Available online at www.sciencedirect.com
ScienceDirect
C. R. Mecanique 334 (2006) 707–712
COMPTES RENDUS
MECANIQUE
http://france.elsevier.com/direct/CRAS2B/

Weighted Korn inequalities for thin-walled elastic structures ☆

O.V. Izotova [a], Serguei A. Nazarov [a], Guido H. Sweers [b,c,*]

[a] *Institute of Mechanical Engineering Problems, V.O., Bolshoi pr., 61, 199178 St.-Petersburg, Russia*
[b] *Delft Institute of Applied Mathematics, TU Delft, P.O. Box 5031, 2600 GA Delft, The Netherlands*
[c] *Mathematisches Institut, Universität zu Köln, Weyertal 86–90, 50931 Köln, Germany*

Received 13 March 2006; accepted 26 September 2006

Available online 7 November 2006

Presented by Évariste Sanchez-Palencia

Abstract

Appropriate weighted norms in H^1 are presented such that the Korn type inequality is asymptotically sharp with respect to relative thickness and stiffness of the elastic plates. The weights depend crucially on the geometric structure of the plates' junction. ***To cite this article: O.V. Izotova et al., C. R. Mecanique 334 (2006).***

Résumé

Inégalités de Korn avec poids pour des structures élastiques de plaques minces. Des normes dans H^1 avec poids appropriés sont présentées tels que l'inégalité du type Korn soit asymptotiquement exacte en ce qui concerne l'épaisseur et la rigidité des plaques élastiques. Les poids dépendent crucialement de la structure géométrique de la jonction des plaques. ***Pour citer cet article : O.V. Izotova et al., C. R. Mecanique 334 (2006).***

Keywords: Computational solid mechanics; Korn's inequality; Elastic junction; Thin plates

Mots-clés : Mécanique des solides numérique ; Inégalité de Korn ; Jonction élastique ; Plaques minces

1. A junction of thin plates

Let $\Pi^0, \Pi^1, \ldots, \Pi^J$ be planes in $\mathbb{R}^3$ and let $\omega^p \subset \Pi^p$ be a Lipschitz domain with boundary $\partial\omega^p$ and compact closure $\overline{\omega}^p$. Each plane we supply with its own system of Cartesian coordinates $x^p = (y^p, z^p)$ where $y^p = (y_1^p, y_2^p)$ and the z^p-axis is perpendicular to the plane Π^p. We do not distinguish in notation between the two-dimensional sets with the superscript p and their immersions into $\mathbb{R}^3$ on the plane Π^p. We introduce the $J+1$ thin plates of variable thickness

$$\Omega_h^p = \left\{x\colon\ y^p \in \omega^p,\ -hH_-^p\big(y^p\big) < z^p < hH_+^p\big(y^p\big)\right\} \tag{1}$$

☆ The authors gratefully acknowledge the support by N.W.O., the Netherlands Organisation for Scientific Research.
* Corresponding author.
E-mail addresses: ovizotova@mail.ru (O.V. Izotova), serna@snark.ipme.ru (S.A. Nazarov), g.h.sweers@ewi.tudelft.nl (G.H. Sweers).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2006.10.002

where $h \in (0,1]$ is a small parameter and $H_\pm^p$ are Lipschitz functions on $\overline{\omega}^p$. For $j=1,\dots,J$ the plate Ω_h^j is rigidly clamped over the part

$$\Gamma_h^j = \{x:\ y^j \in \gamma^j,\ -hH_-^j(y^j) < z^j < hH_+^j(y^j)\} \tag{2}$$

of its lateral surface where γ^j stands for a non-empty open curve on $\partial\omega^j$. Here and later $p=0,\dots,J$ and $j=1,\dots,J$. We assume that $\overline{\gamma}^j \cap \Pi^0 = \emptyset$ and hence, for a small $h_0>0$ and $h \in (0,h_0]$, the sets (2) do not intersect the plate Ω_h^0 which is not clamped at any part of its boundary. In other words, the fixed plates $\Omega_h^1,\dots,\Omega_h^J$ support the plate Ω_h^0 and they together compose the elastic junction $\Xi(h)$. We assume that the set $g_h^j = \Omega_h^j \cap \Omega_h^0$ contains the cylinder

$$Q_h^j = \{x:\ |y_2^j|^2 + |z^j - hz_0^j|^2 < R^2h^2,\ |y_1^j| < L\} \tag{3}$$

with certain numbers R, L and z_0^j independent of h.

Let $u(x) = (u_1(x), u_2(x), u_3(x))^\top$ be a displacement vector at the a point $x \in \overline{\Xi(h)}$ written in the coordinate system $x = (y_1^0, y_2^0, z^0)^\top$ that is related to the supported plate Ω_h^0. The elastic energy type functional that we consider is

$$\mathcal{E}_\mu(u;\Xi(h)) = \sum_{k,l=1}^{3}\left(\int\limits_{\Omega^0(h)}\left|\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right|^2 \mathrm{d}x + \mu\sum_{j=1}^{J}\int\limits_{\Omega^j(h)}\left|\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right|^2 \mathrm{d}x\right) \tag{4}$$

Here $\mu \in (0,\infty)$ is the relative elastic stiffness of the fixed plates and $\Omega^0(h)$, $\Omega^j(h)$ are as follows:

$$\text{if } \mu < 1 \quad \text{then } \Omega^0(h) = \Omega_h^0,\ \Omega^j(h) = \Omega_h^j \setminus \overline{\Omega}_h^0 \tag{5}$$

$$\text{if } \mu > 1 \quad \text{then } \Omega^0(h) = \Omega_h^0 \setminus \overline{\Omega}_h^j,\ \Omega^j(h) = \Omega_h^j \tag{6}$$

In other words, the soft supporting plates must be glued to the surface of the supporting plate while the stiff supporting plates must be inserted into the grooves g_h^j. Of course no need to distinguish for $\mu = 1$.

2. The Korn inequality

Functional (4) vanishes if and only if the displacement field belongs to the 6-dimensional subspace of rigid motions $\mathcal{R} = \{u:\ u(x) = d(x)a,\ a \in \mathbb{R}^6\}$, where

$$d(x) = (d'(x), d''(x)) = \left(\begin{array}{ccc|ccc} 1 & 0 & -x_2 & 0 & 0 & -x_3 \\ 0 & 1 & x_1 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & 1 & x_2 & x_1 \end{array}\right) \tag{7}$$

Since the plates $\Omega_h^1,\dots,\Omega_h^J$ are clamped on the boundaries (2), we deal with the subspace $\mathring{H}^1(\Xi(h);\Gamma(h))^3$ of vector-valued u in $H^1(\Xi(h))^3$ satisfying the boundary condition

$$u(x) = 0, \quad x \in \Gamma(h) := \Gamma_h^1 \cup \cdots \cup \Gamma_h^J \tag{8}$$

It is well-known (see, e.g., [8,2,3]), due to the Dirichlet condition (8), the Korn inequality

$$\|u; H^1(\Xi(h))\|^2 \leqslant C(\Xi(h))\mathcal{E}(u;\Xi(h)) \tag{9}$$

is valid with a constant independent of the vector function $u \in \mathring{H}^1(\Xi(h);\Gamma(h))^3$. At the same time, there is in general no proof of inequality (9) that allows one to find the dependence of Korn's constant $C(\Xi(h))$ on the shape of the domain $\Xi(h)$ and, in particular, on the small parameter h. From our further results, it follows in the case $\mu = 1$ that $C(\Xi(h)) \leqslant ch^{-2}$. However, inequality (9) cannot be recognized as an asymptotically sharp one because, in particular, it does not reflect a distinguishing feature of thin elastic objects, namely, an anisotropic property: it is much easier to bend a plate than to stretch it.

The main goal of this Note is to derive an appropriate form of such Korn type inequalities that indicates how the estimates of the displacements and their derivatives are influenced by the parameters h, μ and the general structure of the elastic junction. Indeed, we will present appropriately defined *weighted anisotropic norms* $|||\cdots|||_{\beta^p}$ for the left-hand side of the inequality

$$|||u^0;\Omega^0(h)|||^2_{\beta^0}+\mu\sum_{j=1}^{J}|||u^j;\Omega^j(h)|||^2_{\beta^j}\leqslant c\,\mathcal{E}_\mu\big(u;\Xi(h)\big) \tag{10}$$

such that the constant c depends on neither the parameters $h\in(0,h_0]$ and $\mu>0$, nor on the vector field $u\in\mathring{H}^1(\Xi(h);\Gamma(h))^3$. Here u^p stands for the restriction of u on Ω^p_h.

It is known due to [10] (see also [1,9,5]) that, for the clamped plate Ω^j_h in the norm

$$\begin{aligned}|||u^p;\Omega^p_h|||_{\beta^p}=\Bigg\{\int_{\Omega^p_h}\Bigg[\sum_{i=1}^{2}\bigg|\frac{\partial u^p_i}{\partial y^p_i}\bigg|^2+\bigg|\frac{\partial u^p_3}{\partial z^p}\bigg|^2+\beta^p_y(h,\mu)\bigg(\bigg|\frac{\partial u^p_1}{\partial y^p_2}\bigg|^2+\bigg|\frac{\partial u^p_2}{\partial y^p_1}\bigg|^2+\sum_{i=1}^{2}|u^p_i|^2\bigg)\\ +\beta^p_z(h,\mu)h^2\bigg(\sum_{i=1}^{2}\bigg(\bigg|\frac{\partial u^p_i}{\partial z^p}\bigg|^2+\bigg|\frac{\partial u^p_3}{\partial y^p_i}\bigg|^2\bigg)+|u^p_3|^2\bigg)\Bigg]\mathrm{d}x\Bigg\}^{1/2}\end{aligned} \tag{11}$$

the weight factors

$$\beta^j_y(h,\mu)=\beta^j_z(h,\mu)=1 \tag{12}$$

are optimal so that the following inequality becomes asymptotically sharp:

$$|||u^j;\Omega^j_h|||^2_{\beta^j}\leqslant c_j\mathcal{E}_1\big(u^j;\Omega^j_h\big),\quad u^j\in\mathring{H}^1\big(\Omega^j_h;\Gamma^j_h\big)^3 \tag{13}$$

3. Geometric and algebraic conditions and the weighted norm

Let us introduce conditions for several configurations. Take $i,j,k>0$ and set $\Lambda^{ij}=\Pi^i\cap\Pi^j$:

1° *There exist three different planes Π^i, Π^j and Π^k such that Λ^{ij} is a line and $\Lambda^{ij}\cap\Pi^k=\emptyset$.*
2° *There exist planes Π^i, Π^j with Λ^{ij} a line and such that $\Lambda^{ij}\subset\Pi^0$.*
3° *There exists a plane Π^j that coincides with Π^0.*
4° *There exist planes $\Pi^i\neq\Pi^j$ that are both perpendicular to Π^0.*

The main result reads as follows:

Theorem 3.1. *The Korn inequality* (10) *is valid with the constant c independent on the parameters $h\in(0,h_0]$, $\mu\in(0,\infty)$ and the field $u\in\mathring{H}^1(\Xi(h);\Gamma(h))^3$, when the norms $|||\cdot;\Omega^p_h|||_{\beta^p}$ are of the form* (11) *with the weight factors for $j=1,\dots,J$ as in* (12) *and*

$$\beta^0_y(h,\mu)=\min\big\{1,h^2\mu\big\},\qquad \beta^0_z(h,\mu)=\min\{1,\mu\} \tag{14}$$

Moreover, if one of the special configurations holds, the weight factors enlarge as follows:

(i) *if* 4° *is met, then*

$$\beta^0_z(h,\mu)=\min\big\{1,h^{-2}\mu\big\} \tag{15}$$

(ii) *if* 2° *or* 3° *is met, then*

$$\beta^0_y(h,\mu)=\min\{1,\mu\} \tag{16}$$

(iii) *if* 1° *is met, then both factors in* (15) *and* (16) *are allowed.*

The geometric conditions 1° to 4° can be reformulated as algebraic ones dealing with the matrix (7) or with its 3×3 blocks. Let the symbol $\mathcal{L}$ further on denote the linear hull of a set of vectors.

Lemma 3.2.

(i) *Condition* 1° *is satisfied if and only if*

$$\mathbb{R}^6 = \mathcal{L}\{d(y,0)b \mid y \in \omega^j \cap \omega^0,\ b \in \mathbb{R}^3 \text{ lies on } \Pi^j,\ j = 1,\dots,J\}$$

(ii) *If condition* 1° *is satisfied then*

$$\mathbb{R}^3 = \mathcal{L}\{d'(y,0)\ (b_1,b_2,0)^\top \mid y \in \omega^j \cap \omega^0,\ (b_1,b_2)^\top \in \mathbb{R}^2 \text{ lies on } \Pi^j \cap \Pi^0,\ j = 1,\dots,J\}$$

(iii) *In the cases* $\Pi^j \cap \Pi^m \subset \Pi^0$, $\Pi^j \neq \Pi^m$ *and* $\Pi^j = \Pi^0$ (*cf.* 2° *and* 3°, *respectively*) *then it holds that*

$$\mathbb{R}^3 = \mathcal{L}\{d'(y,0)\ (b_1,b_2,0)^\top \mid y \in \omega^j \cap \omega^0,\ (b_1,b_2)^\top \in \mathbb{R}^2\}$$

(iv) *Let the planes* $\Pi^1,\dots,\Pi^I$, *but not* $\Pi^{I+1},\dots,\Pi^J$, *be perpendicular to the plane* Π^0. *Condition* 4° *is satisfied if and only if*

$$\mathbb{R}^3 = \mathcal{L}\{d''(y,0)\ (0,0,b_3)^\top \mid y \in \omega^j \cap \omega^0,\ b_3 \in \mathbb{R},\ j = 1,\dots,I\}$$

Variants of the conditions of 'filling an Euclidian space' in the lemma above have appeared as conditions to improve the norms in the Korn inequalities for several types of junctions of thin and massive elastic bodies (see [4,6,7]).

4. Examples

Let us first suppose that the fixed plates are 'very stiff', i.e. $\mu > h^{-2}$, and all factors in (14)–(16) become equal to 1. Then the field u with the components $u^j = 0$ and

$$u_3^0(x) = h^{-1} w_3(y), \quad u_i^0(x) = w_i(y) - \frac{z}{h}\frac{\partial w_3}{\partial y_i}(y), \quad i = 1,2 \tag{17}$$

where $w_k \in C_c^\infty(\omega^0 \setminus (\omega^1 \cup \dots \cup \omega^J))$, satisfies the relations

$$\mathcal{E}(u^0; \Omega_h^0) = \mathrm{O}(h), \quad \mathcal{E}(u^j; \Omega_h^j) = 0, \quad |||u^0; \Omega_h^0|||^2 = \mathrm{O}(h)$$

This shows that zero order for h in the weight factors cannot be diminished.

Let us consider the junction Γ on Fig. 1 which does not meet any of the conditions 1° to 4°. We assume the fixed plate to be 'stiff' ($0 < \mu \leqslant h^{-2}$) and introduce the field u on $\Xi(h)$ as follows:

$$u^0(x) = a_2 e_2 + h^{-1} a_3 e_3 + h^{-1} a_4 (y_2 e_3 - z e_2) \tag{18}$$

$$u^1(x) = a_2\left(e_2 \chi_0(z) - e_3 y_2 \frac{\partial \chi_0}{\partial z}(z)\right) + h^{-1} a_3 e_3 \chi_0(z) + h^{-1} a_4 \left(e_2 \chi_1(z) - e_3 y_2 \frac{\partial \chi_1}{\partial z}(z)\right) \tag{19}$$

where $a_m \in \mathbb{R}$, e_k is the unit vector of the x_k-axis, $\chi_q \in C^\infty(\mathbb{R})$ and $\chi_q(z) = z^q$ for $z \geqslant -2/3$, $\chi_q(z) = 0$ for $z \leqslant -1/3$. One readily gets

$$|||u^0; \Omega_h^0|||^2 = \mathrm{O}\big(h(|a_2|^2 + |a_3|^2 + |a_4|^2)\big) \tag{20}$$

$$\mathcal{E}(u^0; \Omega_h^0) = 0, \quad \mathcal{E}(u^1; \Omega_h^1) = \mathrm{O}\big(h^3|a_2|^2 + h^{-1}|a_3|^2 + h|a_4|^2\big) \tag{21}$$

If $a_2 \neq 0$ and $a_3 = a_4 = 0$, then formulas (20)–(21) show that the factor $\beta_y^0(h,\mu) = h^2\mu$ in (14) cannot be improved. The case $a_3 \neq 0$ and $a_2 = a_4 = 0$ in (20) prove that factor $\beta_z^0(h,\mu) = \min\{1,\mu\}$ in (14) is optimal even for soft

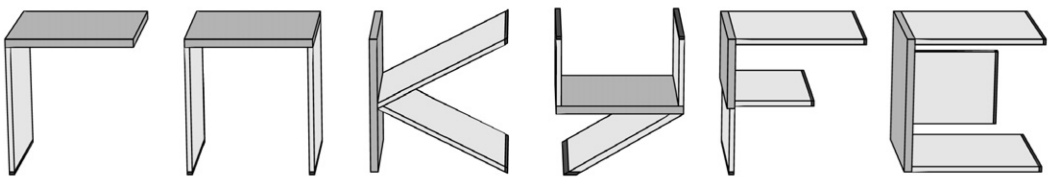

Fig. 1. Junctions Γ, Π, K, Y, F and a strictly 3D; the plate Ω_h^0 appears darker; the clamped edges are in black.

supporting plates ($\mu < 1$). Both the examples indicate the necessity of the additional geometric conditions to get the factors (15) and (16).

We emphasize that in the native coordinates $(y^1, z^1) = (y_1, z, y_2)$ the field (18) keeps the structure of (17) with the functions $w_1^1 = 0$, $w_2^1(y_2^1) = a_2\chi_0(y_2^1) + h^{-1}a_4\chi_1(y_2^1)$ and $w_3^1(y_2^1) = h^{-1}a_3\chi_0(y_2^1) - z^1\partial_{y_2^1} w_2^1(y_2^1)$. The same extension on the plate Ω_h^2 works for the junction Π on Fig. 1 while the formulas (20)–(21) are preserved with $h|a_4|^2$ changed to $h^{-1}|a_4|^2$ due to the addendum $h^{-1}(a_3+a_4)\chi_0(y_2^2)$ in $w_3^2(y_2^1)$. This junction satisfies the condition 3° and thus demonstrates that the factor (15) is exact.

The structure of the field in (19) is the following: a rigid motion u^0 is taken on the plate Ω_h^0 and it is extended to the supporting plates as in formula (17). If a non-trivial vector $u^0(x)$ can be chosen perpendicular to each of the planes $\Pi^1, \dots, \Pi^J$, then the extension makes the energy functional small, otherwise it gets an additional multiplier h^{-2}. Hence, to verify the exactness of the factors found in Theorem 3.1 one has to select a proper rigid motion on Ω_h^0. This way we use the junctions K (2°), Y (2° and 4°), F (2° and 3°) and the strictly three-dimensional junction (1°) on Fig. 1, where the geometrical conditions that are satisfied for these junctions are indicated within brackets.

5. Sketch of the proof

Let u^0 be decomposed in the form $u^0(x) = u^\perp(x) + d(x)a$, where d is matrix (7), $a \in \mathbb{R}^6$ and where $u^\perp$ is subject to an appropriate family of 6 orthogonality conditions (see [6] and [5, §3.1]) which provide the relation

$$|||u^\perp; \Omega_h^0|||^2_{(1,1)} \leqslant c\,\mathcal{E}_1\big(u^\perp; \Omega_h^0\big) = c\,\mathcal{E}_1\big(u^0; \Omega_h^0\big) \leqslant c\,\mathcal{E}_\mu\big(u; \Xi(h)\big) \tag{22}$$

Due to (22) it is sufficient to estimate the components $a_1, \dots, a_6$ of the column a and to use the following inequality in order to choose $\beta_y^0(h,\mu)$ and $\beta_z^0(h,\mu)$:

$$\begin{aligned}|||da; \Omega_h^0|||^2_{\beta^0} \leqslant c\big(&h\,\beta_y^0(h,\mu)\big(|a_1|^2 + |a_2|^2 + h^2(|a_4|^2 + |a_5|^2) + |a_6|^2\big)\\ &+ h^3\beta_z^0(h,\mu)\big(|a_3|^2 + |a_4|^2 + |a_5|^2\big)\big)\end{aligned} \tag{23}$$

To simplify the presentation we assume that, under a proper choice of the coordinates $x = (y, z)$, that the cylinder in (3) can be described by

$$Q_h^j = \big\{x\colon\ |y_2|^2 + |z|^2 < R^2h^2,\ |y_1| < L\big\} \tag{24}$$

By multiplying u^0 with $d(x)^\top$ and integrating over Q_h^j we obtain a system of 6 linear algebraic equations

$$\int\limits_{Q_h^j} d(x)^\top d(x)\,\mathrm{d}x\,a = \int\limits_{Q_h^j} \mathrm{d}(x)^\top\big(u^0(x) - u^\perp(x)\big)\,\mathrm{d}x \tag{25}$$

The right-hand side of (25) has to be estimated using (13), (23) and inequalities of Hardy type which give bounds for the $L^2(Q_h^j)$-norms of $u^0 = u^j$ and $u^\perp$ (cf. [7] and [5, §3.1]). In front of a in (25) stands the Gram matrix $\mathbf{d}(Q_h^j)$ of size 6×6, which is positive definite and, due to the symmetry of the integration set (24), diagonal

$$\mathbf{d}\big(Q_h^j\big) = 2\pi R^2h^2L\,\mathrm{diag}\left\{1, 1, 1, \frac{1}{4}R^2h^2, \frac{1}{6}L^2 + \frac{1}{8}R^2h^2, \frac{1}{6}L^2 + \frac{1}{8}R^2h^2\right\} \tag{26}$$

All entries of the matrix in (26) are of order h^2, except for the fourth one which is $O(h^4)$. This fact leads to a weaker estimate for a_4 than for other components of the column a and results in the weight factors in (15). Under additional geometric conditions we can get a better estimate for a_4 by using the algebraic system of type (25) that is produced on the other cylinders Q_h^m, $m \neq j$, and even to reduce the size of the algebraic systems so that only estimates of the longitudinal components of the vectors u^j with certain indices j are employed that have no additional factor h^2 in the norm $|||u^j; \Omega_h^j|||_{(1.1)}$ (cf. (11)).

Remark. Inequality (10) remains true in the cases $\mu > 1$, (5) and $\mu < 1$, (6) under the additional assumptions that inequalities (13) and (22) are valid with Ω_h^j and Ω_h^0 replaced by $\Omega^j(h)$ and $\Omega^0(h)$, respectively. These inequalities for plates with perturbed edges or grooves need certain geometric assumptions (see, e.g., [7] and [5, Chapter 3]) because in general they can fail, for example due to peak type singularities on the boundary.

References

[1] Ph.G. Ciarlet, Mathematical Elasticity, vol. II. Theory of Plates, North-Holland, Amsterdam, 1997.
[2] G. Duvaut, J.-L. Lions, Inequalities in Mechanics and Physics, Springer-Verlag, Berlin–New York, 1976.
[3] V.A. Kondratiev, O.A. Oleĭnik, Boundary value problems for a system in elasticity theory in unbounded domains. Korn inequalities, Uspekhi Mat. Nauk 43 (5) (1988) 55–98, 239; Russ. Math. Surveys 43 (1988) 65–119.
[4] S.A. Nazarov, Korn's inequalities for junctions of spatial bodies and thin rods, Math. Methods Appl. Sci. 20 (3) (1997) 219–243.
[5] S.A. Nazarov, Asymptotic Theory of Thin Plates and Rods, vol. 1. Dimension Reduction and Integral Estimates, Nauchnaya Kniga, Novosibirsk, 2001 (in Russian).
[6] S.A. Nazarov, Weighted anisotropic Korn's inequality for a junction of a plate and a rod, Mat. Sbornik 191 (4) (2004) 97–126; Sb. Math. 191 (4) (2004).
[7] S.A. Nazarov, Korn's inequalities for junctions of elastic bodies with thin plates, Sibirsk. Mat. Zh. 46 (4) (2005) 876–889; Siberian Math. J. 46 (4) (2005).
[8] J. Nečas, Les méthodes en théorie des équations elliptiques, Masson Academia, Paris/Prague, 1967.
[9] J. Sanchez-Hubert, E. Sanchez-Palencia, Coques élastiques minces: propriétés asymptotiques, Masson, Paris, 1997.
[10] B.A. Shoikhet, On asymptotically exact equations of thin plates of complex structure, Prikl. Mat. Meh. 37 (1973) 914–924; J. Appl. Math. Mech. 37 (1973–1974) 867–877.