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# Limit analysis and conic programming: 'porous Drucker–Prager' material and Gurson's model

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**Abstract**

Extending a previous work on the Gurson model for a 'porous von Mises' material, the present study first focuses on the yield criterion of a 'porous Drucker–Prager' material with spherical cavities. On the basis of the Gurson micro-macro model and a second order conic programming (SOCP) formulation, calculated inner and outer approaches to the criterion are very close, providing a reliable estimate of the yield criterion. Comparison with an analytical criterion recently proposed by Barthélémy and Dormieux—from a nonlinear homogenization method—shows both excellent agreement when considering tensile average boundary conditions and substantial improvement under compressive conditions. Then the results of an analogous study in the case of cylindrical cavities in plane strain are presented. It is worth noting that obtaining these results was made possible by using MOSEK, a recent commercial SOCP code, whose impressive efficiency was already seen in our previous works. ***To cite this article: M. Trillat et al., C. R. Mecanique 334 (2006).***


**Résumé**

**Analyse limite et optimisation conique : étude d'un matériau de Drucker–Prager poreux.** Via l'extension d'un travail précédent sur le modèle micro-macro de Gurson portant sur le cas d'un matériau de von Mises poreux, la présente étude concerne le critère de « Drucker–Prager poreux ». En utilisant le modèle de Gurson et une formulation en optimisation conique du second ordre (SOCP), les approches intérieure et extérieure obtenues sont très proches, donnant ainsi une estimation fiable du critère recherché. La comparaison avec un critère analytique récemment proposé par Barthélémy et Dormieux—via une méthode d'homogénéisation non linéaire—montre à la fois une excellente concordance sous déformation moyenne de traction et une substantielle amélioration dans le cas compressif. Sont donnés ensuite les résultats de la même étude, en déformation plane, pour un matériau à cavités cylindriques. Il faut noter enfin que l'obtention de ces résultats a été rendue possible par l'utilisation de MOSEK, code de SOCP récent et d'une efficacité impressionnante, déjà constatée dans nos travaux précédents. ***Pour citer cet article : M. Trillat et al., C. R. Mecanique 334 (2006).***


*Keywords:* Porous media; Drucker–Prager criterion; Limit analysis; Optimization

*Mots-clés :* Milieux poreux ; Critère de Drucker–Prager ; Analyse limite ; Optimisation

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## 1. Introduction

Gurson's work [1] is concerned with the upper bound limit analysis problem of a hollow von Mises sphere or cylinder under average strains imposed on the boundary in 1977, giving a celebrated criterion for 'porous von Mises' materials. Using his model and appropriate lower/upper numerical limit analysis formulations, we showed that the Gurson criterion is erroneous for a cylindrically porous material (except the axisymmetric loading case) [2,3], but correct for materials with spherical cavities [4,5]. As a natural extension, we propose here to analyze the yield criterion of a 'porous Drucker–Prager' material with spherical cavities, using three-dimensional and plane finite element discretizations.

The porous material is idealized as a single spherical cavity in a homothetic cell of a rigid-plastic Drucker–Prager material, here called 'Elementary Volume' (EV). As is classical in homogenization techniques, the macroscopic stress and strain rates $\Sigma_{ij}$ and $E_{ij}$ are linked to the microscopic ones, $\sigma_{ij}$ and $v_{ij}$, by the averaging relations:

$$\Sigma_{ij} = \frac{1}{V}\int_V \sigma_{ij}\,\mathrm{d}V, \qquad E_{ij} = \frac{1}{V}\int_V v_{ij}\,\mathrm{d}V \tag{1}$$

Due to material isotropy and EV symmetries—resulting in an isotropic criterion—the loading $\underline{\underline{E}}$ is taken as principal, i.e., $E_{ij} = 0$ for $i \neq j$. Therefore, working with an eighth of the spherical volume is sufficient. This eighth of the hollow sphere is meshed into 14 000 tetrahedral elements (see Ref. [5] for more details). The methods used in this Note, i.e., limit analysis and conic optimization, allow us to determine the plastic domain of the corresponding porous material. All present results are obtained by using the recent MOSEK optimizer [6], that is a very impressive commercial second order conic programming (SOCP) code.

Hereafter, we first briefly present both lower/upper limit analysis approaches, essentially recalling the expressions of Drucker–Prager criterion, unit dissipated powers for a 3D loading and their implementation, referring to [5] for other features; in a second step, results are analyzed and compared to those obtained in [7] through a nonlinear homogenization method. Lastly, the criterion for a cylindrically porous material in plane strain is also investigated to improve our previous SOCP results about a 'porous Coulomb' material in [8] and [9].

## 2. Limit analysis: static method

### 2.1. The Drucker–Prager criterion expression

The Drucker–Prager criterion is written as:

$$f(\underline{\underline{\sigma}}) = \sqrt{J_2} + \alpha\,\mathrm{tr}(\underline{\underline{\sigma}}) - k \leqslant 0 \quad \text{with } J_2 = \frac{1}{2}\,\mathrm{tr}(\underline{\underline{s}}^2) \text{ and } \underline{\underline{s}} = \underline{\underline{\sigma}} - \frac{1}{3}\,\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{1}} \tag{2}$$

When $\alpha$ vanishes, the criterion reduces to the von Mises criterion; $k$ is the limit in pure shear. Here, the parameters $\alpha$ et $k$ are chosen to be defined as functions of the internal friction angle $\varphi$ and $c$ cohesion of the Coulomb criterion as in [10]:

$$\alpha = \frac{\sin\varphi}{\sqrt{3(3+\sin^2\varphi)}}, \qquad k = 3\alpha H = 3\frac{\alpha c}{\tan\varphi}, \qquad \text{where } \varphi \in \left[0; \frac{\pi}{2}\right] \text{ and } \alpha \in \left[0; \frac{\sqrt{3}}{6}\right] \tag{3}$$

Then in a $(x, y, z)$ reference frame, the full 3D Drucker–Prager criterion becomes:

$$\sqrt{\left(\frac{2}{\sqrt{3}}\left(\frac{\sigma_x+\sigma_y}{2} - \sigma_z\right)\right)^2 + (\sigma_x-\sigma_y)^2 + (2\tau_{xy})^2 + (2\tau_{xz})^2 + (2\tau_{yz})^2} \leqslant 2k - 2\alpha(\sigma_x+\sigma_y+\sigma_z) \tag{4}$$

It should be noted that (4) can be written, after obvious changes of variables, as a conic constraint for MOSEK:

$$\sqrt{\sum_{j=1}^{n} x_j^2} \leqslant x_{n+1} \tag{5}$$

### 2.2. Implementation

The microscopic stress field is chosen as linearly varying in $x$, $y$, $z$ in each tetrahedral element and it can be discontinuous through any element boundary. We aim at finding the optimal equivalent macroscopic stress $\Sigma_{\text{eqv}}$ defined by:

$$\Sigma_{\text{eqv}}^2 = \Sigma_{\text{gps}}^2 + \Sigma_{\text{ps}}^2 \tag{6}$$

where:

$$\Sigma_m = \frac{1}{3}(\Sigma_x + \Sigma_y + \Sigma_z), \qquad \Sigma_{\text{gps}} = \frac{(\Sigma_x + \Sigma_y)}{2} - \Sigma_z, \qquad \Sigma_{\text{ps}} = \frac{\sqrt{3}}{2}(\Sigma_x - \Sigma_y) \tag{7}$$

To get a microscopic stress field statically and plastically admissible, the following conditions are implemented, briefly:

- definition of macroscopic stresses $\Sigma_{ij}$ as the averages of the corresponding microscopic ones, $\sigma_{ij}$;
- symmetry conditions: the microscopic tangential stresses are null on the three coordinate planes;
- boundary conditions: the stress vector $T_i = \sigma_{ij} n_j$ is null at each apex of the element sides of the boundary cavity. Hence, thanks to linearity, it is zero anywhere on this plane;
- in each element, the equilibrium equations, $\sigma_{ij,j} = 0$, generate three linear equations;
- continuity conditions: the stress vector is continuous across every discontinuity surface; each discontinuity triangular surface generates $3 \times 3$ equations;
- plastically admissible stress field: criterion (4) is written as in (5); it is imposed at each apex of the tetrahedron; hence, due to its convexity, the criterion is verified anywhere in the tetrahedron.

To obtain $\Sigma_{\text{eqv}}$—defined by Eq. (6)—$\Sigma_m$ is fixed to $\Sigma_m^0$ and $\Sigma_{\text{gps}}$ is maximized under $\Sigma_{\text{ps}} = 0$ (axisymmetrical loading); $k$ cohesion is always set equal to 1. After solving the constrained optimization problem using MOSEK, the admissible character of the solution stress field is carefully verified a posteriori.

## 3. Limit analysis: kinematic method

### 3.1. Dissipated powers

According to [10], the expression of the volumic dissipated power is:

$$\pi^{\text{vol}}(\underline{\underline{v}}) = \frac{c}{\tan\varphi}\,\text{tr}(\underline{\underline{v}}) \tag{8}$$

if the strain rate $\underline{\underline{v}}$ is PA (plastically admissible) by verifying the following condition:

$$\text{tr}(\underline{\underline{v}}) \geqslant \sqrt{6\alpha^2\left(3\,\text{tr}\,\underline{\underline{v}}^2 - (\text{tr}\,\underline{\underline{v}})^2\right)} \tag{9}$$

Similarly, on the velocity discontinuity surfaces (whose normal is $\underline{n}$), the unit dissipated power is written as:

$$\pi^{\text{disc}}\big([\underline{u}]\big) = \frac{c}{\tan\varphi}[\underline{u}].\underline{n} \tag{10}$$

if the velocity jump $[\underline{u}]$ is PA by verifying:

$$[\underline{u}].\underline{n} \geqslant \big|[\underline{u}]\big|\sin\varphi \tag{11}$$

### 3.2. Implementation

Under the boundary conditions $u_i = E_{ij} x_j$, from the virtual power principle the total dissipated power $P^{\text{tot}}$ can be written as follows:

$$P^{\text{tot}}/V^{\text{tot}} = \Sigma_m E_m + \Sigma_{\text{ps}} E_{\text{ps}} + \Sigma_{\text{gps}} E_{\text{gps}} \tag{12}$$

where the macroscopic strain rates are defined by:

$$E_m = (E_x + E_y + E_z), \qquad E_{\text{gps}} = \frac{2}{3}\left(\frac{(E_x + E_y)}{2} - E_z\right), \qquad E_{\text{ps}} = \frac{1}{\sqrt{3}}(E_x - E_y) \tag{13}$$

and $P^{\text{tot}}$ is written as:

$$P^{\text{tot}} = P^{\text{vol}} + P^{\text{disc}} \quad \text{with } P^{\text{disc}} = \int_{S_d} \pi\big([\underline{u}]\big)\,\mathrm{d}S \text{ and } P^{\text{vol}} = \int_V \pi(\underline{\underline{v}})\,\mathrm{d}V \tag{14}$$

$S_d$ being the set of discontinuity surfaces.

The displacement velocity field is chosen linear in $x$, $y$, $z$ in each tetrahedral element, and any triangular surface common to two tetrahedrons is a potential surface of velocity discontinuity. To express the admissibility of the displacement field solution, the following conditions are imposed:

- symmetry conditions: the normal velocity is zero on each coordinate plane;
- boundary conditions: $u_{ij} = E_{ij}x_j$ (with $E_{ij} = 0$ for $i \neq j$) which forces the macroscopic strain rates $E_{ij}$ to be the averages of the microscopic $v_{ij}$;
- plastically admissible conditions, Eqs. (9) and (11): these conditions are written as conic inequalities (5) for each tetrahedron and for each apex of a velocity discontinuity triangle, respectively;
- definitions of the dissipated powers, Eqs. (8) and (10), giving linear equations;
- definition of the functional.

More precisely, the volumic PA strain rate condition is written as:

$$(v_x + v_y + v_z) \geqslant \sqrt{6}\alpha\sqrt{(v_x - v_y)^2 + (v_x - v_z)^2 + (v_y - v_z)^2 + 6v_{xy}^2 + 6v_{xz}^2 + 6v_{yz}^2} \tag{15}$$

and the PA velocity jump conditions:

$$[u_n] \geqslant \sqrt{[u_{t1}]^2 + [u_{t2}]^2}.\tan\varphi \tag{16}$$

where $[u_{t1}]$ and $[u_{t2}]$ are orthogonal tangential jumps on the discontinuity interelement side.

Both conditions (15) and (16) are easily converted into conic form (5) for MOSEK. With $\Sigma_{\text{ps}} = 0$ (inducing $E_{\text{ps}}$ free), $\Sigma_m = \Sigma_m^0$, and $E_{\text{gps}} = 1$ the optimization process consists in minimizing $\Sigma_{\text{gps}}$ as:

$$\Sigma_{\text{gps}}^{\text{opt}} = \text{Min}\big\{P^{\text{tot}}/V^{\text{tot}} - \Sigma_m^0 E_m\big\} \tag{17}$$

## 4. Results

In [7], the authors suggest a nonlinear homogenization technique to determine the stress states on the boundary of the macroscopic admissible stress field for a porous Drucker–Prager criterion. Their yield criterion is written as:

$$\frac{2 + 4f/3}{3T^2}\Sigma_{\text{eqv}}^2 + \left(\frac{3f}{2T^2} - 1\right)\Sigma_m^2 + 2(1-f)H\Sigma_m - (1-f)^2H^2 = 0 \tag{18}$$

where $T = 3\sqrt{2}\alpha$. For $\Sigma_{\text{eqv}} = 0$, considering Eq. (18) where $\Sigma_m$ is the unknown, it can be noted that if $f > 2T^2/3$, the two $\Sigma_m$ solution values have opposite signs. As a consequence, failure is possible in compression as well as in tension (Fig. 1, left).

On the contrary, if $f < 2T^2/3$, both $\Sigma_m$ solution values are positive. In this case, failure becomes impossible in compression: their estimation tends to infinity (Fig. 1, right). Hence, a critical porosity rate or a critical angle of internal friction should exist, at least in this solution, as outlined by the authors as a singular value in their method.

Fig. 1 shows that the macroscopic criterion Eq. (18) is very close to our results on the right side of the curve, when the generalized velocity $E_m$ is nonnegative (expansion case). In all our cases, failure is possible in compression as well as in traction, which is not predicted by expression (18) above the critical friction angle $\varphi$ (Fig. 1, right). The Barthélémy and Dormieux criterion provides a reliable estimation of the maximum macroscopic mean stress but the minimum is highly-underestimated.

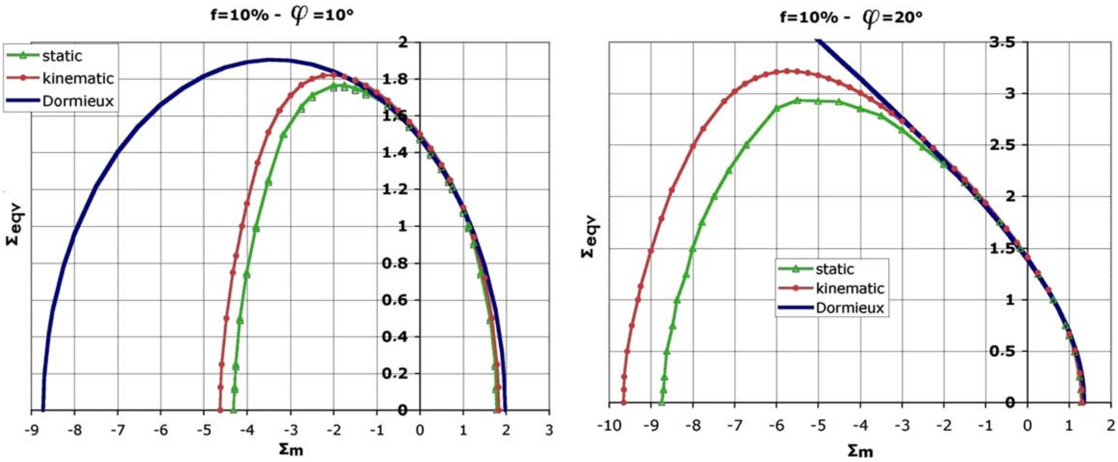


Fig. 1. Present results ($\Sigma_{\text{eqv}}$ versus $\Sigma_m$) compared to the yield criterion of [7]—left: $f = 10\%$ and $\varphi = 10°$—right: $f = 10\%$ and $\varphi = 20°$. Critical angle of internal friction $\varphi \simeq 16°$.

For polymer and geotechnic materials, the Drucker–Prager criterion or modified versions are commonly used; hence we choose a porosity rate of 10%, a realistic value for these materials.

For a 14 000-element problem, the static and kinematic problems have 631 810 and 348 204 rows, 672 009 and 511 706 columns, and 3 266 038 and 2 789 681 nonzero terms, respectively. The optimization is done in roughly 1 h 20 min and 1 h 45 min using MOSEK on a Apple G5-2GHz using 2G of RAM.

## 5. Study in plane strain with cylindrical cavities

Finally, we study a 'porous Drucker–Prager' material with cylindrical cavities in plane strain, to confirm and improve SOCP results first given in [8] and [9]. The EV is now a hollow cylinder. Thanks to symmetries, the study is carried out on one quarter of a hollow ring, using a plane mesh (see [3] for more details). Using (3) the Drucker–Prager criterion is equivalent to the Coulomb criterion, whose expression is:

$$\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2} \leqslant 2c\cos\varphi - (\sigma_x + \sigma_y)\sin\varphi \tag{19}$$

In plane strain expressions (8) and (9) become:

$$\pi(\underline{\underline{v}}) = \frac{c}{\tan\varphi}(v_x + v_y) \quad \text{if } (v_x + v_y) \geqslant \sqrt{3}\alpha\sqrt{(v_x + v_y)^2 + 3(v_x - v_y)^2 + 12v_{xy}^2} \tag{20}$$

Similarly, (10) and (11) give:

$$\pi\big([\underline{u}]\big) = \frac{c}{\tan\varphi}[u_n] \quad \text{if } [u_n] \geqslant \big|[u_t]\big| \tan\varphi \tag{21}$$

It is worth noting that these expressions still hold in plane strain for the Coulomb criterion, as recalled in [11]. Now there are two loading parameters: $\Sigma_{\text{ps}}$ and $\Sigma_h = (\Sigma_x + \Sigma_y)/2$, associated with $E_{\text{ps}} = \frac{1}{\sqrt{3}}(E_x - E_y)$ and $E_h = E_x + E_y$. Applying the static and kinematic methods as above, we represent $\Sigma_{\text{ps}}$ versus $\Sigma_h$ for $f = 10\%$ and $\varphi = 20°$ (Fig. 2).

It can be seen in Fig. 2 that the criterion presents corners on the $\Sigma_h$ axis, as in the 'porous von Mises' case. Note also that getting global incompressibility requires a compressive average stress ranged between three and four times the cohesion.

For a 672-element problem, the static and kinematic problems have, respectively, 11 429 and 8687 rows, 12 100 and 6724 columns, and 55 949 and 46 461 nonzero terms. The optimization is done in a couple of seconds when using MOSEK on the Apple G5-2GHz. Work is in progress to improve the results in the compressive case.

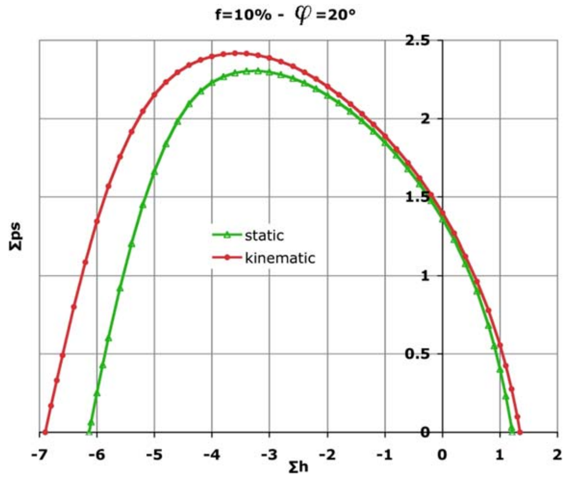


Fig. 2. Limit loading for a Coulomb or Drucker–Prager material with cylindrical cavities in plane strain. $f = 10\%$ and $\varphi = 20^\circ$.

## 6. Concluding remarks

In the 3D case as well as in plane strain, using limit analysis formulated as second order conic programming is confirmed to be very efficient and fast with von Mises and Drucker–Prager materials. In this Note, for predicting the plasticity criterion of a 'porous Drucker–Prager' material, we show that using the Gurson model, in spite of its simplicity, can be as efficient as more recent homogenization theories, for example when predicting compressive failure above a critical friction angle. In plane strain, previous conic programming results have been confirmed and improved in the case of a 'porous Drucker–Prager' material. Work is currently in progress on these subjects.

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