Available online at www.sciencedirect.com

ScienceDirect

C. R. Mecanique 334 (2006) 701–706

COMPTES RENDUS MECANIQUE

http://france.elsevier.com/direct/CRAS2B/

Eshelby generalization for the dynamic J, L, M integrals

Xanthippi Markenscoff

Department of Applied Mechanics and Engineering Sciences, University of California, San Diego, La Jolla, CA 92093-0411, USA

Received 13 September 2006; accepted 26 September 2006

Available online 17 November 2006

Presented by André Zaoui

Abstract

The dynamic generalization—in the presence of inertia forces—of Eshelby's force on an elastic singularity is presented, where the total change of the energy of the system in two different defect motions, differing by an infinitesimal displacement throughout the history of the motion, is computed by considering the difference in the work of the tractions (with inertia forces considered as body forces) on a cut-out surface in Eshelby's thought 'cut and re-insert' experiment needed to realize the shift of the defect in the different motions. This expression, which coincides with a surface-independent obtained by Fletcher (1976) by applying Noether's theorem applied on the Lagrangian, is defined as the dynamic J integral. Changes of the energy of the system as computed by the changes in the work of the tractions (by the same thought experiment) needed to realize the rotation of the defect yield an expression that coincides with another expression obtained by Fletcher, and is defined as the dynamic L integral with meaning of a moment on an elastic singularity, while changes in the work of the tractions with respect to a self-similar scaling parameter coincide with another conserved expression in Fletcher, which is defined as the dynamic M integral. ***To cite this article: X. Markenscoff, C. R. Mecanique 334 (2006).***

Résumé

Généralisation d'Eshelby pour les intégrales dynamiques J, L et M. Nous présentons la généralisation dynamique de la force d'Eshelby agissant sur une singularité élastique en présence des forces d'inertie. Dans le cas étudié, on évalue la variation totale de l'énergie du système avec deux mouvements différents d'un défaut en considérant la différence du travail des forces de traction (en tenant compte de forces inertielles considérées aussi comme des forces appliquées au corps) sur la surface d'un élément découpé dans l'expérience imaginaire d'Eshelby (« découpage et la re-insertion ») permettant d'effectuer le déplacement du défaut dans des mouvements différents. Cette expression, qui coïncide avec l'expression obtenue par Fletcher (1976) et qui ne dépend pas du choix de la surface par l'application du théorème de Noether au Lagrangien du problème, est définie comme l'intégrale dynamique J. Les variations de l'énergie du système considérées comme celles du travail des forces de traction nécessaires pour effectuer une rotation infinitésimale du défaut donnent une expression coïncidant avec une autre formule obtenue par Fletcher, et sont définies comme l'intégrale dynamique L correspondant au moment cinétique de la singularité élastique. Les variations du travail des forces de traction produisant la variation du paramètre d'échelle coïncident avec une autre quantité conservée obtenue dans Fletcher, définie comme l'intégrale dynamique M. ***Pour citer cet article : X. Markenscoff, C. R. Mecanique 334 (2006).***

Keywords: Computational solid mechanics; Elastic singularities; Dynamic integrals; Defect motion

Mots-clés : Mécanique des solides numérique ; Singularités élastiques ; Intégrales dynamiques ; Mouvements des défauts

E-mail address: xmarkens@ucsd.edu (X. Markenscoff).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2006.10.006

1. Introduction

In a history making paper with remarkable intuition and insight, Eshelby [1,2], defined the 'force on an elastic singularity' as the change of the total energy of the system (in the whole body) due to an infinitesimal displacement of the defect, in effect comparing the change of energies in two different configurations and, by a famous thought experiment ingenuously implementing the Lagrangian formulation in which the material displacement is $-\delta\xi_l$, with the defect in effect not shifting, so that finally the defect in physical space shifts by $+\delta\xi_l$. Eshelby's genius was to compute in static elasticity this difference in the work of the surface tractions needed to implement the shift of the defect, which equals the change in the total energy of the body, as a surface integral surrounding the defect, showing in addition that the resulting J surface integral is path independent. In two-dimensional deformation fields, the same result is derived for moving cracks [3]. The physical meaning of the conserved integrals J, L, M has been demonstrated by Budiansky and Rice [4] and Rice [5], for defects in static deformation fields.

Here we present a generalization of the static definition of the force on an elastic singularity for a moving defect, within elastodynamics, by comparing the changes in the total energy of the system in two different defect motions, the histories of which differ by a constant displacement $\delta\xi_l$ for all times. In fact, we follow exactly Eshelby's [2] celebrated cut and re-insert thought experiment (performed quasistatically at a fixed time t) and compute (in the presence of inertia forces considered as body forces) the work of the tractions on the surface S' enclosing the cut-out volume of the replica (due to the incremental displacement because of the shift) from which we subtract the work of the tractions needed to reinsert the cut-out material into the hole S. This is the net increment in the energy of the system to have the defect (shifted in the two different defect motions) and, as in statics, where Eshelby showed that it is independent of the choice of the surface S, we show here that it is "surface independent" in dynamics as well, since the expression coincides with a conservation law obtained by Fletcher [6] on the basis of Noether's theorem. Since the additional volume integral that appears in dynamics contains singularities, it has to be considered as Principal Value and each defect has to be analyzed separately.

By the same procedure of computing the work of the tractions in the thought experiment and comparing the changes in the work of the tractions to cut and re-weld in two different defect motions, the histories of which differ by a constant infinitesimal rotation of the defect, we obtain an expression which coincides with another one of Fletcher [6], which we define as the dynamic L integral having the physical meaning of a 'moment' on an elastic singularity for dynamics. By computing the total energy change of the body with respect to a self-similar scaling parameter in two different defect motions, computed by the same thought experiment that computes the difference of the work of the surface tractions needed to implement the new configuration, we obtain an expression which is shown to coincide with another conserved expression of Fletcher [6], now defined as the dynamic M integral. Eischen and Herrmann [7] stated that the balance laws derived for elastodynamics in [6] "*are not directly related to the energy release rates for defect motion*". Here, we give to these conservation laws physical meaning as changes of total energy of the system with rotation of the defect (L integral) and a self-similar scaling parameter (M integral), by comparing two different motions as described above. While this dynamic force on an elastic singularity as presented here is distinct in concept from the energy-release rate through the tip of the crack [8–11], that considers energy balance through a given moving mass of material (shrinking to zero at the singularity), indeed the two are shown to be equivalent for moving cracks [11–13]. For other defects, such as dislocations, etc., and for the L and M integrals for cracks, separate analyses are needed.

2. Extension of Eshelby's construction of the force on an elastic singularity to elastodynamics

Our definition of the "force on an elastic singularity" in the dynamic case is based on comparing two energy functionals of the system at a time t, for two different motions of the defect $\xi(t)$ and $\xi'(t)$ related to each other by a constant displacement $\delta\xi$: $\xi'(t) = \xi(t) + \delta\xi$, where $\xi(t)$ is a vector in three-dimensional space. The two defect positions are separated initially (at time $t = -\infty$) by $\delta\xi$ and follow identical paths thereafter for all t. The task is how to compute at a time t the *difference in the energy in the whole domain for the two different motions of the defect* $\xi(t)$ and $\xi'(t)$. This is an interpretation that we give to the concept of configurational mechanics, namely, what is the difference in the energies of two systems with different configurations, and by this definition the difference in configuration is uniquely defined. A translation of the defect may be considered to be effected by a 'force', which is defined as a 'dynamic force on an elastic singularity'. The Eshelby thought experiment implements this in statics, by computing the energy change as the difference of work of tractions on a surface surrounding the defect (needed to shift and re-insert a cut-out volume) during a thought experiment that results in the defect translating by $\delta\xi$. In the dynamic case, we repeat Eshelby's thought experiment and compute the work of the same tractions in the presence of inertia forces acting as body forces, performed quasistatically at time t.

We consider the dynamic elasticity equations, with linear momentum equation

$$\sigma_{ij,i} = \partial(\rho\dot{u}_j)/\partial t \tag{1}$$

in Lagrangian coordinates where σ_{ij} denotes the first Piola–Kirchhoff stress tensor, $\sigma_{ij}n_j$ the force per unit reference area ('natural traction'), ρ the density in the reference state, W the strain energy density per unit reference volume with $\sigma_{ij} = \frac{\partial W}{\partial u_{i,j}}$, and $\frac{\partial}{\partial t}$ the material derivative at constant Lagrangian coordinates. As in statics, the dynamic J integral is valid for nonlinear elasticity, while the dynamic integral M is only for linear, and the dynamic L integral is not valid for anisotropy [6].

We follow Eshelby's [6] 'cut and re-insert' thought experiment which we repeat below in its generalization to elastodynamics. These steps are happening quasistatically at time t. We have a loaded body containing a defect enclosed by an arbitrary surface S which encloses a volume V. We also consider a replica of it with S marked out, and also denoting the surface produced by giving S a vector displacement $-\delta\xi_l$, in the *underformed state* (see figure in [2]). This process is along the lines of 'reverse motion' [12]. We denote by V' the volume enclosed by S'which is equal to V. We then do the following operations in the four steps as in Eshelby [2], in order to compute the energy of a system in which the defect has been displaced by $+\delta\xi_l$:

(i) In the original system, cut-out the material inside S and discard it. Apply suitable tractions to the surface of the resulting hole to prevent relaxation.

(ii) Cut out the piece of material inside in the replica and apply surface tractions to prevent relaxation. We denote by u_i' the displacement in the replica and by u_i in the original body, so that the difference for a translation by $-\delta\xi_l$ is

$$u_i' - u_i = \delta u_i = -u_{i,l}\delta\xi_l + O(\delta\xi^2) \tag{2}$$

We will next compute $\int_S n_i\sigma_{ij}\delta u_j\,\mathrm{d}S$, the work of the surface tractions on the surface S' enclosing the cut-out volume $V' = V$ due to the displacement δu_j. We start from the equation of conservation of linear momentum (1), multiply (1) by δu_j and integrate in the volume V (e.g., [14, p. 316]) to obtain

$$\int_V \sigma_{ij,i}\delta u_j\,\mathrm{d}V = \int_V \frac{\partial}{\partial t}(\rho\dot{u}_j)\delta u_j\,\mathrm{d}V \tag{3}$$

We use differentiation properties, and make use of (3) to obtain

$$\int_V \left[(\delta u_j\sigma_{ij})_{,i} - \delta u_{j,i}\sigma_{ij}\right]\mathrm{d}V = -\int_V \rho\dot{u}_j\delta\dot{u}_j\,\mathrm{d}V - \int_V \frac{\partial}{\partial t}(\rho\dot{u}_j u_{j,l})\delta\xi_l\,\mathrm{d}V \tag{4}$$

from which follows, by Gauss' theorem on the first term of the L.H.S. of Eq. (4),

$$\int_S n_i\sigma_{ij}\delta u_j\,\mathrm{d}S = \int_V \sigma_{ij}\delta u_{j,i}\,\mathrm{d}V - \int_V \rho\dot{u}_j\delta\dot{u}_j\,\mathrm{d}V - \int_V \frac{\partial}{\partial t}(\rho\dot{u}_j u_{j,l})\delta\xi_l\,\mathrm{d}V \tag{5}$$

This is the desired expression, but the first two volume integrals in (6), can be further reduced to surface integrals as in Eshelby's articles [1,2]:

$$\int_V \sigma_{ij}\delta u_{j,i}\,\mathrm{d}V = -\int_V \sigma_{ij}u_{j,il}\delta\xi_l\,\mathrm{d}V = -\delta\xi_l\int_S W\,\mathrm{d}S_l \tag{6}$$

$$\int_V \rho\dot{u}_j\delta\dot{u}_j\,\mathrm{d}V = -\delta\xi_l\int_V \rho\dot{u}_j\dot{u}_{j,l}\,\mathrm{d}V = -\delta\xi_l\int_S T\,\mathrm{d}S_l \tag{7}$$

Eq. (7) may be seen if we consider the difference of the kinetic energies in the volume enclosed by and S' and S:

$$\int_{V'} \frac{1}{2}\rho \dot{u}'_j \dot{u}'_j \,\mathrm{d}V - \int_V \frac{1}{2}\rho \dot{u}_j \dot{u}_j \,\mathrm{d}V = \int_{V'} \frac{1}{2}\rho (\dot{u}_j - \dot{u}_{j,l}\delta\xi_l)(\dot{u}_j - \dot{u}_{j,l}\delta\xi_l)\,\mathrm{d}V - \int_V \frac{1}{2}\rho \dot{u}_j \dot{u}_j \,\mathrm{d}V + O\big(\delta\xi_l^2\big)$$
$$= -\delta\xi_l \int_V \rho \dot{u}_j \dot{u}_{j,l}\,\mathrm{d}V + O\big(\delta\xi_l^2\big) \tag{8}$$

because $V = V'$. Eq. (5) is thus written

$$\int_S n_i \sigma_{ij} \delta u_j \,\mathrm{d}S = -\delta\xi_l \int_S W \,\mathrm{d}S_l + \delta\xi_l \int_S T \,\mathrm{d}S_l - \delta\xi_l \int_V \frac{\partial}{\partial t}(\rho \dot{u}_j u_{j,l})\,\mathrm{d}V \tag{9}$$

The last integral in (9) is called pseudomomentum, and the time derivative may be taken outside the integral in Lagrangian coordinates ([12, p. 82], [14], also [6]). This order of differentiation and integration is allowed to be interchanged (and the derivative allowed to be taken outside the volume integral) if in the neighborhood of t the volume integral converges uniformly. As stated by Eshelby [15], when a cut-off radius needs to be introduced because of singularities, atomistics are needed, so for dislocations, separate analysis is required.

(iii) We try to fit the body bounded by S' into the hole S, where n_j is the outward normal to S. For that purpose, since the shape of the surface S' was changed by δu_i we give a displacement $\delta u_i = -u_{i,l}\delta\xi_l$ to the surface of the hole S, which requires *expenditure* of work, so that from (9) and (10)

$$-\int_S \delta u_i \sigma_{ij} \,\mathrm{d}S_j = \delta\xi_l \int_S u_{i,l}\sigma_{ij} n_j \,\mathrm{d}S \tag{10}$$

$$\delta E^{\mathrm{Total}} = -\delta\xi_l \int_S W \,\mathrm{d}S_l + \delta\xi_l \int_S T \,\mathrm{d}S_l - \delta\xi_l \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \dot{u}_j u_{j,l}\,\mathrm{d}V + \delta\xi_l \int_S u_{i,l}\sigma_{ij} n_j \,\mathrm{d}S \tag{11}$$

If there is no singularity, then the difference in the work of the tractions $\int_S n_i \sigma_{ij} \delta u_j \,\mathrm{d}S$ to shift and reinsert would have been zero. However, if there is an elastic singularity, then this difference, which is given by Eq. (11), is defined as a configurational force on the defect:

$$F_1^{\mathrm{dyn}} = \int_S \big[(W - T)\delta_{lj} - \sigma_{ij} u_{i,l}\big]\,\mathrm{d}S_j + \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \dot{u}_j u_{j,l}\,\mathrm{d}V \tag{12}$$

If V is a region of analyticity, expression (12) coincides with the one given by Fletcher [6], in integral form, derived from the local one [6]

$$\frac{\partial}{\partial x_j}\big[(W - T)\delta_{lj} - \sigma_{ij} u_{i,l}\big] + \frac{\partial}{\partial t}(\rho \dot{u}_i u_{i,l}) = 0 \tag{13}$$

which is obtained by application of Noether's theorem on the Lagrangian. If the integral of the pseudomomentum in a volume surrounding the defect as a Cauchy Principal Value, or in the sense of distributions—according to the particular defect—is added to the volume integral obtained from integrating (13) in the region of analyticity between two different surfaces surrounding the defect, then expression (11) is obtained from (13) as 'surface independent'. Each defect has to be analyzed separately.

The relation of expression (11) to the energy release rate for moving cracks has been indicated by Eshelby [15] (by use of the 'transport assumption' at the tip, and the reduction of the volume integral to a surface one of twice the kinetic energy), and analytically treated in [11–13] where it is shown that (12) coincides with the energy-release rate through a surface ∂D surrounding the defect, as $\partial D \to 0$. Indeed, for the near field asymptotic steady-state approximation $\frac{\partial}{\partial t} = -\dot{\xi}_l \frac{\partial}{\partial x_l}$ we have

$$\lim_{D_l, \partial D_l \to 0} \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_l} \rho \dot{u}_i u_{i,l}\,\mathrm{d}V = -\lim_{D_l, \partial D_l \to 0} \frac{\mathrm{d}}{\dot{\xi}_l\,\mathrm{d}t}\int_{D_l} \rho \dot{u}_i u_{i,l}\,\mathrm{d}V = \lim_{D_l, \partial S \to 0}\int_{\partial S} \rho \dot{u}_i u_i n_l \,\mathrm{d}S$$

so that Eq. (12) yields

$$F_l^{\mathrm{dyn}} = \lim_{\partial D \to 0}\int_{\partial D} \big[(W + T)n_l - u_{j,l}\sigma_{jk} n_k\big]\,\mathrm{d}S \tag{14}$$

which is the energy release rate as defined by [8–10].

3. The dynamic "Moment on an Elastic Singularity"

We define the effective 'moment' on the singularity as $-\frac{\partial E}{\partial \omega_i}$, i.e., the change of the total energy of the system with change of angle of rotation $\delta\omega_i$ of the defect by comparing two different motions as differing by a rotation of the defect about a point at a distance $\vec{r}$ so that

$$\delta\xi_l = \varepsilon_{ljk} x_k \delta\omega_j. \tag{15}$$

We compute again the work of the surface tractions on S' with displacement δu_j (given by (2)), and subtract the work of the tractions needed to re-insert the replica into the hole. But these expressions, will be in terms of $\delta\omega_l$ rather than $\delta\xi_j$, and we proceed as before, by writing Eqs. (3) and (4) with $\delta\xi_l$ given by Eq. (15). Now, instead of (5), we have

$$\begin{aligned}\int_V \big[(\delta u_j \sigma_{ij})_{,i} - \delta u_{j,i}\sigma_{ij}\big]\,\mathrm{d}V &= -\int_V \rho \frac{\partial \dot u_j}{\partial t} u_{j,l} \varepsilon_{lkm} x_m \delta\omega_k \,\mathrm{d}V \\ &= -\int_V \frac{\partial}{\partial t}(\rho \dot u_j u_{j,l} \varepsilon_{lkm} x_m)\delta\omega_k \,\mathrm{d}V - \int_V \rho \dot u_j \delta \dot u_j \,\mathrm{d}V\end{aligned} \tag{16}$$

By performing the indicated calculations in (16) and adding (10), the incremental work of the tractions on the surface surrounding the defect (in the cut and re-insert experiment), when the singularity rotates by $\delta\omega_i$, is obtained and is equated to the difference of the energies in the two different configurations, so that we define as the moment on an elastic singularity

$$\begin{aligned}M_i = -\frac{\partial E}{\partial \omega_i} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_V (\rho\varepsilon_{ijk} u_k \dot u_j + \rho \varepsilon_{ijk} x_j \dot u_m u_{m,k})\,\mathrm{d}V \\ &+ \int_S \big(\varepsilon_{imj} u_m \sigma_{jk} - \varepsilon_{imj} x_j u_{l,m} \sigma_{lk} + \varepsilon_{imk} x_m (W - T)\big)\,\mathrm{d}S_k \equiv L_i\end{aligned} \tag{17}$$

which now gives physical meaning to the dynamic L integral. In (17) the time derivative is again taken outside the integral ([12,6]). Expression (17) obtained by the Eshelby thought experiment (by computing the work of the tractions) coincides indeed with the corresponding conservation integral obtained on the basis of Noether's theorem by Fletcher [6].

4. The physical meaning of the dynamic *M* integral

We proceed in an analogous way as with the L integral. In this case

$$\delta u_i = -u_{i,l} x_l \delta\gamma \tag{18}$$

where γ is a self-similar scaling parameter, and we make use of the relations:

$$\begin{aligned}\int_V \sigma_{ij}\delta u_{j,i}\,\mathrm{d}V = -\delta\gamma \int_V \sigma_{ij} u_{j,il} x_l \,\mathrm{d}V &= -\delta\gamma \int C_{ijrs} u_{r,s} u_{j,il} x_l \,\mathrm{d}V \\ &= -\delta\gamma \int_V \frac{\partial}{\partial x_l}\left(\frac{1}{2} C_{ijrs} u_{r,s} u_{j,i} x_l\right)\mathrm{d}V + \delta\gamma \int_V \frac{1}{2} C_{ijkl} u_{r,s} u_{j,i} x_{l,l}\,\mathrm{d}V\end{aligned} \tag{19}$$

and the identity [7]

$$L + \rho \dot u_i \dot u_i = \frac{\partial}{\partial t}[L + \rho t \dot u_i \dot u_i] - t(\sigma_{ij}\dot u_i)_{,j}$$

with $L \equiv W - T$.

Proceeding with the calculation of the change of the work of the tractions while performing the cut and re-insert experiment, as before, according to the Eshelby steps (i) through (iv), and using the identities above, after a semi-involved calculation, which we omit here as straightforward, we obtain in 3-D space ($x_{l,l} = 3$) the dynamic M integral:

$$M \equiv -\frac{\partial E}{\partial \gamma} = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_V \left[\rho \dot{u}_j (u_j + x_m u_{j,m} + t\dot{u}_j) + tL\right] \mathrm{d}X - \int\limits_S \left[\sigma_{jk} n_k (u_j x_m u_{j,m} + t\dot{u}_j) - n_k x_k L\right] \mathrm{d}S \tag{20}$$

which coincides with corresponding expression obtained according to Noether's theorem in [6], valid in the region of analyticity.

Acknowledgements

The contributions of Dr. L. Ni and an anonymous referee are gratefully acknowledged. The support of LMS at the Ecole Polytechnique, Palaiseau, is also gratefully acknowledged, as well as NSF grant CMS-0555280.

References

[1] J.D. Eshelby, The force on an elastic singularity, Phil. Trans. 244 (1951) 87–111.
[2] J.D. Eshelby, The elastic energy-momentum tensor, J. Elasticity 5 (3–4) (1975) 321–335.
[3] J. Rice, Mathematical analysis in the mechanics of fracture, Fracture II (1968) 191–311.
[4] B. Budiansky, J. Rice, Conservation laws and energy release rate, J. Appl. Mech. 40 (1973) 201–203.
[5] J. Rice, Conserved integrals and energetic forces, in: B.A. Bilby, K.J. Miller, J.R. Willis (Eds.), Fundamentals of Deformation and Fracture (Eshelby Memorial Symposium), Cambridge Univ. Press, Cambridge, UK, 1985.
[6] D.C. Fletcher, Conservation laws in linear elastodynamics, Arch. Rat. Mech. Anal. 60 (1976) 329–353.
[7] J.W. Eischen, G. Herrmann, Energy release rates and related balance laws in linear elastic defect mechanics, Trans. ASME 54 (1987) 388–392.
[8] C. Atkinson, J.D. Eshelby, The flow of energy into the tip of a moving crack, Int. J. Fracture 4 (1968) 3.
[9] B.V. Kostrov, L.V. Nikitin, Some general problems of mechanics of brittle fracture, Archivum Mechaniki Stosowanej 22 (1970) 749–775.
[10] L.B. Freund, Energy flux into the tip of an extending crack in an elastic solid, J. Elasticity 2 (1972) 341–349.
[11] C. Dascalu, G.A. Maugin, Forces materielles et taux de restitution de l'énergie dans les corps elastiques homogenes, C. R. Acad. Sci. Paris, Ser. II 317 (1993) 1135.
[12] G.A. Maugin, Material Inhomogeneities in Elasticity, Chapman & Hall, London, 1993.
[13] L.B. Freund, Dynamic Fracture Mechanics, Cambridge Univ. Press, Cambridge, UK, 1990.
[14] Y.C. Fung, Foundation of Solid Mechanics, Prentice-Hall, New York, 1965.
[15] J.D. Eshelby, Energy relations and the energy-momentum tensor in continuum mechanics, in: M.F. Kanninen, et al. (Eds.), Inelastic Behavior of Solids, McGraw–Hill, New York, 1970, pp. 77–115.