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C. R. Mecanique 334 (2006) 311–316

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Exact analytic solutions for the damped Duffing nonlinear oscillator

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Received 16 September 2005; accepted after revision 23 February 2006

Presented by Maurice Roseau

Abstract

We prove that the second-order damped nonlinear Duffing oscillator is reduced to an equivalent equation of the normal Abel form of the second kind. Based on a recently developed mathematical methodology for the construction of exact analytic solutions of Abel's equation, exact analytic solutions are obtained for the nonlinear damped Duffing oscillator obeying the initial conditions adapted to the physical problem. To improve the general developed methodology an application concerning the nonlinear Van der Pol free oscillator is briefly discussed. ***To cite this article: D.E. Panayotounakos et al., C. R. Mecanique 334 (2006).***

Résumé

Les solutions analytiques exactes pour le amortissable non linéaire oscillateur de Duffing. Nous montrons que l'oscillateur de Duffing amorti peut être réduit à une équation équivalente à la forme normale d'équation d'Abel de seconde espèce. Sur la base d'une méthode développée récemment pour construction des solutions analytiques exactes de ce type d'équations d'Abel, des solutions analytiques exactes sont obtenues pour l'oscillateur de Duffing amorti, satisfaisant aux conditions initiales conformes au problème physique sousjacent. Pour illustrer la généralité de la méthode, une application à l'oscillateur de van der Pol est brièvement discutée. ***Pour citer cet article : D.E. Panayotounakos et al., C. R. Mecanique 334 (2006).***

Keywords: Analytical mechanics; Damped Duffing nonlinear oscillator; Abel's equation

Mots-clés : Mécanique analytique ; Non linéaire oscillateur de Duffing ; Équation d'Abel

1. Introduction

Whereas weakly nonlinear oscillators with weak damping can be solved approximately using techniques such as, averaging [1] or multiple-scaling [2], exact solutions of damped oscillators with strong nonlinearities are still lacking. Perturbation problems addressing approximately damped free nonlinear oscillators include [3–5]. In [6] it was proved that there is no exact analytic solution for the damped Duffing oscillator without linear stiffness terms (NLD equation). This unsolvability is due to the fact that the problem leads to an Abel equation of the normal form $yy'_x - y = f(x)$

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doi:10.1016/j.crme.2006.03.009

which does not admit exact analytic solutions in terms of known (tabulated) functions for $f(x)$ arbitrary. Only special cases of equations of this type can be analytically solved [7,8].

In this Note we prove that the damped (NLD) oscillator without linear stiffness terms is reduced to an equivalent Abel equation of the second kind of the normal form $yy'_x - y = f(x)$. Based on a recently developed mathematical methodology [9,10] concerning the construction of exact analytic solutions for this type of Abel's equation, exact analytic solutions are obtained for the nonlinear damped Duffing oscillator obeying initial conditions in accordance with the physical problem. The solving methodology introduced is general, and can be applied to a large class of unsolvable second order nonlinear ordinary differential equations (ODEs) of mathematical physics and nonlinear mechanics including nonlinear waves, nonlinear oscillations, flow in porous media etc. To improve the developed methodology an application to nonlinear oscillations of the Van der Pol type is also briefly discussed.

2. Some results on the class of Abel's equations—a new mathematical construction

Before we address the issue of construction of exact analytic solutions for the nonlinear Duffing oscillator, let us start with a digression on the known admissible functional transformations concerning the class of the Abel nonlinear ODEs of the second and the first kind.

The general form of Abel's equation of the second kind is

$$\big[g_1(x)u + g_0(x)\big]u'_x = f_2(x)u^2 + f_1(x)u + f_0(x) \tag{1}$$

It is well known that there exist admissible functional transformations [8, pp. 46; 50] able to reduce (1) to the normal form

$$yy'_x - y = f(x) \tag{2}$$

In addition, it is also well known that transformation, $u(x) = 1/y(x)$ reduces the restricted form of the Abel equation of the first kind, $u'_x = f_3(x)u^3 + f_2(x)u^2 + f_1(x)u$, to the Abel equation of the second kind (1).

The solution of (2) can be generally obtained in parametric form by differentiation, or through the use of integrating factors ([7, p. 27], [8, pp. 32–35, 45–48]). We stress that for $f(x)$ arbitrary, Eq. (2), and thus (1), are unsolvable in terms of known (tabulated) functions. Special cases of restricted forms for (1) and (2) admitting exact parametric solutions are tabulated in [8, pp. 50–55, 29–45].

However, a mathematical methodology was recently developed in [9] and [10], leading to the construction of exact analytic solutions of the Abel equation of the second kind of the normal form (2), and thus of Eq. (1). Here, we will present the final results of this construction.

Let us consider the Abel ODE (2). According to what was mentioned above, the solution of this equation is given as follows:

$$y(x) = \frac{1}{2}(x + 2\lambda)\left[\bar{N}(x) + \frac{1}{3}\right], \quad \bar{N}'_x = \frac{4(G+2f)}{(x+2\lambda)^2[\bar{N}(x) + \frac{4}{3}]} \tag{3}$$

where $G(x)$ is a subsidiary function defined below, while $\bar{N}(x)$ is one of the roots of the cubic equation of Cardano's form:

$$\bar{N}^3(x) + p\bar{N}(x) + q = 0 \tag{4}$$

that is to say, one of the following six functions:

Case a: $Q < 0$ $(p < 0)$

$$\bar{N}_1(x) = 2\sqrt{-\frac{p}{3}}\cos\frac{a}{3}, \qquad \bar{N}_2(x) = -2\sqrt{-\frac{p}{3}}\cos\frac{a-\pi}{3}, \qquad \bar{N}_3(x) = -2\sqrt{-\frac{p}{3}}\cos\frac{a+\pi}{3}$$
$$\cos a = -\frac{q}{2\sqrt{-(\frac{p}{3})^3}}, \quad 0 < a < \pi \tag{5}$$

Case b: $Q > 0$

$$\bar{N}(x) = \sqrt[3]{-\frac{q}{2} + \sqrt{Q}} + \sqrt[3]{-\frac{q}{2} - \sqrt{Q}} \tag{6}$$

Case c: $Q = 0$

$$\bar{N}_1(x) = 2\sqrt[3]{-\frac{q}{2}}, \qquad \bar{N}_2(x) = \bar{N}_3(x) = -\sqrt[3]{-\frac{q}{2}} \tag{7}$$

In these expressions $Q(x)$, $p(x)$ and $q(x)$ are given by

$$Q(x) = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2, \quad p = -\frac{a^2}{3} + b,\ q = 2\left(\frac{a}{3}\right)^3 - \frac{ab}{3} + c \tag{8}$$

where

$$a = -4, \qquad b = 3 + \frac{4[G(x) + f(x)]}{(x + 2\lambda)}, \qquad c = -\frac{4[G(x) + 2f(x)]}{(x + 2\lambda)} \tag{9}$$

In all the above formulae we define:

$$G(\bar{\xi}) = \frac{1}{16} \frac{[(\bar{\xi}\sin\bar{\xi} + \cos\bar{\xi})A(\bar{\xi}) + \cos^2\bar{\xi}][4\bar{\xi}A(\bar{\xi}) + \cos\bar{\xi}]}{[\bar{\xi}A(\bar{\xi})]^3} \mathrm{e}^{-\bar{\xi}} - 2f(\bar{\xi}) \tag{10}$$

where:

$$\bar{\xi} = \ln|x + 2\lambda|$$

$$A(\bar{\xi}) = \mathrm{ci}(\bar{\xi}) = \text{the cosine integral} = \mathbf{C} + \ln\bar{\xi} + \sum_{v=1}^{\infty}(-1)^v \frac{\bar{\xi}^{2v}}{(2v)(2v)!}$$

$$\mathbf{C} = \text{Euler's number} = 0.5772156649015325\ldots$$

$$\lambda = \text{integration constant}$$

$$f(\bar{\xi}) = \text{the second member of the original Abel equation}$$

Summarizing, the solution of the Abel equation (2) is given by Eqs. (3) to (10). We must underline here that, because of the various types that function $\bar{N}(x)$ admits (formulae (5) to (7)), it is possible that the solution of the prescribed equation (2) is not unique inside the main interval $[x_0, x_1]$ (or $[\bar{\xi}_0, \bar{\xi}_1]$); in other words, it can be divided into several branches of solutions valid separately inside consecutive subintervals. In this case matching of the corresponding solutions must be performed in all values of the main interval that solution changes.

3. Reduction of the (NLD) free oscillator

The governing equation of the damped Duffing oscillator is given by [11,2]:

$$y''_{xx} + \lambda_1 y'_x + \lambda_2 y^3 = 0, \quad -\infty < x < +\infty,\ -\infty < y < +\infty,\ \lambda_1, \lambda_2 > 0 \tag{11}$$

where λ_1 and λ_2 are fixed parameters. This equation is complemented by the initial conditions

$$\text{for } x = 0, \quad y = y_0 \text{ and } y'_x = y'_0 \tag{12}$$

Here, the notation $y'_x = \mathrm{d}y/\mathrm{d}x$, $y''_{xx} = \mathrm{d}^2y/\mathrm{d}x^2, \ldots$ is used for the total derivatives.

The classical transformation

$$y'_x = p(y), \quad y''_{xx} = p'_x = pp'_y \tag{13}$$

reduces (11) into the nonlinear ODE of the first order

$$pp'_y + \lambda_1 p + \lambda_2 y^3 = 0, \quad -\infty < y < +\infty,\ -\infty < p < +\infty \tag{14}$$

Also, by means of the coordinate transformations

$$p(y) = -\alpha n(r), \quad r = \beta y \tag{15}$$

Eq. (14) becomes

$$\alpha^2\beta n n'_r - \alpha\lambda_1 n + \frac{\lambda_2}{\beta^3} r^3 = 0$$

with the following substitutions:

$$\alpha^2\beta = \alpha\lambda_1 = \frac{\lambda_2}{\beta^3} \quad \Rightarrow \quad \alpha = \frac{\lambda_1^2}{\sqrt{\lambda_2}},\ \beta = \frac{\sqrt{\lambda_2}}{\lambda_1} \tag{16}$$

we arrive at the following Abel equations of the second kind in the normal form:

$$nn'_r - n = -r^3, \quad -\infty < r < +\infty,\ -\infty < n < +\infty \tag{17}$$

For the initial conditions $x = 0$, $y = y_0$ and $y'_x = y'_0$, by means of (13) we conclude that $p(y_0) = p_0 = y'_0$ and thus, the corresponding initial conditions for the Abel equation (17) are transformed as follows:

$$\text{for } x = 0, \quad y = y_0, \quad y'_x = y'_0 \quad \Rightarrow \quad r = r_0 = \frac{\sqrt{\lambda_2}}{\lambda_1} y_0 \quad \text{and} \quad n = n_0 = -\frac{\sqrt{\lambda_2}}{\lambda_1^2} y'_0 \tag{18}$$

It was proved in [6] that Eq. (17) does not admit exact analytic solutions in terms of known (tabulated) functions. This unsolvability is due to the special form of the right-hand side of the Abel equation (17) including cubic nonlinearities.

In what follows, using the results presented in Section 2, we provide an exact analytic solution for the problem under consideration, that is for the (NLD) equation (11) through the already constructed exact analytic solutions of the Abel equation of the second kind in its normal form.

3.1. Exact analytic solutions of the (NLD) equation

We now consider the damped Duffing oscillator (Eq. (11)), which, as it was proved in Section 3, can be reduced to the Abel equation of the second kind in the normal form (17). We shall apply the already developed mathematical methodology (Section 2), in order to formulate the exact analytic solution of Eq. (17), which constitutes the intermediate integral of the original Duffing problem (11). We should underline here that in [12] the authors performed analytical solutions of the (NLD) equation (11) by means of an appropriate form of Taylor series applied to the solution of the Abel equation (17), that is the intermediate integral of Eq. (11).

Let us apply the analytic solutions (3) to (10) to the case of the Abel equation

$$nn'_r - n = -r^3, \quad -\infty < r < +\infty,\ -\infty < n < +\infty$$

For $r < 0$ (or $r > 0$) we divide the main interval $(-\infty, 0]$ (or $[0, +\infty)$) to consecutive subintervals $[0, r_1]$, $(r_1, r_2], \ldots,$ $(r_{n-1}, r_n], \ldots$ in each of which we suppose that a different solution of the above Abel equation (according to the formulae (10) in combination with (5) to (7)) holds true. All quantities corresponding in the ith subinterval $(i, i+1]$ are characterized by the upper-right index in parenthesis (i).

Using the analytical process developed here, the solution of Eq. (17) can be constructed by the following procedure:

Step 1: For the initial condition $r = r_0 = 0$, $n(r_0) = n^{(0)} = \mu > 0$, $\mu =$ given constant, Eq. (17) satisfies $n'_r(0) = n'^{(0)}_r = 1$, while the second of (3) together with the expression

$$n'_r = \frac{1}{2}\left(\bar{N} + \frac{1}{3}\right) + \frac{1}{2}(r + 2\lambda)\bar{N}'_{\bar{\xi}}\bar{\xi}'_r = \frac{1}{2}\left(\bar{N} + \frac{1}{3}\right) + \frac{2(G - 2r^3)}{(\bar{N} + \frac{4}{3})(r + 2\lambda)}$$

and for $r = r_0 = 0$, it satisfies

$$1 = \frac{1}{2}\left(\bar{N}^{(0)} + \frac{1}{3}\right) + \frac{G^{(0)}}{\lambda^{(0)}(\bar{N}^{(0)} + \frac{4}{3})} \tag{19}$$

In addition, the solution (10) in case of the prescribed initial conditions, satisfies the equations:

$$\lambda^{(0)}\left(\bar{N}^{(0)} + \frac{1}{3}\right) = \mu$$
$$\frac{G^{(0)}}{\lambda^{(0)}} = \frac{\{[\ln|2\lambda^{(0)}|\sin(\ln|2\lambda^{(0)}|) + \cos(\ln|2\lambda^{(0)}|)]A^{(0)} + \cos^2(\ln|2\lambda^{(0)}|)\}(4\ln|2\lambda^{(0)}|A^{(0)} + \cos(\ln|2\lambda^{(0)}|))}{16[\ln|2\lambda^{(0)}|A^{(0)}]^3} \tag{20}$$

where

$$A^{(0)} = \mathrm{ci}\big(\ln\big|2\lambda^{(0)}\big|\big) = \mathbf{C} + \ln\big(\ln\big|2\lambda^{(0)}\big|\big) + \sum_{v=1}^{\infty}(-1)^{v}\frac{[\ln|2\lambda^{(0)}|]^{2v}}{(2v)(2v)!}$$

Eqs. (19) and (20) constitute a nonlinear (transcendental) system by the solution of which one estimates $\bar{N}^{(0)}$, $G^{(0)}$ and $\lambda^{(0)}$ in terms of the initial parameters. Therefore, since $\bar{N}^{(0)}$ is known, $Q(x)$ defined in (8) is also known, fact that permits us to define the type of the solution $\bar{N}^{(0)}(r)$ valid inside the first subinterval $[0, r_1]$ (Eqs. (5) to (7)).

Step 2: At the unknown point $(r_1, n_{r_1}^{(1)})$, where the solution changes, we provide the following equations:

(i) *Common solution*

$$\left(\bar{N}_{r_1}^{(1)} + \frac{1}{3}\right)\big(r_1 + 2\lambda^{(1)}\big) = \left(\bar{N}_{r_1}^{(0)} + \frac{1}{3}\right)\big(r_1 + 2\lambda^{(0)}\big) \tag{21}$$

(ii) *Common derivative*

$$\left(\bar{N}_{r_1}^{(1)} + \frac{1}{3}\right) + \frac{2(G_{r_1}^{(1)} - 2r_1^3)}{(r_1 + 2\lambda^{(1)})(\bar{N}_{r_1}^{(1)} + \frac{4}{3})} = \left(\bar{N}_{r_1}^{(0)} + \frac{1}{3}\right) + \frac{2(G^{(0)} - 2r_1^3)}{(r_1 + 2\lambda^{(0)})(\bar{N}_{r_1}^{(0)} + \frac{4}{3})} \tag{22}$$

(iii) *Validity of Eq.* (10)

$$\frac{4(G_{r_1}^{(0)} - 2r_1^3)}{r_1 + 2\lambda^{(1)}} = \frac{[(\bar{\xi}_{r_1}^{(1)}\sin\bar{\xi}_{r_1}^{(1)} + \cos\bar{\xi}_{r_1}^{(1)})A_{r_1}^{(1)} + \cos^2\bar{\xi}_{r_1}^{(1)}](4\bar{\xi}_{r_1}^{(1)}A_{r_1}^{(1)} + \cos\bar{\xi}_{r_1}^{(1)})}{4(\bar{\xi}_{r_1}^{(1)}A_{r_1}^{(1)})^3}$$

$$A_{\bar{\xi}_1}^{(1)} = \mathrm{ci}\big(\bar{\xi}_{r_1}^{(1)}\big), \quad \bar{\xi}_{r_1}^{(1)} = \ln\big|r_1 + 2\lambda^{(1)}\big| \tag{23}$$

(iv) *Validity of Eq.* (4)

The nonlinear (transcendental) system of Eqs. (4) and (21) to (23) enables us to estimate r_1, $\bar{N}_{r_1}^{(1)}$, $G_{r_1}^{(1)}$ and $\lambda^{(1)}$ in terms of the initial parameters as well as the known parameters $\bar{N}_{r_1}^{(0)}$, $G_{r_1}^{(0)}$ and $\lambda^{(0)}$. Thus, at the point $(r_1, n_{r_1}^{(1)})$, where the solution changes, and for the consecutive subinterval $(r_1, r_2]$, $Q_{r_1}^{(1)}$ given in Eq. (8) is known. This enables us to define the type of the solution valid inside $(r_1, r_2]$ (Eqs. (5) to (7)).

The prescribed analysis demands successive solutions of nonlinear (transcendental) systems which are must be satisfied for each of the above mentioned consecutive subintervals $(r_i, r_{i+1}]$, where the solution of the problem under consideration in the phase plane changes according to the formulae (5) to (7) .The above analytical technique completes the solution of the (NLD) damped oscillator. We underline that the solution in the physical plane is obtained by a direct integration through transformation (13).

3.2. Discussion and conclusions

By a series of admissible functional transformations we reduce the nonlinear (NLD) equation without linear terms to an Abel equation of the second kind of the normal form. This equation does not admit exact analytic solutions in terms of known (tabulated) functions. This unsolvability is due to the fact that only very special forms of this kind of equation can be solved in parametric form [8]. Our goal is the development of the construction of exact analytic solution of the (NLD) equation based on a mathematical technique leading to the derivation of exact analytic solutions of the Abel equation of the second kind of the normal form (see [9,10]).

The reduction procedure introduced in the paper and the constructed solutions are very general, and can be applied to a large number of nonlinear ODEs in mathematical physics and nonlinear mechanics including the Van der Pol nonlinear oscillator, the Blasius equation in fluids [10], the Langmuir equation in current flow, the Kidder equation in porous media, etc.

For the strengthening of the above contentions, we shall examine briefly one of the above equations and compare its solution methodology with the already developed Duffing's solution procedure.

The Van der Pol free nonlinear oscillator [11] is governed by the following nonlinear ODE

$$y''_{xx} - \varepsilon\big(1 - y^2\big)y'_x + y = 0; \quad -\infty < x < +\infty \tag{24}$$

where ε is a real positive parameter. By the substitution

$$y'_x = w(y) \quad \Rightarrow \quad y''_{xx} = w'_y y'_x = w w'_y \tag{25}$$

Eq. (24) is reduced to the following Abel equation of the second kind:

$$w w'_y - \varepsilon\left(1 - y^2\right) w + y = 0, \quad -\infty < y < +\infty \tag{26}$$

Introducing the coordinate transformations

$$w(y) = \omega(s), \quad s = \varepsilon y - \varepsilon \frac{y^3}{3} \tag{27}$$

we obtain the Abel equation of the second kind of the normal form

$$\omega\omega'_s - \omega = -\frac{1}{\varepsilon}\frac{y}{1-y}, \quad y^3 - 3y + \frac{3}{\varepsilon}s = 0; \quad \varepsilon > 0, \ -\infty < s < \infty, \ -\infty < x < +\infty \tag{28}$$

Eqs. (28) are of the Abel normal form and they are similar to the Eq. (17) of the Duffing problem with different right-hand sides. With a set of similar initial conditions as in Duffing's case, the mathematical methodology for the exact analytic solutions of the above equations follows step by step that prescribed in Sections 3 and 4.

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