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Boundary conditions for elastic beam bending

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Abstract

For beam bending problem, the reciprocal theorem and P–N solution are applied in a novel way to obtain the appropriate stress and mixed boundary conditions accurate to all order. Through generalizing the method proposed by Gregory and Wan, a set of necessary conditions on the edge-data for the existence of a rapidly decaying solution is established. When stress and mixed conditions are imposed on the beam edge, these decaying state conditions are derived explicitly, and they are used for the correct formulation of boundary conditions for the interior solution. For the stress data, our boundary conditions coincide with those obtained in conventional forms of beam theories. More importantly, the appropriate boundary conditions with two different sets of mixed edge-data are obtained for the first time. ***To cite this article: Y. Gao et al., C. R. Mecanique 335 (2007).***

Résumé

Conditions aux limites pour une flexion élastique des poutres. Dans le problème de la flexion des poutres nous utilisons le théorème de réciprocité et la solution de Papkovich–Neuber pour trouver, de manière inédite, les conditions mixtes de tension aux limites avec une exactitude appropriée à tous les ordres d'approximation. En généralisant la méthode proposée par Gregory et Win, on établit l'ensemble des conditions nécessaires pour les données sur les cotés assurant l'existence des solutions évanescentes. Dans le cas où les conditions sur les tensions mixtes sont imposées sur le coté d'une poutre, les conditions assurant l'apparition d'états évanescents sont dérivées explicitement. Nous les utilisons par la suite dans une formulation correcte des conditions aux limites pour la solution intérieure. Nos conditions sur les tensions coïncident avec celles obtenues par la théorie de la flexion des poutres usuelle. Le résultat le plus important est l'obtention, pour la première fois, des conditions aux limites appropriées avec deux ensembles différents des données mixtes sur le bord de la poutre. ***Pour citer cet article : Y. Gao et al., C. R. Mecanique 335 (2007).***

Keywords: Computational solid mechanics; Boundary conditions; Beam bending; Decaying states; Regular states

Mots-clés : Mécanique des solides numérique ; Conditions aux limites ; Fléxion des poutres ; États évanescents ; États réguliers

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Il est bien connu que la solution exacte d'un problème d'élasticité statique pour des poutres minces et plates est composée de deux parties, la composante intérieure et la composante dans la couche extérieure (sous forme évanescente). La détermination de la composante extérieure nécessaire pour satisfaire les conditions aux limites, s'avère souvent très difficile, sauf les cas particuliers d'une symétrie prononcée ou une géométrie très simple.

En appliquant le théorème de réciprocité de Betti-Rayleigh, Gregory et Wan ont développé une technique d'analyse permettant de trouver la solution intérieure explicite grâce à une formulation correcte des conditions aux limites pour les données sur le bord arbitraires [1–8]. Ils ont aussi démontré que le principe de Saint-Venant ne pourrait être appliqué que pour le terme dominant dans la solution extérieure, c'est-à-dire dans le cadre de la théorie classique des plaques uniquement. Nous examinons une formulation des conditions pour les poutres élastiques permettant les applications aux problèmes plus généraux.

Nous prenons pour point de départ les équations d'équilibre d'élasticité statique reliant les déplacements u_x et u_z au tenseur des contraintes σ_{ij} :

$$\nabla^2 u_x + (1+\nu)/(1-\nu)\partial_x(\partial_x u_x + \partial_z u_z) = 0, \qquad \nabla^2 u_z + (1+\nu)/(1-\nu)\partial_z(\partial_x u_x + \partial_z u_z) = 0$$

Ces équations décrivent les déformations d'une poutre mince et longue d'une épaisseur unitaire le long de l'axe y.

Ensuite nous généralisons la méthode proposée par Gregory et Wan [2] et, en invoquant la solution de Papkovich et Neuber (P–N), les conditions nécessaires à l'existence des solutions rapidement évanescentes sont exprimées par un ensemble des conditions aux limites imposées sur les bords de la poutre.

Le théorème de réciprocité pour une poutre mince peut être exprimé par l'intégrale sur la surface :

$$\oiint_S \left(\sigma_{ij}^{(1)} u_j^{(2)} - \sigma_{ij}^{(2)} u_j^{(1)}\right) n_i \, \mathrm{d}S = 0$$

À partir de cette intégrale, en la transformant selon l'exemple de Gregory et Wan, nous établissons les conditions nécessaires assurant l'existence des solutions évanescentes dans quatre cas différents correspondant aux situations géométriques différentes. Ensuite, les solutions intéreures peuvent être déterminées de façon unique pour chacun de ses cas particuliers.

1. Introduction

It is generally known that the exact solution of linear elastostatic problems for slender and thin elastic bodies consists of an interior component and a boundary layer component (in a decaying form). Near a lateral edge, the interior solution is supplemented by boundary layer solution component which becomes insignificant away from the edge. The admissible boundary conditions can be satisfied only by a combination of these components. However, the boundary layer solution, even just a leading term approximation, needed to fit the edge-data is rather intractable except for cases with simple geometries and load symmetries. This and the fact that the solution behavior near the edges is often not needed from practical viewpoint have driven people to take efforts to formulate the interior solution, by assigning an appropriate portion of the prescribed edge-data to it, without any reference to the boundary layer solution.

By an application of the Betti-Rayleigh reciprocal theorem, Gregory and Wan developed a decay analysis technique determining the interior solution successfully and effectively and provided the results for several plate problems, and derived a set of correct boundary conditions for arbitrarily prescribed admissible edge-data [1–8]. From these results, they have now explicit examples showing that the higher order accuracy offered by the governing differential equations of a higher order plate theory may not be attained unless commensurate boundary conditions are developed and used for these equations. These general results also show that, to be strictly correct, Saint-Venant's principle should be applied only to the leading term outer solution, i.e. the classical plate theory.

Otherwise, relevant boundary conditions for elastic beam have not been attempted. Due to its importance, a parallel development of boundary conditions for elastic beam which are formulated in the present paper should be allowed for applications to a broader class of problems. In the following sections, we obtain a set of necessary conditions on the edge-data for the existence of a rapidly decaying solution of the beam problems. Through generalizing the method brought forward by Gregory and Wan [2] and by invoking Papkovich–Neuber (P–N) solution, these necessary conditions are then translated into the desired set of boundary conditions for the interior expansion.

Our results extend the known results to beam problems, which enable us to formulate the correct edge conditions for one-dimensional beam theories with stress and mixed edge-data. For the stress data, our boundary conditions on edge-data for a decaying state are consistent with conventional boundary conditions of beam theories. For the mixed edge-data, the appropriate edge conditions for the interior solution, not previously known in the literature, are also obtained for two different sets.

2. Necessary conditions for a decaying state

Let us consider a homogeneous, isotropic and linearly elastic beam as a plane stress problem. In a fixed rectangular coordinate system, z is the coordinate normal to the neutral surface (x–y plane) of the beam. We assume the beam length in x-direction is l, beam width in y-direction is assumed 1, beam height in z-direction is $2h$, and $l \gg 2h \gg 1$. In the absence of body force, the equilibrium equations of elasticity plane stress problem expressed by displacements u_x and u_z are

$$\nabla^2 u_x + (1+\nu)/(1-\nu)\partial_x(\partial_x u_x + \partial_z u_z) = 0, \qquad \nabla^2 u_z + (1+\nu)/(1-\nu)\partial_z(\partial_x u_x + \partial_z u_z) = 0 \tag{1}$$

where $\nabla^2 = \partial_x^2 + \partial_z^2$ is two-dimensional Laplacian operator, ν, E and μ are Poisson's ratio, the Young's modulus and the shear modulus, respectively. By taking advantage of P–N solution for beam problem, the expressions of displacements and stresses can be obtained as

$$\begin{aligned} &u_x = P_1 - (1+\nu)/4\partial_x e, \qquad u_z = P_3 - (1+\nu)/4\partial_z e \\ &\sigma_{xx} = \mu\big[(2+\nu)\partial_x P_1 + \nu\partial_z P_3 - (1+\nu)/2\partial_x^2 e\big], \qquad \sigma_{xz} = \mu\big(\partial_z P_1 + \partial_x P_3 - (1+\nu)/2\partial_x\partial_z e\big) \\ &\sigma_{zz} = \mu\big[\nu\partial_x P_1 + (2+\nu)\partial_z P_3 - (1+\nu)/2\partial_z^2 e\big] \end{aligned} \tag{2}$$

where $e = P_0 + xP_1 + zP_3$, P_0, P_1 and P_3 are harmonic functions.

The top and bottom faces of the beam are taken to be traction free, so that

$$\sigma_{xz} = \sigma_{zz} = 0 \quad (z = \pm h) \tag{3}$$

The presence of any body or surface loads may be removed by a particular solution. Then the only forcing terms in the problem are prescribed on the end $x = 0$ in terms of stress or displacement edge-data in the form of one of the following four admissible combinations,

$$\text{Case (A):} \quad \sigma_{xx}(0,z) = \bar{\sigma}_{xx}(z), \quad \sigma_{xz}(0,z) = \bar{\sigma}_{xz}(z) \tag{4}$$

$$\text{Case (B):} \quad u_x(0,z) = \bar{u}_x(z), \quad \sigma_{xz}(0,z) = \bar{\sigma}_{xz}(z) \tag{5}$$

$$\text{Case (C):} \quad \sigma_{xx}(0,z) = \bar{\sigma}_{xx}(z), \quad u_z(0,z) = \bar{u}_z(z) \tag{6}$$

$$\text{Case (D):} \quad u_x(0,z) = \bar{u}_x(z), \quad u_z(0,z) = \bar{u}_z(z) \tag{7}$$

In generalization of analogous statements for elastic plates [2], we introduce two definitions for two equilibrium states as follows:

Definition 1. The displacement fields $\boldsymbol{u}$ and stress fields $\boldsymbol{\sigma}$ induced by the prescribed edge-data are said to give rise to a *decaying state* within the beam if they satisfy the condition

$$\{\boldsymbol{u}, \boldsymbol{\sigma}\} = \mathrm{O}\big(M_1 \mathrm{e}^{-\gamma d/h}\big) \quad \text{as } h \to 0 \tag{8}$$

where d is the minimum distance of the observation point from the edge of the beam, M_1 are the maximum modulus of the prescribed edge-data for the decaying state, M_1 and γ are positive constants.

Definition 2. The displacement fields $\boldsymbol{u}$ and stress fields $\boldsymbol{\sigma}$ are said to be a *regular state* within the beam if they satisfy the condition

$$\{\boldsymbol{u}, \boldsymbol{\sigma}\} = \mathrm{O}\big(M_2 h^{\alpha}\big) \quad \text{as } h \to 0 \tag{9}$$

where M_2 are the maximum modulus for the regular state, $M_2 > 0$ and $\alpha \geqslant 0$. That means the fields have at most an algebraic growth as $h \to 0$.

Supposing that the stress data does give rise to the decaying state in the beam, we now apply the reciprocal theorem for a beam, which takes the form

$$\oiint_S \left(\sigma_{ij}^{(1)} u_j^{(2)} - \sigma_{ij}^{(2)} u_j^{(1)}\right) n_i \, \mathrm{d}S = 0 \tag{10}$$

where S is the surface of the beam which consists of two end planes and a lateral surface, n_i is the direction cosine of the outward normal to S. With the foregoing two definitions in mind, now we take the state with a superscript “(1)” to be the exact solution of beam bending problem, and the decaying state induced by the prescribed edge-data $\bar{\sigma}_{xx}$, $\bar{\sigma}_{xz}$, $\bar{u}_x$, $\bar{u}_z$. For the auxiliary state, denoted by a superscript “(2)”, we take any regular state which fulfills load-free conditions on S. Similar to the derivation of necessary conditions for a decaying state in the plate, generalizing Gregory and Wan’s decay analysis technique to a beam, we finally obtain the necessary conditions for a decaying state

$$\text{Case (A):} \quad \int_{-h}^{h} \left[\bar{\sigma}_{xx}(z) u_x^{(2)} + \bar{\sigma}_{xz}(z) u_z^{(2)}\right]_{x=0} \mathrm{d}z = 0 \tag{11}$$

$$\text{Case (B):} \quad \int_{-h}^{h} \left[\bar{u}_x(z) \sigma_{xx}^{(2)} - \bar{\sigma}_{xz}(z) u_z^{(2)}\right]_{x=0} \mathrm{d}z = 0 \tag{12}$$

$$\text{Case (C):} \quad \int_{-h}^{h} \left[\bar{u}_z(z) \sigma_{xz}^{(2)} - \bar{\sigma}_{xx}(z) u_x^{(2)}\right]_{x=0} \mathrm{d}z = 0 \tag{13}$$

$$\text{Case (D):} \quad \int_{-h}^{h} \left[\bar{u}_x(z) \sigma_{xx}^{(2)} + \bar{u}_z(z) \sigma_{xz}^{(2)}\right]_{x=0} \mathrm{d}z = 0 \tag{14}$$

These necessary conditions for the edge-data to induce only a decaying elastostatic state will be translated into appropriate boundary conditions for the beam later in next section.

3. Boundary conditions for the beam

The main difficulty in performing the preceding process lies in obtaining suitable regular states which satisfy the appropriate boundary conditions. However, for the case of elastic beam bending, the necessary regular states can be explicitly determined as follows, at least for edge-data in Cases (A), (B) and (C).

3.1. Case (A)

Our main task lies in obtaining accurate solutions for these regular states. We can take a rigid body translation in the z-direction as the first auxiliary regular state. Then $u_x^{(2)} \equiv 0$ and $u_z^{(2)} =$ constant so that Eq. (11) gives the first condition for the stress data

$$\int_{-h}^{h} \bar{\sigma}_{xz} \, \mathrm{d}z = 0 \tag{15}$$

Now we look for the second auxiliary regular state with the use of P–N solution for beam problem. According to the characteristics of bending problem, we assume

$$P_0 = C_1\left(z^3 - 3x^2 z\right) + C_2 xz, \qquad P_1 = 0, \qquad P_3 = C_3\left(z^2 - x^2\right) + C_4 x \tag{16}$$

where C_i $(i = 1, 2, 3, 4)$ are unknown constants to be determined later.

As the procedure in the preceding section indicates, any candidate for regular state (2) must meet the requirements stipulated below

$$\sigma_{xz}^{(2)} = \sigma_{zz}^{(2)} = 0 \quad (z = \pm h), \qquad \sigma_{xz}^{(2)} = \sigma_{xx}^{(2)} = 0 \quad (x = 0) \tag{17}$$

On substituting Eq. (16) into Eq. (2), then the results into Eq. (17), we can determine the relationship among these unknown constants as

$$C_1 = C_3 = 0, \qquad C_2 = (1-\nu)/(1+\nu)\cdot C_4 \tag{18}$$

which corresponds to a rigid body rotation. Inserting this auxiliary regular state (16) into Eq. (11) with the use of Eq. (15), after taking account of the relationship (18), we obtain the second necessary condition for a decaying state when $\bar{\sigma}_{xx}$ is prescribed

$$\int_{-h}^{h} \bar{\sigma}_{xx} z\,\mathrm{d}z = 0 \tag{19}$$

The necessary conditions (15) and (19) are conventional forms of elastic beam theories [9], although they are formulated explicitly by an application of the reciprocal theorem and P–N solution.

3.2. Case (B)

As in Case (A), selecting a rigid body translation as the first auxiliary regular state, we certainly must have the corresponding necessary conditions

$$\int_{-h}^{h} \bar{\sigma}_{xz}\,\mathrm{d}z = 0 \tag{20}$$

To obtain the second auxiliary regular state, the regular state (2) must meet the requirements stipulated as follows

$$\sigma_{xz}^{(2)} = \sigma_{zz}^{(2)} = 0 \quad (z=\pm h), \qquad \sigma_{xz}^{(2)} = 0, \quad u_x = 0 \quad (x=0) \tag{21}$$

Taking the potential functions P_0, P_1 and P_3 in Case (B) to be of the same form in Case (A), after using the condition (21) leads to

$$C_1 = (1-\nu)/\big[3(1+\nu)\big]\cdot C_3, \qquad C_2 = C_4 = 0 \tag{22}$$

With the help of Eqs. (20) and (22), the second necessary condition for a decaying state is obtained from Eq. (12) when $\bar{u}_x$ and $\bar{\sigma}_{xz}$ are prescribed

$$\int_{-h}^{h} \left(\bar{u}_x z + \frac{\nu}{2E}\bar{\sigma}_{xz} z^2\right)\mathrm{d}z = 0 \tag{23}$$

3.3. Case (C)

Similar to Case (A), we take the regular state (2) as a rigid body rotation, then obtain

$$\int_{-h}^{h} \bar{\sigma}_{xx} z\,\mathrm{d}z = 0 \tag{24}$$

With the features of bending problem, we take the harmonic functions as

$$P_0 = 0, \qquad P_1 = D_1\big(z^3 - 3x^2 z\big) + D_2 z, \qquad P_3 = D_3\big(x^3 - 3xz^2\big) + D_4 x \tag{25}$$

where D_i $(i = 1,2,3,4)$ are unknown constants yet to be determined. By noting that

$$\sigma_{xz}^{(2)} = \sigma_{zz}^{(2)} = 0 \quad (z=\pm h), \qquad \sigma_{xx}^{(2)} = 0, \quad u_z^{(2)} = 0 \quad (x=0) \tag{26}$$

we have the relationship among these unknown constants

$$D_1 = -(1-\nu)/(1+3\nu)\cdot D_3, \qquad D_2 + D_4 = -\big[24\nu(1+\nu)h^2\big]/\big[(1-\nu)(1+3\nu)\big]\cdot D_3 \tag{27}$$

On substituting the expressions (25) into Eq. (13), we obtain, after using Eqs. (24) and (27)

$$\int_{-h}^{h}\left[\bar{u}_z\left(h^2-z^2\right)+\frac{2+\nu}{3E}\bar{\sigma}_{xx}z^3\right]\mathrm{d}z=0 \tag{28}$$

4. Discussion and conclusions

These aforementioned necessary conditions for a decaying state (boundary layer solution) can then be converted into a set of boundary conditions appropriate for the interior solution or its various approximate beam theories, which do not involve the boundary layer solution components. As the preceding discussion in Introduction, the difference between the exact solution and the interior one is a decaying state. Then the above necessary conditions applied to the edge-data at $x=0$ of a beam

$$\bar{u}_x=\left[u_x-u_x^I\right]_{x=0},\quad \bar{u}_z=\left[u_z-u_z^I\right]_{x=0},\quad \bar{\sigma}_{xx}=\left[\sigma_{xx}-\sigma_{xx}^I\right]_{x=0},\quad \bar{\sigma}_{xz}=\left[\sigma_{xz}-\sigma_{xz}^I\right]_{x=0} \tag{29}$$

where u_x^I, u_z^I, σ_{xx}^I and σ_{xz}^I are interior solutions, give

$$\begin{gathered}\int_{-h}^{h}\left[\sigma_{xz}^I\right]_{x=0}\mathrm{d}z=\int_{-h}^{h}\hat{\sigma}_{xz}\,\mathrm{d}z,\qquad \int_{-h}^{h}\left[\sigma_{xx}^I z\right]_{x=0}\mathrm{d}z=\int_{-h}^{h}\hat{\sigma}_{xx}z\,\mathrm{d}z\\ \int_{-h}^{h}\left[u_x^I z+\frac{\nu}{2E}\sigma_{xz}^I z^2\right]_{x=0}\mathrm{d}z=\int_{-h}^{h}\left[\hat{u}_x z+\frac{\nu}{2E}\hat{\sigma}_{xz}z^2\right]\mathrm{d}z\\ \int_{-h}^{h}\left[u_z^I\left(h^2-z^2\right)+\frac{2+\nu}{3E}\sigma_{xx}^I z^3\right]_{x=0}\mathrm{d}z=\int_{-h}^{h}\left[\hat{u}_z\left(h^2-z^2\right)+\frac{2+\nu}{3E}\hat{\sigma}_{xx}z^3\right]\mathrm{d}z\end{gathered} \tag{30}$$

where $\hat{u}_x$, $\hat{u}_z$, $\hat{\sigma}_{xx}$ and $\hat{\sigma}_{xz}$ are the actually prescribed edge-data. The first two conditions in Eq. (30) are for Case (A), the first and third ones for Case (B), while the second and fourth ones correspond to Case (C). By superposition, the boundary conditions for bending deformation of the beam are formed. Thus a portion of the edge-data is effectively allocated to the interior solution, which is analogous to the assignment of edge-data in the form of resultant force and moment by Saint-Venant's principle. Similar as those pointed out by Gregory and Wan [2], the results reveal that indiscriminate extension of Saint-Venant's principle is not justified in general, which may lead to erroneous solution for the beam's deformation even away from the beam edge.

The results of the present paper extend the known results to beam problems, which enable us to establish a set of correct boundary conditions with stress and mixed edge-data. However, attempts to derive similar results on boundary conditions for pure displacement edge-data case have not been successful.

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