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Numerical approximation of a viscoelastic frictional contact problem

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Abstract

We consider a fully discrete scheme for a quasistatic frictional contact problem between a viscoelastic body and an obstacle. The contact is bilateral, the friction is modeled with Tresca's law and the behavior of the material is described with a viscoelastic constitutive law with long memory. We state an existence and uniqueness result for the discrete solution, followed by error estimate results. Then, we present numerical simulations in the study of a two-dimensional test example. ***To cite this article: Á. Rodríguez-Arós et al., C. R. Mecanique 334 (2006).***

Résumé

Approximation numérique d'un problème viscoélastique de contact avec frottement. Nous considérons un schéma totalement discrétisé pour un problème quasistatique de contact avec frottement entre un corps viscoélastique et un obstacle. Le contact est bilatéral, le frottement est modélisé à l'aide de la loi de Tresca et le comportement du matériau est décrit à l'aide d'une loi viscoélastique à mémoire longue. Nous présentons un résultat d'existence et d'unicité pour la solution discrète, suivi des résultats d'estimation de l'erreur. Nous présentons aussi des simulations numériques dans l'étude d'un exemple test en dimension deux. ***Pour citer cet article : Á. Rodríguez-Arós et al., C. R. Mecanique 334 (2006).***

Keywords: Friction; Viscoelastic frictional contact; Fully discrete scheme; Finite element method; Error estimates; Duality-penalization algorithm; Numerical simulations

Mots-clés : Frottement ; Contact viscoélastique avec frottement ; Schéma totalement discrétisé ; Méthode des élements finis ; Estimation de l'erreur ; Algorithme de dualité-pénalisation ; Simulations numériques

Version française abrégée

Nous considérons un corps viscoélastique occupant un ouvert $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, de frontière Γ suffisamment régulière, divisée en trois parties disjointes et mesurables Γ_1, Γ_2 et Γ_3, telle que $\operatorname{meas}(\Gamma_1) > 0$. Soit $\boldsymbol{\nu}$ le vecteur

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unitaire de la normale sortante à Γ et soit $[0, T]$ un intervalle de temps, $T > 0$. Le corps est supposé fixé sur la partie Γ_1 de sa frontière alors que des forces volumiques et surfaciques de densités $\boldsymbol{f}_0$ et $\boldsymbol{f}_2$ agissent respectivement dans Ω et sur Γ_2. Sur la partie Γ_3 le corps est en contact bilatéral avec un obstacle. Le contact est frottant et modélisé à l'aide de la loi de frottement de Tresca. Ce type de conditions sont rencontrées dans certains mécanismes lorsqu'on sait à priori que le contact est maintenu tout au long du processus et on a connaissance, au moins d'une façon approchée, de la contrainte normale. Nous supposons que le processus est quasistatique et le comportement du matériau est viscoélastique à mémoire longue. Sous ces hypothèses, le problème mécanique considéré peut être formulé de la façon suivante :

Problème $\boldsymbol{P}$. Trouver un champ des déplacements $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$ ainsi qu'un champ contraintes $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$ satisfaisant aux relations (1)–(6).

Ici et partout dans ce travail le point au dessus représente la dérivée par rapport au temps, $\|\boldsymbol{v}\|$ est la norme du vecteur $\boldsymbol{v}$ et les indices ν et τ indiquent les composantes normales et tangentielles des tenseurs et des vecteurs. L'Éq. (1) est la loi de comportement où $\boldsymbol{\varepsilon}(\boldsymbol{u})$ dénote le tenseur des déformations linéarisé, $\mathcal{A}$ est le tenseur des coéfficients élastiques et $\mathcal{B}$ est le tenseur de relaxation. L'Éq. (2) représente l'équation d'équilibre, (3) et (4) sont respectivement les conditions aux limites de déplacement-traction et (5) représentent les conditions de contact bilatéral avec frottement de Tresca, g étant le seuil de frottement. Enfin, le champ $\boldsymbol{u}_0$ intervenant dans (6) représente le déplacement initial.

Soient Q et V les espaces de Hilbert définis dans (7) et soient $\boldsymbol{f}$ et j les applications définies dans (8). La formulation variationnelle du Problème P, en déplacements, est la suivante :

Problem $\boldsymbol{P_V}$. Trouver un champ des déplacements $\boldsymbol{u} : [0, T] \to V$ satisfaisant l'inéquation variationnelle (9) avec la condition initiale $\boldsymbol{u}(0) = \boldsymbol{u}_0$.

L'existence et l'unicité de la solution du Problème P_V a été obtenue dans [5]. La difficulté majeure dans la démonstration de ce résultat réside dans le fait que le problème est d'évolution, d'une part, ainsi que dans la présence du terme intégral, d'autre part. Cette dernière difficulté a été surmontée en utilisant une méthode de point fixe.

Dans ce travail nous nous intéressons à l'approximation numérique du Problème P_V, en supposant que les hypothèses listées dans [5] sont satisfaites. Nous utilisons des espaces d'éléments finis $V^h \subset V$ ainsi qu'une partition uniforme de pas k pour l'intervalle de temps $[0, T]$, i.e., $[0, T] = \bigcup_{n=1}^{N}[t_{n-1}, t_n]$ où $t_n = nk$, $k = T/N$, N étant un entier strictement positif. Nous considérons les problèmes discrétisés, notés P_V^{hk}, et nous prouvons que, si le pas de temps k est suffissament petit, ces problèmes possèdent une solution unique $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^{N} \subset V^h$. Pour cela, nous utilisons à nouveau une méthode de point fixe, affin de gérer le terme intégral discrétisé. Par ailleurs, sous des hypothèses supplémentaires de régularité, nous prouvons qu'il existe une constante $c > 0$, indépendante des paramètres de discrétisation h et k, telle que l'estimation (10) est vraie. La démonstration de cette inégalité est basée sur une version discrète du lemme de Gronwall. En utilisant l'inégalité (10) ainsi que des résultats classiques d'interpolation, nous prouvons que si la solution du problème continu a la régularité $\dot{\boldsymbol{u}} \in C([0, T]; [H^2(\Omega)]^d)$ et $\boldsymbol{u}_0 \in [H^2(\Omega)]^d$, alors l'estimation (11) est satisfaite. Les démonstrations de ces résultats peuvent être trouvées dans [10].

Afin de vérifier la précision et l'efficacité de la méthode numérique ci-dessus, nous avons mis en oeuvre quelques experiences numériques en dimension 2 que nous résumons dans la Fig. 1. Nous précisons que dans la résolution du problème P_V^{hk} nous avons utilisé, à chaque pas de temps, une méthode itérative associée à un algorithme de dualité-pénalisation (voir [11]). Bien que la présence du terme de mémoire dans le loi de comportement conduit à des difficultées supplémentaires liées au stockage des données, notre conclusion est que l'algorithme utilisé s'est bien comporté, sa convergence est rapide et les résultats obtenus sont fiables.

1. Introduction

Contact phenomena involving deformable bodies abound in industry and everyday life. For this reason, considerable progress has been made in their modeling and analysis, and the engineering literature concerning this topic is rather extensive (see [1–4]). The present Note is devoted to the numerical analysis of a problem of bilateral frictional contact which leads to a new an interesting mathematical model. The process is quasistatic and the friction is modeled with the well known Tresca's law. The behavior of the material is described with a linear viscoelastic constitutive law with long memory. The variational analysis of the problem was provided in [5], where the existence of a unique weak solution to the model was proved. In the present paper we describe a fully discrete scheme for the problem involving finite difference discretization in time and finite element discretization in space, then we implement it in a computer code and provide numerical simulations. The Note is organized as follows. In Section 2 we present the contact problem, together with its unique weak solvability. Our main interest is in Section 3, where we state new results in the study of the fully discrete approximation of the model, including the existence of a unique discrete solution and error estimates. In Section 4 we present numerical simulations in the study of a two-dimensional test problem. To this end, we use an algorithm which combines a fixed point strategy with a method of duality-penalization. This algorithm represents an alternative to the Lagrangian multipliers method used in the study of various frictional contact problems (see [6]).

2. The model and its well-posedness

The physical setting is the following. A viscoelastic body occupies a regular domain Ω of $\mathbb{R}^d$ $(d = 2, 3)$ with boundary Γ partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\mathrm{meas}(\Gamma_1) > 0$. The body held fixed on Γ_1, is acted upon by body forces and tractions on Ω and Γ_2, respectively, and it is in bilateral contact on Γ_3 with an obstacle, the so-called foundation. We are interested in the evolution process of the mechanical state of the body in the time interval $[0, T]$ with $T > 0$. The model for this mechanical problem is the following.

Problem P. Find a displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$ such that, for all $t \in [0, T]$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big)\,\mathrm{d}s \quad \text{in } \Omega \tag{1}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in } \Omega \tag{2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1 \tag{3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2 \tag{4}$$

$$\begin{aligned} &u_\nu(t) = 0, \quad \big\|\boldsymbol{\sigma}_\tau(t)\big\| \leqslant g \\ &\quad \big\|\boldsymbol{\sigma}_\tau(t)\big\| < g \;\Rightarrow\; \dot{\boldsymbol{u}}_\tau(t) = \boldsymbol{0},\; \big\|\boldsymbol{\sigma}_\tau(t)\big\| = g \;\Rightarrow\; \exists\lambda \geqslant 0 \text{ s.t. } \boldsymbol{\sigma}_\tau(t) = -\lambda\dot{\boldsymbol{u}}_\tau(t) \text{ on } \Gamma_3 \end{aligned} \tag{5}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega \tag{6}$$

Here and below $\boldsymbol{\nu}$ denotes the unit outer normal on Γ, the subscripts ν and τ represent the *normal* and *tangential* components of vectors or tensors, respectively, and the dot above indicates the derivative with respect to the time; $\mathbb{S}^d$ is the space of second order symmetric tensors on $\mathbb{R}^d$, while "$\cdot$" and $\|\cdot\|$ represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively. Eq. (1) is the viscoelastic constitutive law where $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor, $\mathcal{A} = (\mathcal{A}_{ijkl})$ represents the fourth order tensor of elastic coefficients and $\mathcal{B} = (\mathcal{B}_{ijkl})$ is the relaxation tensor, see [1,7], for instance. Eq. (2) represents the equilibrium equation where $\boldsymbol{f}_0$ is the density of volume forces. Relations (3) and (4) are the displacement and traction boundary conditions, respectively, in which $\boldsymbol{f}_2$ is the density of surface tractions acting on Γ_2. Conditions (5) are the frictional contact conditions. Equality $u_\nu(t) = 0$ on Γ_3 shows that there is no loss of the contact during the process, that is, the contact is bilateral. Such kind of conditions may be found in many machines and in moving parts and components of mechanical equipment. The rest of conditions in (5) represent Tresca's law of dry friction where $g \geqslant 0$ is a given function, the friction bound. Considering g as given simplifies the analysis considerable and allows to prove the uniqueness of the solution. The relationship between the Coulomb and Tresca condition, including points out to a possible transition from the first to the second one can be found in [8]. Finally, (6) is the initial condition in which $\boldsymbol{u}_0$ is given.

We turn now to the variational formulation of Problem P. To this end, we use the Hilbert spaces

$$Q = \big\{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega),\; i, j = \overline{1, n}\big\}, \quad V = \big\{\boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1,\; v_\nu = 0 \text{ on } \Gamma_3\big\} \tag{7}$$

together with their canonical inner products $(\cdot,\cdot)_Q$, $(\cdot,\cdot)_V$ and associate norms $\|\cdot\|_Q$, $\|\cdot\|_V$, respectively.

Let $\boldsymbol{f}:[0,T]\to V$ and $j:V\to\mathbb{R}_+$ be defined by

$$\big(\boldsymbol{f}(t),\boldsymbol{v}\big)_V=\int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}+\int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,\mathrm{d}a,\qquad j(\boldsymbol{v})=\int_{\Gamma_3} g\|\boldsymbol{v}_\tau\|\,\mathrm{d}a\quad\forall\boldsymbol{v}\in V,\ t\in[0,T]\tag{8}$$

Proceeding in a standard way we obtain the following variational formulation of the contact problem (1)–(6), in terms of displacement.

Problem P_V. Find a displacement field $\boldsymbol{u}:[0,T]\to V$ such that $\boldsymbol{u}(0)=\boldsymbol{u}_0$ and

$$\begin{aligned}&\left(\mathcal{A}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)+\int_0^t\mathcal{B}(t-s)\boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big)\,\mathrm{d}s,\boldsymbol{\varepsilon}\big(\boldsymbol{v}-\dot{\boldsymbol{u}}(t)\big)\right)_Q+j(\boldsymbol{v})-j\big(\dot{\boldsymbol{u}}(t)\big)\\&\quad\geqslant\big(\boldsymbol{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t)\big)_V\quad\forall\boldsymbol{v}\in V,\text{ a.e. }t\in(0,T)\end{aligned}\tag{9}$$

The existence of a unique solution $\boldsymbol{u}\in W^{1,2}(0,T;V)$ of Problem P_V was proved in [5] by using an abstract result for a class of evolutionary variational inequalities with Volterra-type integral term. The main difficulty in studying this problem arise from the fact that the problem is evolutionary and, on the other hand, it contains a Volterra integral term. This last difficulty was treated by using a fixed point method. In the rest of the Note we assume the conditions stated in [5] which implies that Problem P_V has a unique solution.

3. Fully discrete approximation

We now consider a family of fully discrete schemes to approximate Problem P_V. We assume that Ω is a polyhedron. Let $\mathcal{T}_h$ be a finite element triangulation of $\overline{\Omega}$ composed by d-simplex, compatible with the boundary decomposition $\Gamma=\overline{\Gamma}_1\cup\overline{\Gamma}_2\cup\overline{\Gamma}_3$, i.e., any point where the boundary condition type changes is a vertex of the triangulation. We denote by $h>0$ the maximum diameter of triangles of $\mathcal{T}^h$ and we introduce the following finite element space associated to $\mathcal{T}^h$:

$$V^h=\left\{\boldsymbol{v}^h=(v_i^h)\in\left[C\big(\overline{\Omega}\big)\right]^d,\boldsymbol{v}^h_{|T^h}\in\left[P^1\big(T^h\big)\right]^d\ \forall T^h\in\mathcal{T}^h,\ \boldsymbol{v}^h=\boldsymbol{0}\text{ on }\overline{\Gamma}_1,\ v^h_\nu=0\text{ on }\overline{\Gamma}_3\right\}$$

Here $P^m(T^h)$ is the space of polynomials of degree less or equal to m on d variables. Also, we denote by $\mathcal{P}^h:V\to V^h$ the operator given by $(\mathcal{P}^h\boldsymbol{v},\boldsymbol{v}^h)_V=(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\varepsilon}(\boldsymbol{v}^h))_Q\ \forall\boldsymbol{v}\in V,\ \boldsymbol{v}^h\in V^h$.

In addition to the finite-dimensional subspace V^h, we use a uniform partition of the time interval: $[0,T]=\bigcup_{n=1}^N[t_{n-1},t_n]$ with $t_n=nk$, where $k=T/N$ is the step-size. Below, the symbol $\delta\boldsymbol{u}_n$ denotes the backward divided differences of $\boldsymbol{u}$. For each time step t_n, the constants $\alpha_j^n>0$ $(0\leqslant j\leqslant n)$ denote the weights of the composed trapezoidal quadrature formula of $n+1$ points in $[0,t_n]$. With this notation, a family of fully discrete approximation schemes to Problem P_V is the following.

Problem P_V^{hk}. Find $\boldsymbol{u}^{hk}=\{\boldsymbol{u}_n^{hk}\}_{n=0}^N\subset V^h$ such that $\boldsymbol{u}_0^{hk}=\boldsymbol{u}_0^h=\mathcal{P}^h\boldsymbol{u}_0$ and, for $n=1,2,\dots,N$,

$$\left(\mathcal{A}\boldsymbol{\varepsilon}\big(\boldsymbol{u}_n^{hk}\big)+\sum_{j=0}^n\alpha_j^n\mathcal{B}(t_n-t_j)\boldsymbol{\varepsilon}\big(\boldsymbol{u}_j^{hk}\big),\boldsymbol{\varepsilon}\big(\boldsymbol{v}^h-\delta\boldsymbol{u}_n^{hk}\big)\right)_Q+j\big(\boldsymbol{v}^h\big)-j\big(\delta\boldsymbol{u}_n^{hk}\big)\geqslant\big(\boldsymbol{f}_n,\boldsymbol{v}^h-\delta\boldsymbol{u}_n^{hk}\big)_V\quad\forall\boldsymbol{v}^h\in V^h$$

By using arguments similar to those used in [9], we obtain the following result:

Theorem 3.1. *Assume that k is sufficiently small. Then, there exists a unique solution $\boldsymbol{u}^{hk}=\{\boldsymbol{u}_n^{hk}\}_{n=0}^N\subset V^h$ to Problem P_V^{hk}. In addition, assume that $\boldsymbol{u}\in W^{2,\infty}(0,T;V)$, $\mathcal{B}\in W^{1,\infty}(0,T;[L^\infty(\Omega)]^{d^4})$, and assume also that $\mathcal{B}$ and $\dot{\mathcal{B}}$ are Lipschitz continuous functions on $[0,T]$ with values to $[L^\infty(\Omega)]^{d^4}$. Then the following error estimate holds*:

$$\max_{1\leqslant n\leqslant N}\big\|\boldsymbol{u}(t_n)-\boldsymbol{u}_n^{hk}\big\|_V^2\leqslant c\Big(k^2+\big\|\boldsymbol{u}_0-\boldsymbol{u}_0^{hk}\big\|_V^2+\max_{1\leqslant n\leqslant N}\Big\{\inf_{\boldsymbol{v}^h\in V^h}\big\{\big\|\dot{\boldsymbol{u}}(t_n)-\boldsymbol{v}^h\big\|_V\big\}\Big\}\Big)\tag{10}$$

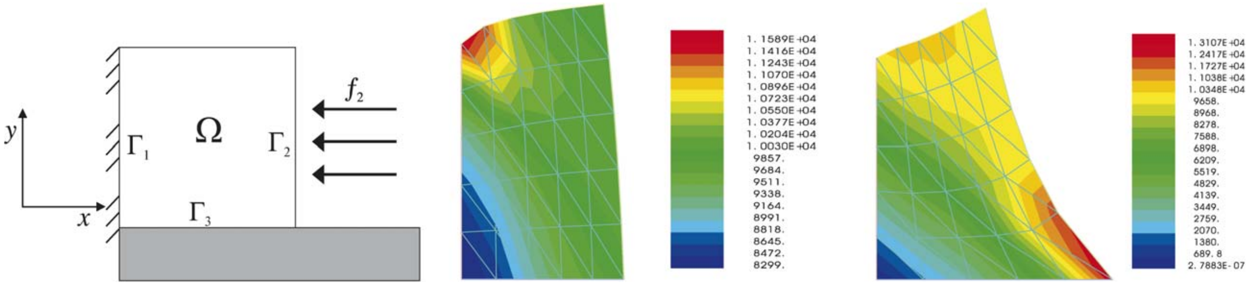

Fig. 1. Physical setting (left), deformed configuration and stress distribution in Von Mises norm for g *small* (centre) and g *large* (right).

Fig. 1. Le problème physique (gauche), configuration deformée et distribution des contraintes dans la norme de Von Mises pour g *petit* (centre) et g *grand* (droite).

where c is a positive constant which depends on data, but is independent on the discretization parameters h and k.

We refer to [10] for the proof of Theorem 3.1. There, the unique solvability of Problem P_V^{hk} is obtained by using a fixed point method related to the discretized integral term and (10) is obtained by using a discrete version of Gronwall's lemma. Inequality (10) is the basis for error estimates. Indeed, for example, it can be shown that if $\dot{\boldsymbol{u}} \in C([0,T];[H^2(\Omega)]^d)$ and $\boldsymbol{u}_0 \in [H^2(\Omega)]^d$, then (10) implies

$$\max_{1\leqslant n\leqslant N} \left\| \boldsymbol{u}(t_n) - \boldsymbol{u}_n^{hk} \right\|_V \leqslant c\left(k + h^{1/2}\right) \tag{11}$$

4. Numerical simulations

To show the performance of the numerical method described in the previous section we have developed a FORTRAN-based software to solve variational inequalities of the second kind involving the functional $j : V \to \mathbb{R}$ given in (8). It combines a fixed point strategy with a method of duality-penalization based on the algorithm in [11]. Its main ingredient consists in solving a sequence of elastic problems where the stiffness matrix and the force term are conveniently updated. When $\mathcal{B}$ is time dependent we have to stock the elastic solutions in order to update the discrete memory term which provides an additional difficulty as the size of the problem grows, i.e., as $h, k \to 0$. When $\mathcal{B}$ is time-independent, updating is performed just by adding the present solution to the accumulated one. Despite this data storage problem, the algorithm was found to perform well, the convergence was rapid and the computations were reliable.

We tested it in a number of problems and we present here two numerical simulations obtained in the study of a two-dimensional test problem. The physical setting is presented in Fig. 1-left and the results for a *small* and *large* value of the friction bound are provided in Fig. 1-center and Fig. 1-right, respectively. The deformations are amplified for the sake of a better visual analysis. For the small value of g, we observe that the horizontal displacements are quite similar either the node is far or near to the contact zone, whereas for the large value of g, the nodes close to contact zone are subjected to displacements which are much smaller than those of the nodes far from the contact zone. Also, for the large value of g, the stresses on the nodes close to contact zone are higher, as well.

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