Available online at www.sciencedirect.com
ScienceDirect
C. R. Mecanique 334 (2006) 725–731
COMPTES RENDUS MECANIQUE
http://france.elsevier.com/direct/CRAS2B/

Computing Green's function of elasticity in a half-plane with impedance boundary condition

Mario Durán [a,*], Eduardo Godoy [a], Jean-Claude Nédélec [b]

[a] *Facultad de Ingeniería, Pontificia Universidad Católica de Chile, Casilla 306, Santiago 22, Chile*
[b] *CMAP, École polytechnique, 91128 Palaiseau cedex, France*

Received 6 September 2006; accepted 19 September 2006

Available online 1 November 2006

Presented by Évariste Sanchez-Palencia

Abstract

This Note presents an effective and accurate method for numerical calculation of the Green's function G associated with the time harmonic elasticity system in a half-plane, where an impedance boundary condition is considered. The need to compute this function arises when studying wave propagation in underground mining and seismological engineering. To theoretically obtain this Green's function, we have drawn our inspiration from the paper by Durán et al. (2005), where the Green's function for the Helmholtz equation has been computed. The method consists in applying a partial Fourier transform, which allows an explicit calculation of the so-called spectral Green's function. In order to compute its inverse Fourier transform, we separate $\widehat{G}$ as a sum of two terms. The first is associated with the whole plane, whereas the second takes into account the half-plane and the boundary conditions. The first term corresponds to the Green's function of the well known time-harmonic elasticity system in $\mathbb{R}^2$ (cf. J. Dompierre, Thesis). The second term is separated as a sum of three terms, where two of them contain singularities in the spectral variable (pseudo-poles and poles) and the other is regular and decreasing at infinity. The inverse Fourier transform of the singular terms are analytically computed, whereas the regular one is numerically obtained via an FFT algorithm. We present a numerical result. Moreover, we show that, under some conditions, a fourth additional slowness appears and which could produce a new surface wave. ***To cite this article: M. Durán et al., C. R. Mecanique 334 (2006).***

Résumé

Calcul de la fonction de Green de l'élasticité dans un demi-plan avec une condition aux limites d'impédance. Dans cette Note, nous présentons une méthode numérique effective et précise de calcul de la fonction de Green G du système de l'élasticité en régime harmonique. Le problème est posé dans un demi-plan et les conditions aux limites sur la frontière de ce demi-plan sont du type impédance. Ce problème se rencontre dans l'exploitation des mines souterraines et en séismologie. La méthode de calcul est inspirée du travail de Duran et co-auteurs (2005) qui traite du calcul de la fonction de Green du problème de Helmholtz dans un demi-plan. Elle consiste à utiliser la transformée de Fourier partielle en la variable tangentielle, ce qui permet un calcul explicite de la fonction de Green spectrale $\widehat{G}$. Pour calculer sa transformée de Fourier inverse, nous décomposons $\widehat{G}$ en une somme de deux termes. Le premier est associé à l'espace entier, tandis que le second prend en compte le demi-plan et les conditions aux limites. Le premier terme n'est autre que la fonction de Green du plan pour le système de l'élasticité en régime harmonique (cf. la thèse de J. Dompierre). Le second terme se sépare en trois termes, dont deux contiennent les singularités de la fonction spectrale (les pôles

* Corresponding author.
E-mail addresses: mduran@ing.puc.cl (M. Durán), egodoy@ing.puc.cl (E. Godoy), nedelec@cmap.polytechnique.fr (J.-C. Nédélec).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2006.09.003

et les pseudo-pôles) tandis que le dernier est régulier et décroissant à l'infini. Nous calculons analytiquement la transformée de Fourier des deux termes singuliers et celle du dernier terme est obtenue par un algorithme type FFT. Nous présentons un résultat numérique. En outre, nous montrons que, sous certaines conditions, il est possible d'avoir une quatrièmme lenteur, qui pourrait propager une nouvelle onde de surface. ***Pour citer cet article : M. Durán et al., C. R. Mecanique 334 (2006).***

Keywords: Computational solid mechanics; Spectral Green's function; Seismology

Mots-clés : Mécanique des solides numérique ; Fonction de Green spectrale ; Séismologie

1. Basic mathematical model

Let us consider the upper half-plane $\mathbb{R}^2_+ = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 \mid y > 0\}$, whose boundary is given by $\Gamma = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 \mid y = 0\}$. Let $\mathbf{x} = (x_0, y_0) \in \mathbb{R}^2_+$ be a source point (because of the half-plane's symmetry, we can assume $x_0 = 0$) and let $\mathbf{y} = (x, y) \in \mathbb{R}^2_+$ be a variable point. The Green's function of the time-harmonic elasticity system in $\mathbb{R}^2_+$, denoted by $G = G(\mathbf{x}, \mathbf{y})$, is a 2×2 matrix function of complex values. Its column vectors are denoted by $\mathbf{g}_j = \mathbf{g}_j(\mathbf{x}, \mathbf{y})$, $j = 1, 2$, and they must satisfy the next differential system:

$$\rho\,\omega^2 \mathbf{g}_j(\mathbf{x},\mathbf{y}) + \operatorname{div}\sigma\big(\mathbf{g}_j(\mathbf{x},\mathbf{y})\big) = -\delta_{\mathbf{x}}(\mathbf{y})\mathbf{e}_j \quad \text{in } \mathbb{R}^2_+ \tag{1a}$$

$$\sigma\big(\mathbf{g}_j(\mathbf{x},\mathbf{y})\big)\mathbf{n}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma \tag{1b}$$

$$\big(\sigma\big(\mathbf{g}_j(\mathbf{x},\mathbf{y})\big)\mathbf{n} - Z\mathbf{g}_j(\mathbf{x},\mathbf{y})\big)\cdot\mathbf{t} = 0 \quad \text{on } \Gamma \tag{1c}$$

$$+\ \text{Radiation conditions at infinity} \tag{1d}$$

where ρ is the elastic solid density, $\omega > 0$ is the pulsation frequency assumed to be known, $\delta_{\mathbf{x}}(\mathbf{y})$ is the Dirac mass evaluated at $\mathbf{y}$ and centered at $\mathbf{x}$, $\mathbf{e}_j$ is the jth canonic vector of $\mathbb{R}^2$, $\mathbf{n} = (0, -1)$ is the exterior unitary normal vector to $\mathbb{R}^2_+$, $\mathbf{t} = (1, 0)$ is the unitary tangent vector at Γ, Z is an impedance constant and σ is the stress tensor, which evaluated at a displacement field $\mathbf{u}(\mathbf{y})$ is given by the Hooke law:

$$\sigma\big(\mathbf{u}(\mathbf{y})\big) = \lambda\big(\operatorname{div}\mathbf{u}(\mathbf{y})\big)I + \mu\big(\nabla\mathbf{u}(\mathbf{y}) + \big(\nabla\mathbf{u}(\mathbf{y})\big)^T\big) \tag{2}$$

where λ and μ are positive real numbers known as *Lamé constants* and I is the 2×2 identity matrix. The boundary condition (1b) means that the normal stresses vanish on Γ, whereas the boundary condition (1c) represents a proportionality relation between sheer stresses and tangential displacements on Γ. On the other hand, radiation conditions at infinity must be posed in such a way that $|G|$ decreases when $|\mathbf{x}|$ or $|\mathbf{y}|$ go to infinity. For a broader framework about the Green's functions and their use in integral equations for time-harmonic problems, see Nédélec [1], Colton and Kress [2], Bonnet [3] and Linkov [4].

2. Spectral Green's function

The problem (1) is solved by applying the following partial Fourier transform in the x variable:

$$\hat{\mathbf{g}}_j(\mathbf{x}, s, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{g}_j(\mathbf{x},\mathbf{y})\, \mathrm{e}^{-\mathrm{i}\omega s(x-x_0)}\, \mathrm{d}x \tag{3}$$

where for convenience reasons, a Fourier variable s normalized by ω has been considered instead of the standard one. We obtain a matrix system of ordinary differential equations in y with constant coefficients, where s is considered as a parameter. The Fourier transform of G, denoted by $\widehat{G} = \widehat{G}(\mathbf{x}, s, y)$, is called *spectral Green's function*. In order to compute $\widehat{G}$, we solve the matrix system by standard methods. The general solution has both exponentially increasing terms and exponentially decreasing terms in y. We deal separately with the bounded region $\{0 < y < y_0\}$ and with the unbounded region $\{y > y_0\}$. Within the bounded region, $\widehat{G}$ is expressed as a linear combination of both terms, but on the contrary, within the unbounded region, $\widehat{G}$ cannot have exponentially increasing terms so that the radiation conditions at infinity holds, therefore, $\widehat{G}$ is expressed as a multiple of the exponentially decreasing term. Next, we impose continuity at $y = y_0$ and we achieve an expression for $\widehat{G}$ that we separate as a sum of two terms:

$$\widehat{G}(\mathbf{x}, s, y) = \widehat{G}^{\infty}(\mathbf{x}, s, y) + \widehat{G}^{\Gamma}(\mathbf{x}, s, y) \tag{4}$$

where $\widehat{G}^{\infty}(\mathbf{x}, s, y)$ is a symmetric matrix that we call *whole plane term* and $\widehat{G}^{\Gamma}(\mathbf{x}, s, y)$ is a non-symmetric matrix that we call *boundary term*. In order to write the expressions of these terms, we define two physical constants which are called *transversal slowness* s_T and *longitudinal slowness* s_L as follows:

$$s_T = \sqrt{\frac{\rho}{\mu}}, \qquad s_L = \sqrt{\frac{\rho}{\lambda + 2\mu}} \tag{5}$$

and the components of $\widehat{G}^{\infty}(\mathbf{x}, s, y)$ are:

$$\widehat{G}^{\infty}_{11}(\mathbf{x}, s, y) = \frac{1}{\sqrt{8\pi}\,\rho\,\omega}\big(s^2\theta_L(s)^{-1}\mathrm{e}^{-\omega\theta_L(s)|y-y_0|} - \theta_T(s)\,\mathrm{e}^{-\omega\theta_T(s)|y-y_0|}\big) \tag{6a}$$

$$\widehat{G}^{\infty}_{12}(\mathbf{x}, s, y) = \frac{\mathrm{i}s}{\sqrt{8\pi}\,\rho\,\omega}\,\mathrm{sign}(y - y_0)\big(\mathrm{e}^{-\omega\theta_L(s)|y-y_0|} - \mathrm{e}^{-\omega\theta_T(s)|y-y_0|}\big) \tag{6b}$$

$$\widehat{G}^{\infty}_{22}(\mathbf{x}, s, y) = \frac{1}{\sqrt{8\pi}\,\rho\,\omega}\big(s^2\theta_T(s)^{-1}\mathrm{e}^{-\omega\theta_T(s)|y-y_0|} - \theta_L(s)\,\mathrm{e}^{-\omega\theta_L(s)|y-y_0|}\big) \tag{6c}$$

whereas the components of $\widehat{G}^{\Gamma}(\mathbf{x}, s, y)$ are:

$$\begin{aligned}\widehat{G}^{\Gamma}_{11}(\mathbf{x}, s, y) = -\frac{1}{\sqrt{8\pi}\,\rho\,\omega}\Big[&\big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)\big)s^2\theta_L(s)^{-1}\mathrm{e}^{-\omega\theta_L(s)(y+y_0)}\\ &+ \big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) - \eta z\theta_T(s)\big)\theta_T(s)\,\mathrm{e}^{-\omega\theta_T(s)(y+y_0)}\\ &+ 4s^2\theta_T(s)\varphi(s)\big(\mathrm{e}^{-\omega(\theta_T(s)y+\theta_L(s)y_0)} + \mathrm{e}^{-\omega(\theta_L(s)y+\theta_T(s)y_0)}\big)\Big]\\ &\times \big(\varphi(s)^2 - 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)\big)^{-1}\end{aligned} \tag{7a}$$

$$\begin{aligned}\widehat{G}^{\Gamma}_{12}(\mathbf{x}, s, y) = \frac{\mathrm{i}s}{\sqrt{8\pi}\,\rho\,\omega}\Big[&4\varphi(s)\big(\theta_T(s)\theta_L(s)\mathrm{e}^{-\omega(\theta_T(s)y+\theta_L(s)y_0)} + s^2\mathrm{e}^{-\omega(\theta_L(s)y+\theta_T(s)y_0)}\big)\\ &+ \big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) - \eta z\theta_T(s)\big)\mathrm{e}^{-\omega\theta_T(s)(y+y_0)}\\ &+ \big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)\big)\mathrm{e}^{-\omega\theta_L(s)(y+y_0)}\Big]\\ &\times \big(\varphi(s)^2 - 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)\big)^{-1}\end{aligned} \tag{7b}$$

$$\begin{aligned}\widehat{G}^{\Gamma}_{21}(\mathbf{x}, s, y) = -\frac{\mathrm{i}s}{\sqrt{8\pi}\,\rho\,\omega}\Big[&4\varphi(s)\big(s^2\mathrm{e}^{-\omega(\theta_T(s)y+\theta_L(s)y_0)} + \theta_T(s)\theta_L(s)\mathrm{e}^{-\omega(\theta_L(s)y+\theta_T(s)y_0)}\big)\\ &+ \big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) - \eta z\theta_T(s)\big)\mathrm{e}^{-\omega\theta_T(s)(y+y_0)}\\ &+ \big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)\big)\mathrm{e}^{-\omega\theta_L(s)(y+y_0)}\Big]\\ &\times \big(\varphi(s)^2 - 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)\big)^{-1}\end{aligned} \tag{7c}$$

$$\begin{aligned}\widehat{G}^{\Gamma}_{22}(\mathbf{x}, s, y) = -\frac{1}{\sqrt{8\pi}\,\rho\,\omega}\Big[&\big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) - \eta z\theta_T(s)\big)s^2\theta_T(s)^{-1}\mathrm{e}^{-\omega\theta_T(s)(y+y_0)}\\ &+ \big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)\big)\theta_L(s)\,\mathrm{e}^{-\omega\theta_L(s)(y+y_0)}\\ &+ 4s^2\theta_L(s)\varphi(s)\big(\mathrm{e}^{-\omega(\theta_T(s)y+\theta_L(s)y_0)} + \mathrm{e}^{-\omega(\theta_L(s)y+\theta_T(s)y_0)}\big)\Big]\\ &\times \big(\varphi(s)^2 - 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)\big)^{-1}\end{aligned} \tag{7d}$$

where $\eta = \rho/\mu^2$ and $z = Z/\omega$ is the impedance normalized by the pulsation. The functions $\theta_T(\cdot)$, $\theta_L(\cdot)$ and $\varphi(\cdot)$ are given by:

$$\theta_T(s) = \sqrt{s^2 - s_T^2}, \qquad \theta_L(s) = \sqrt{s^2 - s_L^2}, \qquad \varphi(s) = s_T^2 - 2s^2 \tag{8}$$

These square roots are complex maps, hence an exact meaning must given to them. We consider these maps as the products between $\sqrt{s+s_T}$ and $\sqrt{s-s_T}$, and between $\sqrt{s+s_L}$ and $\sqrt{s-s_L}$, respectively. Each one of these roots is defined in particular analytical roots of the logarithm on the complex plane, in such a way that the real parts of $\theta_T(s)$ and $\theta_L(s)$ are always positive for $s \in \mathbb{R}$ (see Durán et al. [5] for details).

3. Effective computation of Green's function

In order to make an effective computation of Green's function $G(\mathbf{x},\mathbf{y})$, it is necessary to calculate the inverse Fourier transform of the spectral Green's function $\widehat{G}(\mathbf{x},s,y)$. Due to Eq. (4), we deal separately with terms $\widehat{G}^{\infty}(\mathbf{x},s,y)$ and $\widehat{G}^{\Gamma}(\mathbf{x},s,y)$. We finally obtain that the respective inverse Fourier transforms, which we note $G^{\infty}(\mathbf{x},\mathbf{y})$ and $G^{\Gamma}(\mathbf{x},\mathbf{y})$, are associated with the whole plane $\mathbb{R}^2$ and with the boundary Γ, respectively.

3.1. The whole plane term

We proceed analytically to calculate the inverse Fourier transform of $\widehat{G}^{\infty}(\mathbf{x},s,y)$, which expression is given by the system (6). For do this, we utilize some integral formulas taken from Bateman [6]. We obtain that the matrix $G^{\infty}(\mathbf{x},\mathbf{y})$ can be written as:

$$G^{\infty}(\mathbf{x},\mathbf{y}) = \frac{\mathrm{i}}{4\mu}\big(a(r)I + b(r)E(\mathbf{x},\mathbf{y})\big) \tag{9}$$

where

$$r = |\mathbf{x}-\mathbf{y}| = \sqrt{(x-x_0)^2+(y-y_0)^2} \tag{10}$$

$$E(\mathbf{x},\mathbf{y}) = \frac{1}{r^2}\begin{bmatrix} (x-x_0)^2 & (x-x_0)(y-y_0) \\ (x-x_0)(y-y_0) & (y-y_0)^2 \end{bmatrix} \tag{11}$$

$$a(r) = H_0^{(1)}(k_T r) - \frac{1}{k_T r}\big(H_1^{(1)}(k_T r) - \beta H_1^{(1)}(k_L r)\big) \tag{12}$$

$$b(r) = H_2^{(1)}(k_T r) - \beta^2 H_2^{(1)}(k_L r) \tag{13}$$

and $k_T = \omega s_T$, $k_L = \omega s_L$ are the *transversal* and the *longitudinal wave numbers*, respectively, and $\beta = k_L/k_T$. Moreover, $H_0^{(1)}$, $H_1^{(1)}$ and $H_2^{(1)}$ denote the first kind Hankel functions of order 1, 2 and 3, respectively (see Bell [7] for their definition and properties). The expression (9) corresponds to the Green's function of the time-harmonic elasticity system in the whole plane $\mathbb{R}^2$ (see Dompierre [8] for details).

3.2. The boundary term

The inverse Fourier transform of $\widehat{G}^{\Gamma}(\mathbf{x},s,y)$, whose components are given by (7), is computed by combining analytical and numerical techniques. The treatment of singularities is fundamental in order to carry out this computation. Hence, we separate $\widehat{G}^{\Gamma}(\mathbf{x},s,y)$ as the following sum:

$$\widehat{G}^{\Gamma}(\mathbf{x},s,y) = \widehat{G}^{\Gamma,\mathrm{r}}(\mathbf{x},s,y) + \widehat{G}^{\Gamma,\mathrm{sp}}(\mathbf{x},s,y) + \widehat{G}^{\Gamma,\mathrm{p}}(\mathbf{x},s,y) \tag{14}$$

where $\widehat{G}^{\Gamma,\mathrm{r}}$ corresponds to a regular part, $\widehat{G}^{\Gamma,\mathrm{sp}}$ is a part containing pseudo-pole type singularities and $\widehat{G}^{\Gamma,\mathrm{p}}$ is a part containing poles. If we are able to compute the inverse Fourier transforms of each one of these three terms, we will achieve the term $G^{\Gamma}(\mathbf{x},\mathbf{y})$ of the Green's function. Therefore, we individually specify each term, as well as the method to compute each one of their inverse Fourier transforms.

3.2.1. Pseudo-poles

In the first place, the term $\widehat{G}^{\Gamma}$ has pseudo-poles, that is, half-order poles. Specifically, as $\theta_L(\pm s_L) = \theta_T(\pm s_T) = 0$, the diagonal components $\widehat{G}_{11}^{\Gamma}$ and $\widehat{G}_{22}^{\Gamma}$ have two pseudo-poles at $s = \pm s_L$ and at $s = \pm s_T$, respectively. The terms causing these singularities are as follows:

$$\widehat{G}_{11}^{\Gamma,\mathrm{sp}}(\mathbf{x},s,y) = -\frac{s^2}{\sqrt{8\pi}\rho\omega}\theta_L(s)^{-1}\mathrm{e}^{-\omega\theta_L(s)(y+y_0)} \tag{15a}$$

$$\widehat{G}_{22}^{\Gamma,\mathrm{sp}}(\mathbf{x},s,y) = -\frac{s^2}{\sqrt{8\pi}\rho\omega}\theta_T(s)^{-1}\mathrm{e}^{-\omega\theta_T(s)(y+y_0)} \tag{15b}$$

therefore, the term $\widehat{G}^{\Gamma,\,\mathrm{sp}}$ in (14) is the diagonal matrix whose components are given by (15a) and (15b). The associated inverse Fourier transform $G^{\Gamma,\,\mathrm{sp}}(\mathbf{x},\mathbf{y})$ is analytically computed using an integral formula taken from [6]. We obtain the following expressions for its diagonal components:

$$G_{11}^{\Gamma,\mathrm{sp}}(\mathbf{x},\mathbf{y}) = -\frac{\mathrm{i}}{4\mu}\beta^2\left(\frac{1}{k_L\tilde{r}}H_1^{(1)}(k_L\tilde{r}) - H_2^{(1)}(k_L\tilde{r})\frac{(x-x_0)^2}{\tilde{r}^2}\right) \tag{16a}$$

$$G_{22}^{\Gamma,\mathrm{sp}}(\mathbf{x},\mathbf{y}) = -\frac{\mathrm{i}}{4\mu}\left(\frac{1}{k_T\tilde{r}}H_1^{(1)}(k_T\tilde{r}) - H_2^{(1)}(k_T\tilde{r})\frac{(x-x_0)^2}{\tilde{r}^2}\right) \tag{16b}$$

where

$$\tilde{r} = \sqrt{(x-x_0)^2 + (y+y_0)^2} \tag{17}$$

On the other hand, we must eliminate the pseudo-poles from $\widehat{G}^{\Gamma}$, for which we subtract $\widehat{G}^{\Gamma,\mathrm{sp}}$ from $\widehat{G}^{\Gamma}$. It is possible to show that the term $(\widehat{G}^{\Gamma} - \widehat{G}^{\Gamma,\mathrm{sp}})$ can be written as follows:

$$\big(\widehat{G}^{\Gamma} - \widehat{G}^{\Gamma,\mathrm{sp}}\big)(\mathbf{x},s,y) = -\frac{1}{\sqrt{8\pi}\rho\omega}\big(\varphi(s)^2 - 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)\big)^{-1}\begin{bmatrix}\psi_{11}(s) & \mathrm{i}\psi_{12}(s)\\ \mathrm{i}\psi_{21}(s) & \psi_{22}(s)\end{bmatrix} \tag{18}$$

where $\psi_{jk}(\cdot),\ j,k = 1,2$, are the next functions:

$$\begin{aligned}\psi_{11}(s) = \theta_T(s)\Big[&\big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) - \eta z\theta_T(s)\big)\mathrm{e}^{-\omega\theta_T(s)(y+y_0)}\\ &+ 8s^4\mathrm{e}^{-\omega\theta_L(s)(y+y_0)} + 4s^2\varphi(s)\big(\mathrm{e}^{-\omega(\theta_T(s)y+\theta_L(s)y_0)} + \mathrm{e}^{-\omega(\theta_L(s)y+\theta_T(s)y_0)}\big)\Big]\end{aligned} \tag{19a}$$

$$\begin{aligned}\psi_{12}(s) = -s\Big[&4\varphi(s)\big(\theta_T(s)\theta_L(s)\mathrm{e}^{-\omega(\theta_T(s)y+\theta_L(s)y_0)} + s^2\mathrm{e}^{-\omega(\theta_L(s)y+\theta_T(s)y_0)}\big)\\ &+ \big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) - \eta z\theta_T(s)\big)\mathrm{e}^{-\omega\theta_T(s)(y+y_0)}\\ &+ \big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)\big)\mathrm{e}^{-\omega\theta_L(s)(y+y_0)}\Big]\end{aligned} \tag{19b}$$

$$\begin{aligned}\psi_{21}(s) = s\Big[&4\varphi(s)\big(s^2\mathrm{e}^{-\omega(\theta_T(s)y+\theta_L(s)y_0)} + \theta_T(s)\theta_L(s)\mathrm{e}^{-\omega(\theta_L(s)y+\theta_T(s)y_0)}\big)\\ &+ \big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) - \eta z\theta_T(s)\big)\mathrm{e}^{-\omega\theta_T(s)(y+y_0)}\\ &+ \big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)\big)\mathrm{e}^{-\omega\theta_L(s)(y+y_0)}\Big]\end{aligned} \tag{19c}$$

$$\begin{aligned}\psi_{22}(s) = \theta_L(s)\Big[&\big(\varphi(s)^2 + 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)\big)\mathrm{e}^{-\omega\theta_L(s)(y+y_0)}\\ &+ 8s^4\mathrm{e}^{-\omega\theta_T(s)(y+y_0)} + 4s^2\varphi(s)\big(\mathrm{e}^{-\omega(\theta_T(s)y+\theta_L(s)y_0)} + \mathrm{e}^{-\omega(\theta_L(s)y+\theta_T(s)y_0)}\big)\Big]\\ &- 2s^2\eta z\,\mathrm{e}^{-\omega\theta_T(s)(y+y_0)}\end{aligned} \tag{19d}$$

3.2.2. Poles

In the second place, the term $\widehat{G}^{\Gamma}$ has poles due to the common denominator in (7). Actually, we deal with the term without pseudo-poles $(\widehat{G}^{\Gamma} - \widehat{G}^{\Gamma,\mathrm{sp}})$ given by (18). Therefore, in order to find these poles, we must solve the next equation:

$$\varphi(s)^2 - 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s) = 0 \tag{20}$$

The method for solving this equation depends on the value of z. If $z = 0$, and only in this case, we can lead Eq. (20) to a third degree equation in s^2 and solve analytically. On the other hand, if $z \neq 0$ we only are able to lead Eq. (20) to a sixth degree equation in s^2. Hence, at this time it is necessary to solve numerically. Nevertheless, in any of both cases we obtain that Eq. (20) has two real solutions in the form $s = \pm s_R$, where $s_R > s_T$. The quantity s_R is called *Rayleigh slowness* and it is associated with the propagation of *surface waves* along boundary Γ, and its value depends on z, that is, the quotient between the impedance and the pulsation. It is possible to show that $(\widehat{G}^\Gamma - \widehat{G}^{\Gamma,\mathrm{sp}})$ has two first order poles at s_R and $-s_R$. So, we define the next quantities associated with each one of both poles:

$$a_{jk} = \lim_{s\to +s_R} (s - s_R)\left(\frac{\psi_{jk}(s)}{\varphi(s)^2 - 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)}\right) \tag{21a}$$

$$a^*_{jk} = \lim_{s\to -s_R} (s + s_R)\left(\frac{\psi_{jk}(s)}{\varphi(s)^2 - 4s^2\theta_T(s)\theta_L(s) + \eta z\theta_T(s)}\right) \tag{21b}$$

where $1 \leqslant j, k \leqslant 2$. It can be proved that these quantities satisfy the following equalities:

$$a^*_{11} = -a_{11}, \qquad a^*_{12} = a_{12}, \qquad a^*_{21} = a_{21}, \qquad a^*_{22} = -a_{22} \tag{22}$$

Hence, we define the next matrices:

$$A^+ = \begin{bmatrix} a_{11} & \mathrm{i}a_{12} \\ \mathrm{i}a_{21} & a_{22} \end{bmatrix}, \qquad A^- = \begin{bmatrix} -a_{11} & \mathrm{i}a_{12} \\ \mathrm{i}a_{21} & -a_{22} \end{bmatrix} \tag{23}$$

and the term containing the first order poles of $(\widehat{G}^\Gamma - \widehat{G}^{\Gamma,\mathrm{sp}})$ at $s = \pm s_R$ is:

$$\widehat{G}^{\Gamma,\mathrm{p}}(\mathbf{x}, s, y) = -\frac{1}{\sqrt{8\pi}\,\rho\,\omega}\big((s - s_R)^{-1}A^+ + (s + s_R)^{-1}A^-\big) \tag{24}$$

We proceed analytically in order to compute $G^{\Gamma,\mathrm{p}}(\mathbf{x},\mathbf{y})$, that is, the inverse Fourier transform of $\widehat{G}^{\Gamma,\mathrm{p}}$. For this, we use an integral formula taken from [6] for the Fourier transform of a pole. We obtain that the components of $G^{\Gamma,\mathrm{p}}$ are:

$$G^{\Gamma,\mathrm{p}}_{11}(\mathbf{x},\mathbf{y}) = \frac{a_{11}}{2\rho}\,\mathrm{sign}(x - x_0)\sin\big(\omega s_R\,(x - x_0)\big) \tag{25a}$$

$$G^{\Gamma,\mathrm{p}}_{12}(\mathbf{x},\mathbf{y}) = \frac{a_{12}}{2\rho}\,\mathrm{sign}(x - x_0)\cos\big(\omega s_R\,(x - x_0)\big) \tag{25b}$$

$$G^{\Gamma,\mathrm{p}}_{21}(\mathbf{x},\mathbf{y}) = \frac{a_{21}}{2\rho}\,\mathrm{sign}(x - x_0)\cos\big(\omega s_R\,(x - x_0)\big) \tag{25c}$$

$$G^{\Gamma,\mathrm{p}}_{22}(\mathbf{x},\mathbf{y}) = \frac{a_{22}}{2\rho}\,\mathrm{sign}(x - x_0)\sin\big(\omega s_R\,(x - x_0)\big) \tag{25d}$$

Thus, in order to eliminate the poles from $(\widehat{G}^\Gamma - \widehat{G}^{\Gamma,\mathrm{sp}})$, we subtract the term $\widehat{G}^{\Gamma,\mathrm{p}}$ given by (24).

Furthermore, we have found by numerical evidence that there is a unique positive value of z for which Eq. (20) has two additional real solutions. These new solutions have the form $s = \pm s_A$, where $s_L \leqslant s_A \leqslant s_T < s_R$. The quantity s_A corresponds to an additional slowness that influences the propagation of the surface waves along boundary Γ. In this case, it can be shown that $(\widehat{G}^\Gamma - \widehat{G}^{\Gamma,\mathrm{sp}})$ has two additional first order poles at s_A and $-s_A$ and they must be eliminated. This is done in the same the other two poles were eliminated.

3.2.3. Regular part

Once we have subtracted the terms corresponding to pseudo-poles and poles of $\widehat{G}^\Gamma$, we obtain the next term:

$$\widehat{G}^{\Gamma,\mathrm{r}}(\mathbf{x}, s, y) = \widehat{G}^{\Gamma}(\mathbf{x}, s, y) - \widehat{G}^{\Gamma,\mathrm{sp}}(\mathbf{x}, s, y) - \widehat{G}^{\Gamma,\mathrm{p}}(\mathbf{x}, s, y) \tag{26}$$

This term is regular, because all its singularities in s were eliminated. Furthermore, $\widehat{G}^\Gamma$ decreases fast at infinity in s, since $(\widehat{G}^\Gamma - \widehat{G}^{\Gamma,\mathrm{sp}})$ is exponentially decreasing (see (18)) and $\widehat{G}^{\Gamma,\mathrm{p}}$ is square integrable far from the poles (see (24)). Hence, we are able to numerically compute its inverse Fourier transform $G^{\Gamma,\mathrm{r}}(\mathbf{x},\mathbf{y})$. This computation is carried out by an Fast Fourier Transform (FFT) algorithm, from which we obtain a numerical approximation to term $G^{\Gamma,\mathrm{r}}(\mathbf{x},\mathbf{y})$ in a bounded region of $\mathbb{R}^2_+$. Furthermore, note that from (26) we obtain the expression (14) for the boundary term $\widehat{G}^\Gamma$.

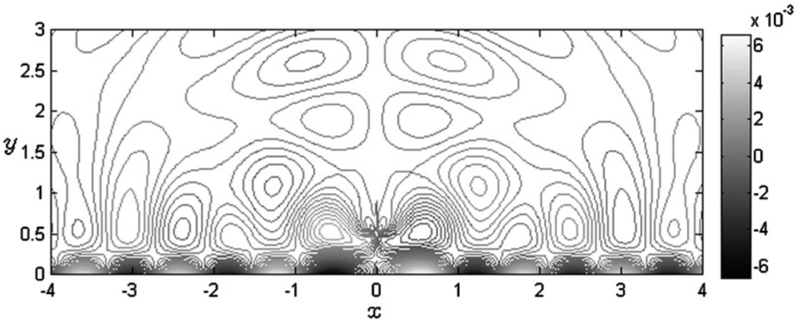

Fig. 1. Real part of the computed component G_{12}.

4. A numerical result

Here we show a numerical result of the Green's function obtained by the method described above. We have considered the following numerical values for the variables: $\omega = 1$, $\rho = 500$, $\lambda = 16$, $\mu = 25$, $Z = 50$. The considered source point is $\mathbf{x} = (x_0, y_0) = (0, 0.5)$. Fig. 1 shows a contour plot in $\mathbf{y} = (x, y)$ of the off-diagonal component $G_{12}(\mathbf{x}, \cdot)$ (real part). Notice that a surface wave appears.

Acknowledgements

This work was supported by the Fondecyt Project # 1030480, the ECOS/Conicyt Project # C03-E08 and the PUC fellowship for doctorate student.

References

[1] J.-C. Nédélec, Acoustic and Electromagnetic Equations. Integral Representations for Harmonic Problems, Applied Mathematical Sciences, vol. 144, Springer-Verlag, Berlin/New York, 2001.
[2] P. Colton, R. Kress, Integral Equations Methods in Scattering Theory, John Wiley, New York, 1983.
[3] M. Bonnet, Boundary Integral Equation Methods for Solids and Fluids, John Wiley & Sons Ltd., Chichester, 1995.
[4] A.M. Linkov, Boundary Integral Equations in Elasticity Theory, Kluwer Academic Publishers, Dordrecht, Boston, 2002.
[5] M. Durán, I. Muga, J.-C. Nédélec, The Helmholtz equation with impedance in a half-plane, C. R. Acad. Sci. Paris, Ser. I 340 (2005) 483–488.
[6] H. Bateman, Tables of Integral Transformations, vol. I, McGraw–Hill Book Company, Inc., New York, 1954.
[7] W.W. Bell, Special Functions for Scientists and Engineers, Dover Publications, Inc., New York, 1968.
[8] J. Dompierre, Équations intégrals en axisymétrie généralisée, application à la sismique entre puits, Thesis, École Centrale de Paris, 1993.