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Combining Approach in Stages with Least Squares for fits of data in hyperelasticity

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Abstract

The present work concerns a method of continuous approximation by block of a continuous function; a method of approximation combining the Approach in Stages with the finite domains Least Squares. An identification procedure by sub-domains: basic generating functions are determined step-by-step permitting their weighting effects to be felt. This procedure allows one to be in control of the signs and to some extent of the optimal values of the parameters estimated, and consequently it provides a unique set of solutions that should represent the real physical parameters. Illustrations and comparisons are developed in rubber hyperelastic modeling. ***To cite this article: T. Beda, C. R. Mecanique 334 (2006).***

Résumé

Approximation par sous-domaines combinant les moindres carrés en hyperélasticité. Les présents travaux portent sur une méthode d'approximation combinant une approche d'identification par sous domaines et la méthode des moindres carrés. C'est une approximation continue par morceau d'une fonction continue. Une méthode qui linéarise une procédure non linéaire tout en contrôlant, par conséquent orientant, les signes des solutions obtenues. Cette possibilité permet l'obtention d'un ensemble unique de solutions qui vérifie les critères des paramètres mécaniques des matériaux. En plus, elle minimise l'erreur pour les faibles déformations, ce qui améliore l'estimation des paramètres en déformations infinitésimales, déduites des relations non linéaires. Des études comparatives en hyperélasticité de certains matériaux classiques sont illustrées. ***Pour citer cet article : T. Beda, C. R. Mecanique 334 (2006).***

Keywords: Computational solid mechanics; Approximation; Stage approach; Finite Least Squares; Rubber behavior; Fitting hyperelastic models

Mots-clés : Mécanique des solides numérique ; Approximation ; Approche par sous domaines ; Moindres carrés ; Caoutchouc ; Modèle hyperélastique ; Caractérisation

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1. Introduction

Often in material strain energy modeling, the hypotheses laid down provide conditions on the signs of the model's parameters; for example in rubber hyperelastic modeling [1]. These conditions eliminate, in certain cases, the use of the ordinary procedure of approximation such as the single Least Squares process. These methods can provide negative values or/and multiple sets of optimal numerical solutions [2] which might not correspond to material parameters. This work concerns a methodology that introduces a multi-stage-of-identification process that should permit one to estimate parameters by finite domains called 'stages of identification' while taking care of the signs of the solutions by a linearization of a nonlinear procedure.

2. The Approach in Stages procedure

Beda and Chevalier [3] have shown how to approximate a continuous function generated by basic functions; one of the outstanding features of the method is the fact that it provides a unique set of numerical solutions using a nonlinear procedure. The solutions are of form [3]:

$$y(x)=\sum_{j=1}^{N} a_j x^{\beta_j};\quad a_j,\beta_j \text{ real numbers} \tag{1}$$

The Approach in Stages method [3] identifies monomials $a_j x^{\beta_j}$ generating the function $y(x)$ in a decreasing order and the identification sub-domains are ranked in a decreasing values of x [3,4]. To begin with small stretches, one should instead identify parameters in increasing values of x, that is, instead of plotting the curve $(\varphi_j(x), f(x))$, one rather plots $(1/\varphi_j(x), 1/f(x))$. The solution is obtained when the curve is linear, and the slope corresponds to $1/a_j$. Table 1 presents a comparison of parameters values [4] obtained for the Ogden model [2] from Treloar data [5], and Fig. 1 depicts the relative Mooney plot, which gives, at the unstrained state (stretch = 1), the value of the material Young modulus. From this figure, one sees that the Approach in Stages gives the most appropriate parameter values than the usual Least Squares. The present work aims at transforming a nonlinear approximation to a linear approximation, enabling to avoid non-integer-power functions and provide solely integer-power functions (simple polynomials).

3. Combining Approach in Stages with Least Squares

3.1. Uniqueness of positive-weighted binomial approximating a positive power function

Considering a discrete power function, defined by $y(x)=Ax^{\beta}$, with $A>0$ and $\beta>0$, and given by the data (x_i, y_i), $i=1,2,3,\ldots,N_d$, with $1\leqslant x_i<x_{i+1}$ and $y_i=y(x_i)=Ax_i^{\beta}$, to be estimated by taking as the base of approximation a binomial $\langle P(x)\rangle=\langle x^k, x^{k+1}\rangle$. The approximating function is given by $z(x)=c_1x^k+c_2x^{k+1}$, where c_1 and c_2 are the parameters to be evaluated. By the Least Squares method, one obtains $c_1=n_1/D$ and $c_2=n_2/D$ where D, a positive value, is given by: $D=\sum_{i=1,j>i}^{N_d} x_i^{2k}x_j^{2k}(x_i-x_j)^2$, and n_1 and n_2 are:

$$n_1=\sum_{i=1,\,j>i}^{N_d} x_i^k x_j^k\big(x_i^{k+1}y_j-x_j^{k+1}y_i\big)(x_i-x_j)\quad\text{and}\quad n_2=\sum_{i=1,\,j>i}^{N_d} x_i^k x_j^k\big(x_j^k y_i-x_i^k y_j\big)(x_i-x_j)$$

Table 1
Comparison of Ogden model parameters for Treloar rubber-1

Tableau 1
Comparaison des paramètres évalués du modèle de Ogden pour le caoutchouc-1 de Treloar

	μ_1	μ_2	μ_3	μ_4	α_1	α_2	α_3	α_4
(LS):	−0.011229 MPa	16.169 MPa	0.77817 MPa	1.269×10^{-7} MPa	−2.7188	0.33382	2.7971	10.505
(AS):	−0.011 MPa	0.6276 MPa	0.0013 MPa	3.0×10^{-26} MPa	−2.0	1.30	5.0	29.7

Least Squares (LS) values are from [2], Approach in Stages (AS) values from [4].

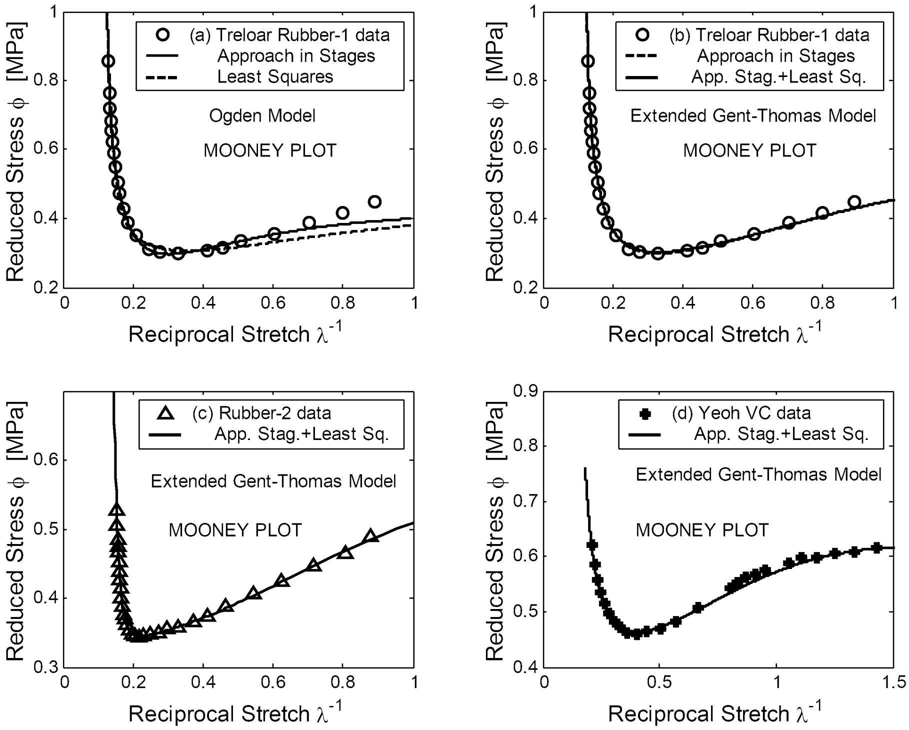

Fig. 1. Predictive curves from different methods in comparison with data.

Fig. 1. Courbes estimées à l'aide des différentes méthodes comparées aux acquisitions.

Having $y_i = Ax_i^{\beta}$, let us take $\beta = m + \xi$, m being the integer part of β; $m \geqslant 1$ and $0 < \xi < 1$.

Let one take the case $k = m$, one has:

$$n_1 = A \sum_{i=1,\, j>i}^{N_d} x_i^{2k} x_j^{2k} \big(x_i x_j^{\xi} - x_j x_i^{\xi}\big)(x_i - x_j) \quad \text{and} \quad n_2 = A \sum_{i=1,\, j>i}^{N_d} x_i^{2k} x_j^{2k} \big(x_i^{\xi} - x_j^{\xi}\big)(x_i - x_j)$$

So,

$$1 < \left(\frac{x_j}{x_i}\right)^{\xi} < \frac{x_j}{x_i} \quad \Rightarrow \quad n_1 > 0 \text{ and } n_2 > 0 \quad \Rightarrow \quad c_1 > 0 \text{ and } c_2 > 0$$

For the case $k \neq m$, one has:

$$n_1 = A \sum_{i=1,\, j>i}^{N_d} x_i^{2k} x_j^{2k} \big(x_i x_j^{m-k+\xi} - x_j x_i^{m-k+\xi}\big)(x_i - x_j) \quad \text{and}$$

$$n_2 = A \sum_{i=1,\, j>i}^{N_d} x_i^{2k} x_j^{2k} \big(x_i^{m-k+\xi} - x_j^{m-k+\xi}\big)(x_i - x_j)$$

Thus, if $k < m$, one has $1 < \frac{x_j}{x_i} < (\frac{x_j}{x_i})^{m-k+\xi} \Rightarrow n_1 < 0$ and $n_2 > 0 \Rightarrow c_1 < 0$ and $c_2 > 0$. Else if $k > m$, $0 < (\frac{x_j}{x_i})^{m-k+\xi} < 1 \Rightarrow n_2 < 0$ and $n_1 > 0 \Rightarrow c_2 < 0$ and $c_1 > 0$.

3.2. Controlling the signs of the solutions

The previous demonstration indicates that one can control the signs of the solutions by considering various bases of approximation. There exists a unique pair of positive coefficients c_1 and c_2 that approximates binomially a positive power function. It occurs when the order of approximation is taken equal to the integer part of the exponent of the power function. To get c_1 negative and c_2 positive, one needs to take the order k of approximation greater than the integer part m of the exponent β of the power function. Conversely to get c_1 positive and c_2 negative, k must be taken less than m. It is worth noting that there exists multiple couples of solutions among which a unique couple of positive solutions. The ability to control the signs of the parameters estimated and the uniqueness of positive solutions show how the usefulness of the technique for an approximation under sign constraints on unknown parameters.

3.3. Combining the Approach in Stages with the Least Squares

In order to obtain simple polynomials as basic generating functions, one combines the Approach in Stages with the Least Squares method. One has to binomially approximate, in a Least Squares sense, each fractional power function generated by the Approach in Stages, see relation (1). Thus, the new expression of the solution is of form:

$$y(x) = \sum_{j=1}^{N} \left(b_j x^{k_j} + c_j x^{k_j+1}\right) \tag{2}$$

k_j is the integer part of β_j given in relation (1), b_j and c_j being positive real numbers.

4. Application in hyperelastic behavior modeling

In Fig. 1, from the experimental curve and by the Approach in Stages method [3], it sets out that the reduced stress depends on both the stretch and the reciprocal stretch. The proposed methodology of resolution can be explained through an identification of the extended Gent–Thomas model [6]. In simple tension, the reduced stress ϕ of the extended Gent–Thomas model [6] for an incompressible material is given by:

$$\phi(I_1, I_2) = 2\left(\frac{K}{\lambda I_2} + \sum_{p=1}^{M} C_p (I_1 - 3)^{p-1}\right) \tag{3}$$

$I_1 = \lambda^2 + 2/\lambda$ and $I_2 = 2\lambda + 1/\lambda^2$ are invariants of the Cauchy–Green strain tensor and λ the stretch. By the Approach in Stages method, the general solution can further be written as:

$$\frac{1}{2}\phi = C_1 + \frac{K}{\lambda I_2} + \sum_{j=1}^{N} a_j (I_1 - 3)^{\beta_j} \tag{4}$$

This is not exactly the expression given in relation (3). The use of the linear binomial Least Squares on each $a_j(I_1 - 3)^{\beta_j}$ term of relation (4) leads to the sought-after form:

$$\frac{1}{2}\phi = C_1 + \frac{K}{\lambda I_2} + \sum_{j=1}^{N} \left[C_{k_j+1}(I_1 - 3)^{k_j} + C_{k_j+2}(I_1 - 3)^{k_j+1}\right], \quad k_j \geqslant 1 \tag{5}$$

Table 2 presents parameters estimated for Treloar rubber-1. In Fig. 1, good predictions are observed. From Table 3, which presents parameters for the Yeoh vulcanizate C [7] and the Treloar rubber-2, it sets out that the latter is a

Table 2
Evaluated parameters from Treloar rubber-1 data for the extended Gent–Thomas model
Tableau 2
Paramètres estimés du caoutchouc-1 de Treloar pour le modèle de Gent–Thomas généralisé

(AS + LS):					
$C_1 = 0.127$ MPa	$K = 0.3$ MPa	$C_2 = 3.066 \times 10^{-4}$ MPa	$C_3 = 7.63 \times 10^{-5}$ MPa	$C_{15} = 2.672 \times 10^{-25}$ MPa	$C_{16} = 1.48 \times 10^{-26}$ MPa
(AS):					
$C_1 = 0.127$ MPa	$K = 0.3$ MPa	$a_1 = 2.2 \times 10^{-4}$ MPa	$\beta_1 = 1.745$	$a_2 = 3.5801 \times 10^{-27}$ MPa	$\beta_2 = 14.13$

LS: Least Squares, AS: Approach in Stages.

Table 3
Evaluated parameters for the extended Gent–Thomas model

Tableau 3
Paramètres estimés du modèle de Gent–Thomas généralisé

Yeoh vulcanizate C. *Caoutchouc C de Yeoh*				
(AS + LS):	$C_1 = 0.203$ MPa	$K = 0.25$ MPa	$C_2 = 1.6 \times 10^{-3}$ MPa	$C_3 = 1.7 \times 10^{-4}$ MPa
(AS):	$C_1 = 0.203$ MPa	$K = 0.25$ MPa	$a_1 = 2.1 \times 10^{-2}$ MPa	$\beta_1 = 1.3$
Treloar rubber-2. *Caoutchouc-2 de Treloar*				
(AS + LS):	$C_1 = 0.16475$ MPa	$K = 0.27$ MPa	$C_6 = 1.917 \times 10^{-10}$ MPa	$C_7 = 1.8815 \times 10^{-11}$ MPa
(AS):	$C_1 = 0.16475$ MPa	$K = 0.27$ MPa	$a_1 = 5.578 \times 10^{-11}$ MPa	$\beta_1 = 5.77$

LS: Least Squares, AS: Approach in Stages.

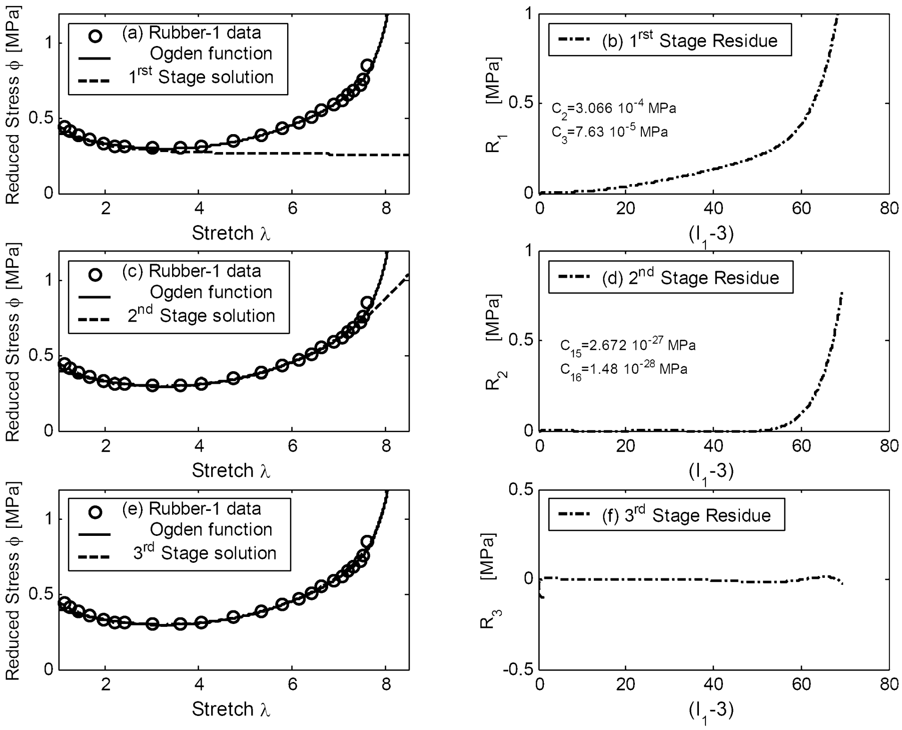

Fig. 2. Different identification stages of the procedure for the extended Gent–Thomas model.

Fig. 2. Différentes étapes d'identification de la procédure pour le modèle de Gent–Thomas généralisé.

high-order material. Fig. 2(a), (c), (e) shows the different stages of identification and the corresponding sub-domains. Fig. 2(b), (d), (f) depicts the relative residue functions, that is, the errors. From Fig. 1(a), (b), the various procedures provide different values of the Young modulus. That estimated by the proposed method is the more appropriate. It should be noted that the ordinary Least Squares procedure quasi-systematically provides negative-valued parameters. It is for this reason that one faces difficulties combining all requirements necessary for hyperelastic modeling [2,4,8].

5. Conclusions

An advantage of the procedure of the Approach in Stages combining Least Squares lies on the fact that it could provide solutions under constraint. The method first of all transforms a nonlinear procedure to a linear problem through which a unique and most appropriate set of solutions is obtained. It permits the control of solutions' signs and consequently their values.

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