-wave perspectives, it is noted that a ban for more than two exceptional waves proved in [24] becomes readily evident from the fact that there are three fundamental plate branches and one of them (flexural) is always dispersive.

[5] It can also be a *one-partial degenerate* bulk mode travelling along an acoustic axis of the tangent type, rather than a *single* mode as specified in this theorem formulation, see Section 3.5 of [11].

5. Non-monotonicity in the velocity spectrum and the long wave onset of the fundamental branches

5.1. Auxiliary notations

In this section, we denote the velocities of fundamental plate waves in the long wave limit $\omega, k = 0$ by $v_\alpha(0) \equiv v_\alpha^{(0)}$ ($\alpha = 1, 2, 3$), reserving $\alpha = 1$ for the flexural branch ($v_1^{(0)} = 0$). In the case of symmetric $\mathbf{R_t}$, the velocities $v_2^{(0)}, v_3^{(0)}$ are specified as $v_{\text{ext}}^{(0)}$ for the inplane (quasi) extensional branch (S_0 if $\mathbf{R_n}$ is also symmetric) and $v_{SH}^{(t.s.)}$ for the SH_0 non-dispersive branch.

5.2. Non-monotonicity

Let us return to the issue of non-monotonicity and extreme points of velocity branches. Recall first the state of affairs for an isotropic plate. In this case, the branches $v(k)$ are strictly decreasing. In view of the identity $\mathrm{d}v/\mathrm{d}k = g(\mathrm{d}v/\mathrm{d}\omega)$, where $g = \mathrm{d}\omega/\mathrm{d}k \gtreqless 0$ is the (inplane) group velocity which is finite, the branches $v(\omega)$ admit an increasing trend due to $g < 0$ and so may be non-monotonic, but they cannot have extreme points. Switching monotonicity of velocity branches via a horizontal tangent (g stays positive) is thus an anisotropy-stipulated feature. Fig. 1 provides its evident examples related to the weaving pattern of the fundamental branches and to the concave TS. Another possible reason may be a flexural branch tending from below to the symmetrical SSW velocity (Section 4): on turning $\mathbf{R_t}$ away from the symmetry plane, the break-up of the flexural branch yields an envelope of increasing regions of the otherwise decreasing upper branches. These examples are not at all the only possibilities. At the same time, the velocity interval, where $v(k)$ and $v(\omega)$ may have extrema, is bounded.

According to [42], the branches $v(k)$ must be uniformly decreasing (the sign of derivative of $v(\omega)$ must be inverse to the sign of g) above the horizontal level V, which is laid by the long wave limit of one of the fundamental wave velocities: it is $\max(v_2^{(0)}, v_3^{(0)})$ if $\mathbf{R_t}$ is not a symmetry plane, and $v_{\text{ext}}^{(0)}$ ($\leqslant v_{SH}^{(t.s.)}$) if $\mathbf{R_t}$ is a symmetry plane; in other words, V is the largest of $v_\alpha^{(0)}$, which is not itself the SH_0-branch velocity $v_{SH}^{(t.s.)}$. Thus the curves $v(k)$ and $v(\omega)$ admit extreme points and resulting changes of monotonicity only for $v \leqslant V$ or, more precisely, for $v < V$ except at the point $\omega, k = 0$ itself and except in the particular case when V is set by a non-dispersive branch (which is not SH_0).

One of the immediate consequences is that the long wave onset of the fastest fundamental velocity branch cannot bend upwards—either it bends downwards or it is non-dispersive (see [20]). The existence of the monotonicity bound V also underlies the spectral feature remarked in Section 3 and highlighted by Fig. 1(b). For the plate material and geometry in hand, this bound $V = v_{\text{ext}}^{(0)} = \sqrt{(c_{11} - c_{13}^2/c_{33})/\rho}$ lies below the concave TS $v^{(t.s.)} = v_{SV}^{(\mathbf{m})} = \sqrt{c_{55}/\rho}$ (see the shaded area in Fig. 1(b)). Hence, the dispersion branches within the terracing pattern, ascending to the concave TS $v^{(t.s.)} > V$, must switch to the decreasing trend above V.

Interestingly, subjecting the plate to fluid loading may cause the real part of the fastest dispersive fundamental branch, which is now complex, to change its trend and to bend upwards (see [43]). It is also noteworthy that the aforementioned bound of $v(k)$ branches non-monotonicity is no longer unreservedly valid for piezoelectric plates.

5.3. Can all three fundamental branches start in the subsonic range?

Consider the long wave onset $\omega, k \to 0$ of the fundamental velocity branches $v_\alpha(\omega)$ or $v_\alpha(k)$. The flexural branch $v_1(\omega)$ emerges from zero velocity as a subsonic wave ($v < v_L$), thus consisting of the inhomogeneous modes only. What can be said in this regard about the two upper fundamental branches? For an isotropic plate, the S_0 extensional branch always starts off as a supersonic wave: its origin point given by the beam velocity $v = 2v_t\sqrt{1 - v_t^2/v_l^2}$ lies above the SH_0 non-dispersive branch $v = v_t$, which represents the bulk-wave threshold v_L (the first TS). Provided $\mathbf{R_t}$ is a symmetry plane in an anisotropic plate, the origin $v_{\text{ext}}^{(0)}$ of the (quasi) extensional branch may lie above or below the SH_0 branch $v_{SH}^{(t.s.)}$. For any one of these options, the SH_0 branch may be supersonic ($v_{SH}^{(t.s.)} > v_L$) so that the first TS is associated with an inplane mode, or the SH_0 velocity may be the first TS itself ($v_{SH}^{(t.s.)} = v_L$). Assume the latter case and also $v_{\text{ext}}^{(0)} < v_L$, which is a fairly common setup. Now let us perturb the orientation of $\mathbf{R_t}$ away from the symmetry plane so that the inplane/SH coupling is on. Among other possibilities, one may expect that the onset of the now dispersive quasi-SH_0 branch could slip under v_L and thus that all three fundamental branches would originate below v_L as packets of inhomogeneous partial modes. This speculation leads us to the general question—in principle, can all three fundamental waves at $\omega, k \to 0$ be subsonic, i.e., can $\max(v_2^{(0)}, v_3^{(0)})$ be strictly less than v_L?

The answer to this question is negative. Here is a simple proof, based on the existence of monotonicity bound V. The presence of an exceptional wave, at v_L or above, and/or of the one-partial, hence supersonic, surface wave W1A (see Section 4) a priori negates the conjecture. Consider a generic case when there is neither exceptional wave nor W1A, i.e., all three fundamental velocity branches are dispersive. Assume that $v_L > \max(v_2^{(0)}, v_3^{(0)})$. Non-existence of an exceptional wave guarantees a subsonic Rayleigh wave by the Barnett–Lothe theorem, hence at $k \to \infty$ two fundamental branches $v_\alpha(k)$ tend to $v_{\mathrm{R}} < v_L$ and the third one tends to v_L. The branch tending to v_L would have to reach $\max(v_2^{(0)}, v_3^{(0)})$, assumed below v_L, and to increase above this level. But any branch $v(k)$ must be decreasing in the range $v > \max(v_2^{(0)}, v_3^{(0)})$ $(= V$ for the case in hand). Thus here is a contradiction which proves that the initial assumption is false and so $v_L \leqslant \max(v_2^{(0)}, v_3^{(0)})$.

5.4. Relation of $v_\alpha^{(0)}$ to Rayleigh and bulk-wave velocities

Explicitly, the velocities and polarizations of the free-plate fundamental waves in the limit $\omega, k = 0$ are defined [20] by the eigenspectrum of the left off-diagonal block $\mathbf{N}_3$ of the Stroh matrix $\mathbf{N}$, which plays a profound role in anisotropic elasticity (see [28]). This link enables some useful inequalities for $v_\alpha^{(0)}$. Ting [28, p. 472] has presented an identity showing that ρv_{R}^2 is less than the largest eigenvalue of $\mathbf{N}_3$. The eigenvalues of $\mathbf{N}_3$ are equal to $\rho v_\alpha^{(0)2}$; thus it immediately follows that the Rayleigh velocity is less than the largest of the long-wave plate velocities: $v_{\mathrm{R}} < \max(v_2^{(0)}, v_3^{(0)})$ (note a misprint of formulation in [20]). In the case of a symmetric sagittal plane $\mathbf{R_t}$, Ting's identity leads to the especially simple relation $v_{\mathrm{R}} = v_{\mathrm{ext}}^{(0)} |\mathbf{A}_{\mathrm{R}} \cdot \mathbf{m}|$, where $\mathbf{A}_{\mathrm{R}}$ is the unit-normalized polarization of Rayleigh wave. Another observation involves the velocities $v^{(\mathbf{m})}$ of three bulk waves travelling along $\mathbf{m}$: the largest of them is always greater or equal than $\max(v_2^{(0)}, v_3^{(0)})$ [20]. Equality in the latter statement is a special option exemplified by the longitudinal exceptional wave in Section 4.

By evaluating the long wave limit $\omega, k = 0$ of free-plate waves, the eigenspectrum of $\mathbf{N}_3$ certainly incorporates the possibilities of non-dispersive branches related to the exceptional waves or, theoretically, to the W1A surface wave. For any orientation of $\mathbf{m}$ and $\mathbf{n}$, both eigenvectors of $\mathbf{N}_3$ associated with nonzero eigenvalues are normal to $\mathbf{n}$, hence so is the polarization of exceptional waves and of W1A, which is in agreement with the well-known result of the surface-wave theory.

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