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A method for the calculation of nonsymmetric steady periodic capillary–gravity waves on water of arbitrary uniform depth

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Abstract

In this Note the method developed by Aider and Debiane (2004) for the calculation of nonsymmetric water waves on infinite depth is extended to finite depth. The water-wave problem is reduced to a system of nonlinear algebraic equations which is solved by using Newton's method. Solutions are computed up to their limiting forms by decrementing the depth from the infinity to a value of the depth-wavelength ratio h/λ less than 0.025. It is found that the waves become symmetric when the depth becomes very small. Relations giving some integral properties are derived. ***To cite this article: R. Aider, M. Debiane, C. R. Mecanique 334 (2006).***

Résumé

Une méthode de calcul des ondes de gravité–capillarité périodiques et non-symétriques en profondeur arbitraire. La méthode développée par Aider et Debiane (2004) pour le calcul des ondes de gravité-capillarité non-symétriques en profondeur infinie est étendue au cas d'une profondeur arbitraire. Les équations classiques des vagues sont réduites à un système d'équations algébriques non-linéaires, résolu par la méthode de Newton. Des solutions ont été calculées jusqu'à leur forme limite en décrémentant le paramètre lié à la profondeur de la valeur infinie jusqu'à une valeur telle que le rapport h/λ, de la profondeur sur la longueur d'onde, devient inférieur à 0,025. On a trouvé que les ondes deviennent symétriques lorsque la profondeur devient très faible. Les relations donnant quelques propriétés intégrales ont été dérivées. ***Pour citer cet article : R. Aider, M. Debiane, C. R. Mecanique 334 (2006).***

Keywords: Fluid mechanics; Nonsymmetric water waves; Finite depth

Mots-clés : Mécanique des fluides ; Vagues non-symétriques ; Profondeur finie

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1. Introduction

A wave is said to be symmetric when, if $f(x)$ represents the profile of the wave, the origin of the horizontal axis can be chosen such that $f(x) = f(-x)$. The problem of symmetric periodic capillary–gravity waves has been the subject of a substantial literature. However, much less work appears to have been done on the corresponding problem when the profile is nonsymmetric. Zufiria [1,2] is the first to calculate nonsymmetric periodic water waves. He found that for gravity waves on shallow and deep water, the symmetry can be broken via a spontaneous symmetry-breaking bifurcation. He found that class 6 wave (six crests per period) is the minimum class to have symmetry-breaking. In an other work, using a weakly nonlinear Hamiltonian model, Zufiria [3] found that for water of finite and uniform depth, families of non-symmetric pure gravity waves are possible. They appear via spontaneous symmetry-breaking bifurcation from families of symmetric waves. He concluded that the symmetric solutions are just a subset of the solutions coming from the complex bifurcation structure of the Hamiltonian systems. Recently Aider and Debiane [4] have given numerical evidence that, on deep water, a class 1 gravity wave (one crest per period) including very weak surface tension could have asymmetric bifurcation when it is close to the limiting form. They also calculated non-symmetric gravity–capillary waves on infinite depth. The method they used consists in determining the Fourier coefficients in Stokes expansion through a set of integral relations derived by Longuet-Higgins [5] for gravity waves. Then, following Debiane and Kharif [6], the term due to surface tension is reduced to a simple function of the slope of the local tangent to the profile of the free surface. Finally a set of nonlinear algebraic equations is derived and solved by Newton's method. Motivated by the efficiency of this method, we develop here a similar method for the arbitrary uniform depth problem.

2. Form of solutions

We consider periodic capillary–gravity waves on the surface of a frictionless liquid of arbitrary uniform depth h. We assume that the flow is irrotational and the fluid is homogeneous and incompressible. Bidimensional waves propagating at a constant phase velocity c without changing their form are considered. In frame of reference travelling with the phase speed c these waves are steady. We choose axes with x horizontal and z vertically upwards and the origin above a crest at such a distance that Bernoulli's equation, at the free surface, can be written as

$$V^2 + 2\eta - 2\kappa \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = 0 \tag{1}$$

where $\eta(x)$ is the free surface level, V the particle velocity and κ is the dimensionless capillary number defined by $\kappa = \frac{k^2\tau}{\rho g}$. Here g is the acceleration due to the gravity, k the wave number, τ and ρ are the surface tension and the density of the fluid, respectively. The problem is considerably simplified if the potential function $\phi(x,z)$ and the stream function $\Psi(x,z)$ are used as independent variables. The stream function takes the values $\psi = 0$ at the free surface and $\psi = \psi_B$ at the bottom where $z = z_B$. Then following a standard approach we may express the coordinates of a 2π-periodic wave by Stokes expansion:

$$\eta = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{\bar{p}_n \operatorname{sh} n\psi_B \cos\left(n\frac{\Phi}{c}\right) - \bar{q}_n \operatorname{sh} n\psi_B \sin\left(n\frac{\Phi}{c}\right)\right\} \tag{2a}$$

$$x = -\varphi + \sum_{n=1}^{\infty}\left\{-\bar{q}_n \operatorname{ch} n\psi_{\mathrm{B}} \cos\left(n\frac{\Phi}{c}\right) - \bar{p}_n \operatorname{ch} n\psi_B \sin\left(n\frac{\Phi}{c}\right)\right\} \tag{2b}$$

Let $\varphi = \frac{\Phi}{c}$; $a_0 = p_0$; $p_n = \bar{p}_n \operatorname{sh} n\psi_B$; $q_n = \bar{q}_n \operatorname{sh} n\psi_B$; $\gamma_n = \coth n\psi_B$.

Eqs. (2) are rewritten as follows:

$$\eta = \frac{p_0}{2} + \sum_{n=1}^{\infty}\{p_n \cos n\varphi - q_n \sin n\varphi\} \tag{3a}$$

$$x = -\varphi + \sum_{n=1}^{\infty}\{-\gamma_n q_n \cos n\varphi - \gamma_n p_n \sin n\varphi\} \tag{3b}$$

If the wave is symmetric the coefficients q_n are all equal to zero. In the case of infinite depth $\gamma_n = 1$.

3. Calculation of the coefficients

Using the set of integral equations given by Longuet-Higgins:

$$S_j = \int_0^{-2\pi} V^2(x_\varphi + \mathrm{i}\gamma_j \eta_\varphi)\mathrm{e}^{-\mathrm{i}j\varphi}\,\mathrm{d}\varphi = 0 \quad (j = 1, 2, 3, \ldots) \tag{4}$$

and proceeding in the same way as Debiane and Kharif [6], we derive a system of nonlinear algebraic equations solved for the coefficients p_n $(n = 0, 1, 2, \ldots)$ and q_n $(n = 1, 2, \ldots)$ by Newton's method. Using (1) the integral I_j of Eq. (4) can be written as

$$S_j = S_j^{(1)} + S_j^{(2)} \tag{5a}$$

where

$$S_j^{(1)} = -2\int_0^{-2\pi} \eta(x_\varphi + \mathrm{i}\gamma_j \eta_\varphi)\mathrm{e}^{-\mathrm{i}j\varphi}\,\mathrm{d}\varphi \tag{5b}$$

$$S_j^{(2)} = 2\kappa\int_0^{-2\pi} \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}(x_\varphi + \mathrm{i}\gamma_j \eta_\varphi)\mathrm{e}^{-\mathrm{i}j\varphi}\,\mathrm{d}\varphi \tag{5c}$$

Using (3a) and (3b), $S_j^{(1)}$ may be integrated easily. We get the following quadratic relation

$$S_j^{(1)} = \pi\lfloor R_j^{(1)} + \mathrm{i}I_j^{(1)}\rfloor \tag{6}$$

with

$$\begin{aligned} R_j^{(1)} = -2p_j + \sum_{m=1}^{\infty}\big[&-\alpha_m p_m p_{m+j} - \alpha_m p_m p_{m-j} - \alpha_m q_m q_{m+j} - \alpha_m q_m q_{m-j} \\ &+ \gamma_j(m p_m p_{m+j} - m p_m p_{m-j} + m q_m q_{m+j} - m q_m q_{m-j})\big] \end{aligned} \tag{7a}$$

$$\begin{aligned} I_j^{(1)} = -2q_j + \sum_{m=1}^{\infty}\big[&-\alpha_m p_m q_{m+j} + \alpha_m p_m q_{m-j} + \alpha_m q_m p_{m+j} - \alpha_m q_m p_{m-j} \\ &+ \gamma_j(m p_m q_{m+j} + m p_m q_{m-j} - m q_m p_{m+j} - m q_m p_{m-j})\big] \end{aligned} \tag{7b}$$

where $\alpha_m = m\gamma_m$. In order to calculate the remaining integral $S_j^{(2)}$, we introduce the following development:

$$\mathrm{e}^{\mathrm{i}\theta} = \frac{1+\mathrm{i}\eta_x}{(1+\eta_x^2)^{1/2}} = \frac{C_0}{2} + \sum_{n=1}^{\infty}(C_n\cos n\varphi + D_n\sin n\varphi) + \mathrm{i}\left[\frac{E_0}{2} + \sum_{n=1}^{\infty}(E_n\cos n\varphi + F_n\sin n\varphi)\right] \tag{8}$$

where $\theta(\varphi) = \arctan(\eta_x)$ is the slope of the local tangent to the free surface. One can show that:

$$S_j^{(2)} = -2\pi j\kappa\{\gamma_j C_j + F_j + \mathrm{i}(E_j - \gamma_j D_j)\} = \pi\lfloor R_j^{(2)} + \mathrm{i}I_j^{(2)}\rfloor \tag{9}$$

The resulting expression of S_j is then:

$$S_j = \pi\lfloor R_j^{(1)} + R_j^{(2)} + \mathrm{i}(I_j^{(1)} + I_j^{(2)})\rfloor \tag{10}$$

Finally the integral equations (4) can be written as:

$$S_j = 0 \quad (j = 1, 2, 3, \ldots) \tag{11}$$

After the separation of the real and imaginary parts we obtain the following algebraic system:

$$\Re_j = -2p_j + \sum_{m=1}^{\infty}\big[-\alpha_m p_m p_{m+j} - \alpha_m p_m p_{m-j} - \alpha_m q_m q_{m+j} - \alpha_m q_m q_{m-j} + \gamma_j(mq_m q_{m+j} - mq_m q_{m-j} + mp_m p_{m+j} - mp_m p_{m-j})\big] - 2j\kappa(\gamma_j C_j + F_j) = 0, \quad j = 0, 1, 2, \ldots \tag{12a}$$

$$\Im_j = -2q_j + \sum_{m=1}^{\infty}\big[-\alpha_m p_m q_{m+j} + \alpha_m p_m q_{m-j} + \alpha_m q_m p_{m+j} - \alpha_m q_m p_{m-j} + \gamma_j(mp_m q_{m+j} + mp_m q_{m-j} - mq_m p_{m+j} - mq_m p_{m-j})\big] - 2j\kappa(-\gamma_j D_j + E_j) = 0, \quad j = 0, 1, 2, \ldots \tag{12b}$$

The series (3a) and (3b) are truncated to the order N, therefore the number of unknowns is limited to $2N + 1$. The summation in Eqs. (12a) and (12b) are also limited to N. Thus we construct a system of algebraic equations, where the first N equations correspond to:

$$\Re_j = 0, \quad j = 1, N \tag{13a}$$

and the following N are:

$$\Im_j = 0, \quad j = 1, N \tag{13b}$$

These equations could be supplemented by introducing a parameter such as the dimensionless mean depth

$$H = -z_B + \bar{\eta} = d + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_n\big(p_n^2 + q_n^2\big) \tag{13c}$$

Here z_B is the value of z at the bottom, $\bar{\eta}$ the mean level and $\alpha_n = n\gamma_n$.

The system of nonlinear equations (13a), (13b) is solved by a process of Newtonian iteration.

4. Integral properties

The solutions given by the previous method may be used to provide several integral properties of the motion such as phase velocity and kinetic and potential energies:

$$c^2 = -2\eta(0)\big(x_\varphi^2 + \eta_\varphi^2\big)_{\varphi=0} + 2\kappa\big|x_\varphi(0)\big|\big(x_\varphi^2 + \eta_\varphi^2\big)_{\varphi=0}^{-1/2}\left[\frac{\eta_{\varphi\varphi}x_\varphi - \eta_\varphi x_{\varphi\varphi}}{x_\varphi^3}\right]_{\varphi=0} \tag{14}$$

the kinetic energy can be calculated in the form

$$T = \frac{1}{2}c^2\left[\frac{1}{2}\sum_{n=1}^{\infty}n\gamma_n\big(p_n^2 + q_n^2\big)\right] \tag{15}$$

The gravity and capillary components of the potential energy are respectively V_g and V_τ, are given by:

$$V_g = \frac{1}{4}\varsigma_0 + \frac{1}{4}\sum_{m=1}^{\infty}\alpha_m p_m \varsigma_m \tag{16}$$

$$V_\tau = \frac{\kappa}{2}\left\{C_0 + \sum_{n=1}^{\infty}[\alpha_n p_n C_n - \alpha_n q_n D_n + nq_n E_n + np_n F_n] - 2\right\} \tag{17}$$

$$\varsigma_0 = 2b_0^2 + \sum_{n=1}^{\infty}\big(p_n^2 + q_n^2\big)$$

where

$$\varsigma_m = -b_0 p_m + \frac{1}{2}\sum_{n=1}^{\infty} p_n\left(p_{m+n} + p_{|m-n|}\right) + \frac{1}{2}\sum_{n=1}^{\infty} q_n\left(q_{m+n} + q_{n-m}\right)$$

$$b_0 = \frac{1}{2}\sum_{n=1}^{\infty}\alpha_n\left(p_n^2 + q_n^2\right)$$

5. Results

In a previous work (Aider and Debiane [4]), we have calculated a nonsymmetric gravity–capillary wave on infinite depth with $\kappa = 1/12$ (Fig. 1). This solution is used to initialise the iterative Newton process for finite depth; afterwards the resolution of the system (13) is performed by decreasing gradually the mean depth H until the ratio h/λ reaches the value 0.024, corresponding to the Bond number $B = \frac{\tau}{\rho g h^2}$ varying from 0 to 3.6. The Fourier coefficients in (8), are calculated by the use of Fast Fourier Transform (F.F.T.) which requires a number M, of intervals of the sampling of the variable φ, equal to a power of 2. Integer N is increased until convergence of the solutions is reached and is chosen as small as possible provided, that further increase in its value will not significantly change the solutions. The bulk of the computations has been carried out using $N = 150$ and $M = 4096$. Different families of steady nonsymmetric periodic waves are produced and are computed until their maximum height, without any trouble. For high wave steepness 6 to 10 iterations are required to satisfy the system of equations with an error less than 10^{-12}. Fig. 2 displays profiles of

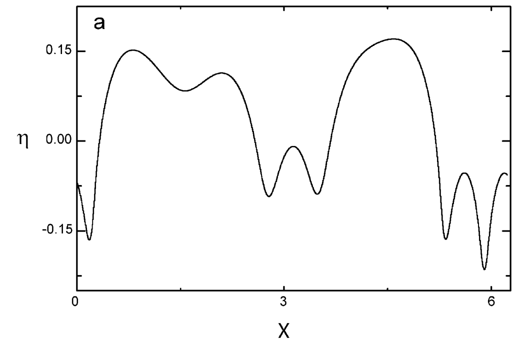

Fig. 1. Profile of periodic and nonsymmetric gravity–capillary wave on infinite depth.

Fig. 1. Profil d'une onde de gravité–capillarité périodique et non-symétrique en profondeur infinie.

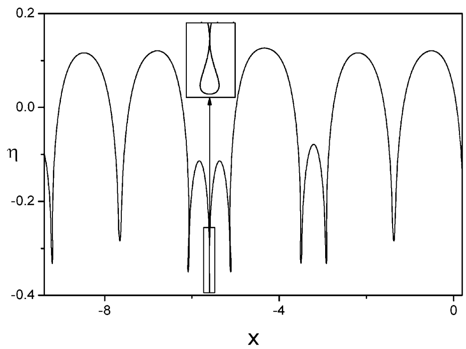

Fig. 2(a). Limiting profile of periodic and nonsymmetric gravity–capillary wave for $h/\lambda = 0.2$.

Fig. 2(a). Profil limite d'une onde de gravité–capillarité périodique et non-symétrique pour $h/\lambda = 0{,}2$.

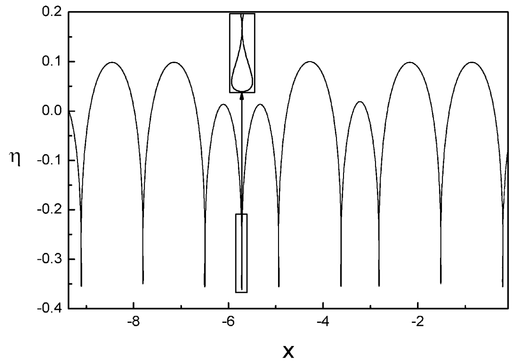

Fig. 2(b). Limiting profile of nonsymmetric gravity–capillary wave for $h/\lambda = 0.1$.

Fig. 2(b). Profil limite d'une onde de gravité–capillarité non-symétrique pour $h/\lambda = 0{,}1$.

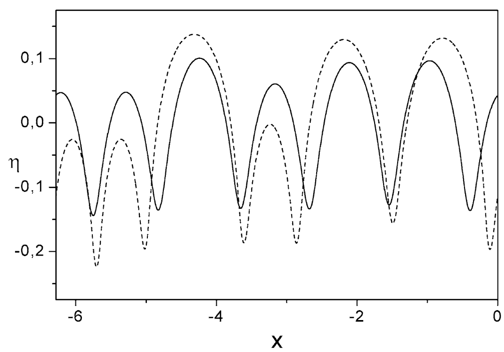

Fig. 3. Evolution of the shape of the wave with the depth; Solid line $h/\lambda = 0.5$: dashed line: $h/\lambda = 0.1$.

Fig. 3. Evolution du profil de l'onde avec la diminution de la profondeur. En trait plein : $h/\lambda = 0{,}5$; en pointillés : $h/\lambda = 0{,}1$.

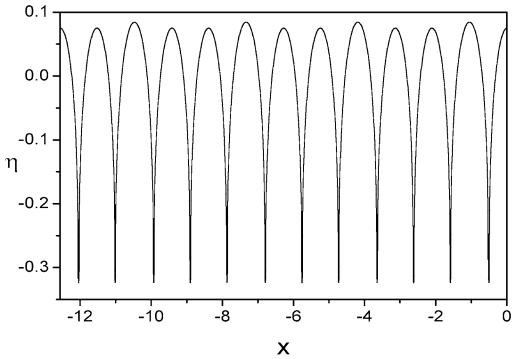

Fig. 4. Limiting profile of the symmetric wave for $h/\lambda = 0.0899$.

Fig. 4. Profile limite de l'onde symetrique pour $h/\lambda = 0{,}0899$.

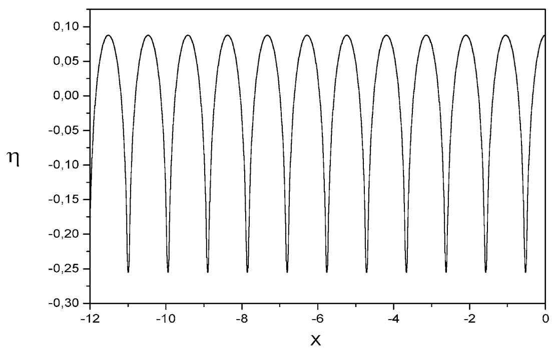

Fig. 5. Profile of the symmetric wave for $h/\lambda = 0.08$.

Fig. 5. Profile de l'onde symetrique pour $h/\lambda = 0{,}08$.

limiting waves, that is to say two adjacent crests are joined to enclose a bubble in the trough, obtained for $h/\lambda = 0.2$ and 0.1, where h and λ are the depth and the wavelength respectively. As the depth decreases the profiles undergo modifications and the difference of level between two neighbouring crests decreases and the coefficients q_i become smaller and smaller (Fig. 3). When h/λ reaches the value $h/\lambda = 0.09$ ($B = 0.26$) these modifications give rise to symmetric waves of wavelength π in which the crests are unequal whereas the troughs are at the same depth (Fig. 4). The symmetries are about principal crests and the troughs located between two secondary crests. As we pursue the decrement of the depth, the difference of level between all crests becomes smaller and smaller and eventually for $h/\lambda \sim 0.08$ ($B = 0.33$), the coefficients q_i are all null and the wave becomes symmetrical about all the troughs and all the crests which are now equal (Fig. 5). For h/λ less than 0.08 ($B > 1/3$) the wave remains symmetric, and no topological changes in the profile were noted. This behaviour, i.e., the bifurcation leading to the disappearance of the asymmetry for B greater than $1/3$, is expected and could be explained by the fact that the nonlinear effects, which are responsible for the symmetry-breaking leading to no symmetric waves, decrease as h/λ diminishes and become insignificant when the depth becomes very small.

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