

Boussinesq equation, surface waves in fluids

J. Boussinesq and the standing water waves problem

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Abstract

In this short Note we present the original Boussinesq's contribution to the nonlinear theory of the two dimensional standing gravity water wave problem, which he defined as 'le clapotis'. **To cite this article:** *G. Iooss, C. R. Mecanique 335 (2007)*.
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R sum 

J. Boussinesq et le probl me du clapotis. Dans cette courte Note on pr sente, en la situant dans le contexte actuel, la contribution originale de Boussinesq sur la th orie non lin aire du «clapotis». **Pour citer cet article :** *G. Iooss, C. R. Mecanique 335 (2007)*.

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The two-dimensional standing gravity water wave problem has only recently been solved (see [1–3]); thus this is an opportunity for pointing out the seminal contribution of J. Boussinesq to this challenging nonlinear fluid dynamics problem.

Let us consider the classical 2-dimensional water wave problem where H is the depth at rest of the perfect incompressible fluid layer, the flow is assumed to be potential, and ϕ is the velocity potential. The free surface is $y = \eta(x, t)$, where y and x are respectively the vertical and horizontal coordinates. Then the problem is ruled by the nonlinear system:

$$\begin{aligned}\Delta\phi &= 0, & x, t \in \mathbb{R}, & -H < y < \eta(x, t) \\ \partial_y\phi &= 0, & x, t \in \mathbb{R}, & y = -H \\ \partial_t\eta + \partial_x\eta\partial_x\phi - \partial_y\phi &= 0, & x, t \in \mathbb{R}, & y = \eta(x, t) \\ \partial_t\phi + \frac{1}{2}(\nabla\phi)^2 + g\eta &= 0, & x, t \in \mathbb{R}, & y = \eta(x, t)\end{aligned}$$

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where g is the acceleration of gravity, which is the only external force acting on the system. The condition on $y = -H$ expresses that the velocity is tangential to the boundary at the bottom; in the case of an infinite depth layer one has to replace the second equation by

$$\nabla\phi \rightarrow 0 \quad \text{as } y \rightarrow -\infty$$

The third condition is kinematic and results from the definition of the free surface; the fourth condition comes from the Bernoulli first integral of the Euler equation for perfect fluids in case of a potential flow, and it expresses the continuity of the pressure in crossing the free surface. Up to differences of notations (specially about partial derivatives) Boussinesq knew this system, and studied the linearized problem on it. In fact, the linearized system was first derived satisfactorily by Poisson in 1818, although Laplace in 1776 had come very close (see [4]).

Considering the linearized system near $(\phi, \eta) = (0, 0)$ (flat free surface, fluid at rest), when we look for nontrivial solutions periodic in t and in x , corresponding to standing waves, under the form of

$$\phi(x, y, t) = A(y) \sin \omega t \cos kx \tag{1}$$

(this also holds for any ϕ obtained from the above in shifting coordinates x and t) one obtains the *dispersion relation*

$$\omega^2 = gk \tan kH \tag{2}$$

with

$$A(y) = \varepsilon \cosh k(y + H) \quad \text{for finite depth}$$

or

$$\omega^2 = gk$$

with

$$A(y) = \varepsilon e^{ky} \quad \text{for infinite depth}$$

As was observed in particular by Boussinesq, linear combinations of solutions (1) shifted in (x, t) may lead to travelling waves with velocity $c = \omega/k$ which may be written as

$$\frac{c^2}{gH} = \frac{\tan kH}{kH}$$

This equation is beautifully described by de Saint Venant in his paper in Comptes Rendus, 21 February 1870 (quoted by Boussinesq in [5] at p. 526), as

« cette équation exprime que le carré de la vitesse de propagation des ondes est à celui de la vitesse qu'un corps acquerrait en tombant en chute libre d'une hauteur égale à la moitié de la profondeur du liquide, comme la tangente hyperbolique du rapport de cette profondeur au rayon d'une circonférence égale à la longueur d'onde est à ce rapport lui-même. »

(This sentence, which was considered as being much clearer than any equation in 1870, is impossible to translate into Queen's English.)

Denoting by $T = 2\pi m/\omega$ and $\lambda = 2\pi n/k$, the time period and the wave length, we observe that *in the infinite depth case*, we obtain a solution of the form (1) for

$$\mu = gT^2/2\pi\lambda = m^2/n$$

where m and n are integers, i.e. when the dimensionless parameter μ takes *any rational value*. This means that in such a case the problem is highly degenerate because every function of the form

$$\phi(x, y, t) = \sin\left(\frac{2m\pi t}{T}\right) \cos\left(\frac{2n\pi x}{\lambda}\right) \exp\left(\frac{2n\pi y}{\lambda}\right)$$

is a solution of the linearized system of period T in t , and λ in x . Thus the set of 'eigenvalues' μ of the linearized problem is \mathbb{Q}^+ , is dense in $[0, \infty)$, and each eigenvalue has an infinite set of linearly independent eigenfunctions. We

refer to this latter property as *complete resonance*, by analogy with the theory of nonlinear oscillators. This fact, which leads to serious complications for the nonlinear study, was overlooked by most authors until Schwartz and Whitney [6] in 1981.

The dispersion equation (2) was obtained by Boussinesq in two different ways, the first way being mainly as above in [7], with an interesting construction of the solution of the Laplace equation in the (x, y) plane. *He did not use this Eulerian form of the system for studying the nonlinear terms.* Let us recall that in 1847 Stokes gave a nonlinear theory of *travelling waves* [8] up to the third order, while Boussinesq in 1877 was the first to deal with nonlinear *standing-waves*. On pages 332–335 and in footnotes (taking 90% of the pages!) in pages 348–353 of [7] he refers to ‘le clapotis’, meaning standing waves, and his treatment, which includes the cases of finite and infinite depth, is a nonlinear theory taken to second order in the amplitude and uses Lagrangian coordinates.

Let us write the Euler system, for perfect fluid flows subjected to gravity, in Lagrangian coordinates, where (X, Y) denotes the position of fluid particles at time $t = 0$:

$$\partial_{tt}x\partial_Xx + (\partial_{tt}y + g)\partial_Xy + \frac{1}{\rho}\partial_Xp = 0, \quad X, t \in \mathbb{R}, -H < Y < 0 \quad (3)$$

$$\partial_{tt}x\partial_Yx + (\partial_{tt}y + g)\partial_Yy + \frac{1}{\rho}\partial_Yp = 0, \quad X, t \in \mathbb{R}, -H < Y < 0 \quad (4)$$

$$\partial_Xx\partial_Yy - \partial_Yx\partial_Xy = 1, \quad X, t \in \mathbb{R}, -H < Y < 0 \quad (5)$$

$$\partial_{tt}x\partial_Xx + (\partial_{tt}y + g)\partial_Xy = 0, \quad X, t \in \mathbb{R}, Y = 0 \quad (6)$$

$$\partial_t y = 0, \quad X, t \in \mathbb{R}, Y = -H \quad (7)$$

where the unknowns are functions $x(X, Y, t)$, $y(X, Y, t)$ which are the coordinates of fluid particles and $p(X, Y, t)$ the pressure, all being in terms of their reference position (X, Y) and time t . The two first equations (3), (4) are the usual Euler equations in Lagrange coordinates, the third equation (5) expresses the volume conservation (incompressibility), Eq. (6) expresses the constantness of the pressure ($p = 0$) on the free surface ($Y = 0$), while (7) expresses the fact that the fluid velocity is tangential to the bottom. Once this system is solved, the shape of the free surface is deduced under parametric form with

$$x = x(X, 0, t)$$

$$y = y(X, 0, t)$$

where X is the parameter. The rest state here is given by

$$x(X, Y, t) = X$$

$$y(X, Y, t) = Y$$

$$p(X, Y, t) = -\rho g Y$$

and introducing the displacement, as Boussinesq did, in setting $x = X + u$, $y = Y + v$, and defining $P = \frac{p}{\rho} + gY$. Let us assume that the solution expands in power series of an amplitude ε as

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \dots$$

$$v = \varepsilon v_1 + \varepsilon^2 v_2 + \dots$$

$$P = \varepsilon P_1 + \varepsilon^2 P_2 + \dots$$

then at order ε we obtain

$$\partial_{tt}u_1 + \partial_X(P_1 + gv_1) = 0, \quad X, t \in \mathbb{R}, -H < Y < 0$$

$$\partial_{tt}v_1 + \partial_Y(P_1 + gv_1) = 0, \quad X, t \in \mathbb{R}, -H < Y < 0$$

$$\partial_Xu_1 + \partial_Yv_1 = 0, \quad X, t \in \mathbb{R}, -H < Y < 0$$

$$\partial_{tt}u_1 + g\partial_Xv_1 = 0, \quad X, t \in \mathbb{R}, Y = 0$$

$$\partial_t v_1 = 0, \quad X, t \in \mathbb{R}, Y = -H$$

and Boussinesq finds easily that $(u_1, v_1) = \nabla_X \Phi_1$ where Φ_1 is a potential, and he finds periodic solutions (linear standing waves) of the form

$$\Phi_1 = A_1(Y) \cos \omega t \cos kX$$

provided that the same dispersion equation (2) as above is satisfied, and

$$A_1(Y) = \cosh k(Y + H)$$

$$P_1(X, Y, t) = -\frac{gk}{\cosh kH} \sinh kY \sin \omega t \cos kX$$

in the finite depth case, and

$$A_1(Y) = e^{kY}, \quad P_1(X, Y, t) = 0$$

in the infinite depth case. Now at order ε^2 , Boussinesq observes that we have again $(u_2, v_2) = \nabla_X \Phi_2$, and thanks to

$$\Delta \Phi_1 = 0, \quad \partial_{tt} \Phi_1 = -\omega^2 \Phi_1$$

he obtains

$$\Delta \Phi_2 = (\partial_{XY} \Phi_1)^2 + (\partial_{XX} \Phi_1)^2, \quad X, t \in \mathbb{R}, \quad -H < Y < 0$$

$$\partial_{tY} \Phi_2 = 0, \quad X, t \in \mathbb{R}, \quad Y = -H$$

$$\partial_{tt} \Phi_2 + g \partial_Y \Phi_2 = \frac{\omega^2}{2} \{(\partial_X \Phi_1)^2 + (\partial_Y \Phi_1)^2\} + f(t), \quad X, t \in \mathbb{R}, \quad Y = 0$$

where $f(t)$ is an arbitrary function of time t . Here he noticed that the dispersion equation (2) should not be exactly satisfied, i.e. gk/ω^2 differs from the value given by (2) (which is 1 in the infinite depth case) at order ε^2 which was computed later by Rayleigh in 1915 [9], and which would have given the relationship between the amplitude ε of the waves and the bifurcation parameter g (or preferably μ , see below). J. Boussinesq gave the order ε^2 for Φ :

$$\Phi_2 = \frac{k^2}{16} (1 + \cos 2\omega t) \{ \cosh 2k(Y + H) - \cos 2kX \}$$

$$+ \frac{k^2}{32} \left(\frac{3}{\sinh^2 kH} \cos 2\omega t - \frac{1}{\cosh^2 kH} \right) \cos 2kX \cosh 2k(Y + H)$$

in the case of finite depth, and

$$\Phi_2 = \frac{k^2}{8} (1 + \cos 2\omega t) e^{2kY}$$

in the case of infinite depth. It should be noticed that the solution given above at order ε^2 has a *zero vorticity ω at this order*, as can be checked with the formula:

$$\omega = \partial_{Xx} \partial_{tY} x - \partial_{Yx} \partial_{tX} x + \partial_{Xy} \partial_{tY} y - \partial_{Yy} \partial_{tX} y$$

The identification at order ε^3 is more tricky. Seemingly unaware of this work, Lord Rayleigh [9] developed much later the third order theory that included travelling and standing waves on infinite depth as special cases, and much later Tadjbaksh and Keller [10] used a different expansion to obtain a third-order theory in the case of finite depth. These theories deal with Eulerian descriptions of the flow, while in 1947 Sekerkh-Zenkovich [11] took the theory to fourth order using Lagrangian coordinates.

Let us consider the order ε^3 in Lagrangian coordinates, for the *infinite depth case* (which gives simpler formulas). Taking into account of

$$u_2 = 0, \quad \partial_X v_2 = 0$$

we obtain the following system

$$\begin{aligned}
\partial_{tt}u_3 + g_0\partial_X(v_3 + P_3) + g_2\partial_X v_1 + \partial_{tt}v_2\partial_X v_1 &= 0, & Y < 0 \\
\partial_{tt}v_3 + g_0\partial_Y(v_3 + P_3) + g_2\partial_Y v_1 + \partial_{tt}v_2\partial_Y v_1 + \partial_{tt}v_1\partial_Y v_2 &= 0, & Y < 0 \\
\partial_X u_3 + \partial_Y v_3 + \partial_X u_1\partial_Y v_2 &= 0, & Y < 0 \\
P_3 &= 0, & Y = 0
\end{aligned}$$

where

$$g = g_0 + \varepsilon^2 g_2 + \dots, \quad \frac{g_0 k}{\omega^2} = 1 \text{ (linear critical value).}$$

We then find

$$\begin{aligned}
u_3 &= -\frac{g_2 k^2}{\omega^2} e^{kY} \cos \omega t \sin kX + g_0 \partial_X \phi_3 \\
v_3 &= \frac{g_2 k^2}{\omega^2} e^{kY} \cos \omega t \cos kX + \frac{k^5}{4} e^{3kY} \cos kX \left(\cos \omega t + \frac{1}{3} \cos 3\omega t \right) + g_0 \partial_Y \phi_3 \\
v_3 + P_3 &= \frac{k^5}{2} e^{3kY} \cos kX (\cos \omega t + \cos 3\omega t) - \partial_{tt} \phi_3 \\
\Delta \phi_3 &= 0, \quad Y < 0
\end{aligned}$$

The boundary condition $P_3 = 0$ on $Y = 0$ leads to

$$\partial_{tt} \phi_3|_{Y=0} + g_0 \partial_Y \phi_3|_{Y=0} = k^5 \left(\frac{1}{4} - \frac{g_2}{\omega^2 k^3} \right) \cos kX \cos \omega t + \frac{5k^5}{12} \cos 3\omega t \cos kX$$

For finding ϕ_3 it is then necessary and sufficient that

$$g_2 = \frac{\omega^2 k^3}{4}$$

which, in using the dimensionless parameter $\frac{gk}{\omega^2}$ of the problem, is better written as (here ε has the dimension of the square of a length):

$$\frac{gk}{\omega^2} = 1 + \frac{\varepsilon^2 k^4}{4} + \dots$$

hence

$$\begin{aligned}
u_3 &= -\frac{k^5}{4} e^{kY} \left(\cos \omega t - \frac{5}{24} \cos 3\omega t \right) \sin kX \\
v_3 &= \frac{k^5}{4} e^{kY} \left(\cos \omega t - \frac{5}{24} \cos 3\omega t \right) \cos kX + \frac{k^5}{4} e^{3kY} \cos kX \left(\cos \omega t + \frac{1}{3} \cos 3\omega t \right)
\end{aligned}$$

The proof of existence of these standing waves is delicate, due to several mathematical difficulties, in particular a *small divisor problem*, and the additional problem of *complete resonance* in the infinite depth case. For the *finite depth case* this was proved using Lagrange variables in 2001 by Plotnikov and Toland in [1]. In the *infinite depth case*, which needs a priori to satisfy infinitely many compatibility conditions, due to the infinite dimension of the linearized system, the proof of eligibility of an asymptotic expansion in powers of the amplitude ε , starting as above, is due to Amick and Toland [12] who answered positively in 1987 to a conjecture made in 1981 [6]. The proof of existence of these standing waves was made with a mixed integral formulation in 2005 by Iooss, Plotnikov and Toland [2] for the simplest monomodal solutions (with only one term at order ε), and by Iooss and Plotnikov in [3] for multimodal solutions (for any allowed combinations of the infinite set of possible linear modes, at order ε). In fact all these results are valid only for suitable values of the parameters, which have asymptotically full measure as one approaches the bifurcation set.

As noticed by Boussinesq in [7] p. 351, another side aspect of the computations above (in the case of infinite depth), is that it provides an easy approximation at order ε^2 of the trajectories of fluid particules since the components of the displacement are given up to order ε^2 by (infinite depth case)

$$u = -\varepsilon k e^{kY} \cos \omega t \sin kX$$

$$v = \varepsilon k e^{kY} \cos \omega t \cos kX + \frac{\varepsilon^2 k^3}{2} e^{2kY} \cos^2 \omega t$$

and after elimination of $e^{kY} \cos \omega t$ we obtain (X is fixed here)

$$\left(u - \frac{\sin 2kX}{2k}\right)^2 = \frac{2 \sin^2 kX}{k} \left(v + \frac{\cos^2 kX}{2k}\right)$$

showing little arcs of *parabolas with vertical axis and with concavity upwards, their shape being independent of the distance to the free surface*. This movement of fluid particules is sufficiently important to justify research in the twenty first century on the best way to collect and exploit the kinetic energy of standing waves.

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