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Homogenization of a class of nonlinear elliptic equations with nonstandard growth

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Abstract

We study the homogenization of the Dirichlet variational problem of a class of nonlinear elliptic equations with nonstandard growth. Such equations arise in many engineering disciplines, such as electrorheological fluids, non-Newtonian fluids with thermo-convective effects, and nonlinear Darcy flow of compressible fluids in heterogeneous porous media. We derive the homogenized model by means of the variational homogenization technique in the framework of Sobolev spaces with variable exponents. This result is then illustrated with a periodic example. ***To cite this article: B. Amaziane et al., C. R. Mecanique 335 (2007).***

Résumé

Homogénéisation d'une classe d'équations elliptiques non linéaires de croissance non standard. On étudie l'homogénéisation du problème variationnel de Dirichlet d'une classe d'équations elliptiques non linéaires de croissance non standard. Ce genre d'équations apparaît dans la modélisation de certains problèmes de l'ingénierie, comme par exemple les fluides électrorhéologiques, les écoulements non Newtoniens thermoconvectifs, et les écoulements non linéaires de Darcy de fluides compressibles en milieux poreux hétérogènes. On obtient le problème homogénéisé par la technique de l'homogénéisation variationnelle dans le cadre des espaces de Sobolev avec des exposants variables. Enfin, on présente un exemple périodique pour illustrer le résultat obtenu. ***Pour citer cet article : B. Amaziane et al., C. R. Mecanique 335 (2007).***

Keywords: Fluid mechanics; Homogenization; Nonlinear variational problem; Nonstandard growth

Mots-clés : Mécanique des fluides ; Homogénéisation ; Problème variationnel non linéaire ; Croissance non standard

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1. Introduction

In this Note we study the homogenization of the following nonlinear Dirichlet variational problem:

$$-\partial_{x_i}\mathsf{G}^{\varepsilon}_{u_{x_i}}\big(x,u^{\varepsilon},\nabla u^{\varepsilon}\big)+\mathsf{G}^{\varepsilon}_{u}\big(x,u^{\varepsilon},\nabla u^{\varepsilon}\big)=0\quad\text{in }\Omega^{\varepsilon}\tag{1}$$

where $\varepsilon>0$, $\Omega^{\varepsilon}=\Omega\setminus\mathcal{F}^{\varepsilon}$ is a perforated domain in $\mathbb{R}^n$, $n\geqslant 2$, $\mathsf{G}^{\varepsilon}(x,u,\xi)$ satisfies some convexity assumptions and its growth, with respect to u and ξ, is an oscillating function $p_{\varepsilon}(x)$ satisfying some conditions which will be specified in Section 3 and $\mathsf{G}^{\varepsilon}_{u_{x_i}}$, $\mathsf{G}^{\varepsilon}_{u}$ denote the partial derivatives of G^{ε}.

In recent years, there has been an increasing interest in the study of such equations (in the case where there is no dependence on the small parameter) motivated by their applications to the mathematical modeling in continuum mechanics. These equations arise, for example, from the modeling of non-Newtonian fluids with thermo-convective effects (see for instance [1]), the modeling of electrorheological fluids (see, e.g., [2]), and the motion of a compressible fluid in a heterogeneous anisotropic porous medium obeying to the nonlinear Darcy law (see, e.g., [3]).

In this Note we deal with the Dirichlet variational problem corresponding to the nonlinear equation (1). The homogenization of the Dirichlet boundary value problem was studied for the first time in [4] and then it was revisited by many authors (see [5–8] and the references therein). Here, problem (1) is stated in the framework of Sobolev spaces with variable exponents which will be briefly described in the following section. Let us mention that the homogenization of variational problems in Sobolev spaces with variable exponents has not been yet studied in the literature except one special case where the growth p_{ε} is a piecewise constant function (see [9]). In [10], the homogenization of a $p_{\varepsilon}(x)$-Laplacian equation was studied. In this Note, we extend these ideas to a class of nonlinear elliptic equations.

Following the approach developed in [6], instead of a classical periodicity assumption on the structure of the perforated domain Ω^{ε}, we impose certain conditions on the so-called local energy characteristics associated with the equation (1). It will be shown that the asymptotic behavior, as $\varepsilon\to 0$, of the solution u^{ε} is described by the Dirichlet variational problem with the integrand $\mathsf{G}^0(x,u,\nabla u)+c(x,u)$, where G^0 is a limit of the sequence $\{\mathsf{G}^{\varepsilon}\}$ and $c(x,u)$ is calculated by the local energy characteristic of Ω^{ε}.

The proof of the main result is based on the variational homogenization technique which is nowadays widely used in the homogenization theory (see, e.g., [8,9,11] and the references therein).

2. Preliminary results and notation

For the reader's convenience, we recall some background facts concerning Sobolev spaces with variable exponents, see for instance [12–15], and introduce some notation.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and a function $p(x)$ satisfying the following conditions: $1<p^{(-)}=\inf_{\Omega}p(x)\leqslant p(x)\leqslant\sup_{\Omega}p(x)=p^{(+)}<\infty$ with $p^{(-)}<n$, and for all $x,y\in\Omega$,

$$\big|p(x)-p(y)\big|\leqslant\omega\big(|x-y|\big)\quad\text{with}\ \varlimsup_{\tau\to 0}\omega(\tau)\ln\left(\frac{1}{\tau}\right)=0\tag{2}$$

(1) Denote by $L^{p(\cdot)}(\Omega)$ the space of measurable functions f such that $A_{p(\cdot)}(f)=\int_{\Omega}|f(x)|^{p(x)}\,\mathrm{d}x<\infty$. The space $L^{p(\cdot)}(\Omega)$ equipped with the norm $\|f\|_{L^{p(\cdot)}(\Omega)}=\inf\{\lambda>0\colon\ A_{p(\cdot)}(\frac{f}{\lambda})\leqslant 1\}$ is a Banach space.

(2) The space $W^{1,\,p(\cdot)}(\Omega)$ is defined as follows: $W^{1,\,p(\cdot)}(\Omega)=\{f\in L^{p(\cdot)}(\Omega)\colon\ |\nabla f|\in L^{p(\cdot)}(\Omega)\}$. Under the assumption (2), $W_0^{1,\,p(\cdot)}(\Omega)$ is the closure of the set $C_0^{\infty}(\Omega)$ with respect to the norm of $W^{1,\,p(\cdot)}(\Omega)$. If the boundary of Ω is Lipschitz-continuous and $p(x)$ satisfies (2), then $C_0^{\infty}(\Omega)$ is dense in $W_0^{1,\,p(\cdot)}(\Omega)$. The norm in $W_0^{1,\,p(\cdot)}$ is defined by $\|u\|_{W_0^{1,\,p(\cdot)}}=\sum_i\|D_iu\|_{L^{p(\cdot)}(\Omega)}+\|u\|_{L^{p(\cdot)}(\Omega)}$. If the boundary of Ω is Lipschitz and $p\in C^0(\Omega)$, then the norm $\|\cdot\|_{W_0^{1,\,p(\cdot)}(\Omega)}$ is equivalent to the norm $\widetilde{\|u\|}_{W_0^{1,\,p(x)}(\Omega)}=\sum_i\|D_iu\|_{L^{p(\cdot)}(\Omega)}$.

(3) If $p\in C^0(\overline{\Omega})$, then $W^{1,p(\cdot)}(\Omega)$ is separable and reflexive.

(4) Let $p,\,q\in C^0(\overline{\Omega})$ and define the function p_* in Ω as follows: $p_*(x)=\frac{p(x)n}{n-p(x)}$ for $p(x)<n$, $p_*(x)=+\infty$ for $p(x)>n$, and $1<q(x)\leqslant\sup_{\Omega}q(x)<\inf_{\Omega}p_*(x)$, then the embedding $W_0^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

3. Statement of the problem and the main result

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with a smooth boundary. Let $\mathcal{F}^\varepsilon$ be a closed subset in Ω. Here ε is a small parameter characterizing the scale of the microstructure. We assume that $\mathcal{F}^\varepsilon$ is distributed in a asymptotically regular way in Ω, i.e., for any ball $V(y,r)$ of radius r centered at $y \in \Omega$ and $\varepsilon > 0$ small enough ($\varepsilon \leqslant \varepsilon_0(r)$), $V(y,r) \cap \mathcal{F}^\varepsilon \neq \emptyset$. We set $\Omega^\varepsilon = \Omega \setminus \mathcal{F}^\varepsilon$.

Let $p_\varepsilon = p_\varepsilon(x)$ be a continuous function defined in $\overline{\Omega}$. We assume that, for any $\varepsilon > 0$, it satisfies the following conditions:

(i) p_ε *is bounded, namely*: $1 < \mathsf{p}^{(-)} \leqslant p_\varepsilon^{(-)} \equiv \min_{x\in\overline{\Omega}} p_\varepsilon(x) \leqslant p_\varepsilon(x) \leqslant \max_{x\in\overline{\Omega}} p_\varepsilon(x) \equiv p_\varepsilon^{(+)} \leqslant \mathsf{p}^{(+)} \leqslant n$ in $\overline{\Omega}$;
(ii) *it satisfies condition* (2);
(iii) p_ε *converges uniformly in* Ω *to a function* p_0, *where* p_0 *satisfies the conditions* (i) *and* (2);
(iv) p_ε *satisfies*: $p_\varepsilon(x) \geqslant p_0(x)$ *in* Ω.

Let $\mathsf{G}^\varepsilon(x,u,\xi)$ be a function that is defined and continuous for $x \in \overline{\Omega}$, $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$. We assume that this function has continuous partial derivatives G^ε_u, $\mathsf{G}^\varepsilon_{\xi_i}$ and, for any $\varepsilon > 0$, satisfies the conditions:

(A.1) G^ε *is a convex function with respect to* ξ: $\mathsf{G}^\varepsilon(x,u,\xi) - \mathsf{G}^\varepsilon(x,u,\eta) - \mathsf{G}^\varepsilon_{\xi_i}(x,u,\xi)(\xi_i - \eta_i) \geqslant 0$;
(A.2) $A_1|\xi|^{p_\varepsilon(x)} - A_2|u|^{p_\varepsilon(x)} \leqslant \mathsf{G}^\varepsilon(x,u,\xi) \leqslant A_3[1 + |\xi|^{p_\varepsilon(x)} + |u|^{p_\varepsilon(x)}]$ with $A_1, A_2, A_3 > 0$;
(A.3) $|\mathsf{G}^\varepsilon(x,u,\xi) - \mathsf{G}^\varepsilon(x,v,\eta)| \leqslant A_4(1 + |\xi| + |u|)^{p_\varepsilon(x)-1}(|u-v| + |\xi - \eta|)$ with $A_4 > 0$;
(A.4) $\mathsf{G}^\varepsilon(x,0,0) = 0$;
(A.5) G^ε *converges to a function* G^0 *in the following sense*: *for any* $w \in C_0^\infty(\Omega)$

$$\lim_{\varepsilon\to 0} \int_\Omega \left|\mathsf{G}^\varepsilon(x,w,\nabla w) - \mathsf{G}^0(x,w,\nabla w)\right| \mathrm{d}x = 0$$

where G^0 *is assumed to be defined and continuous for* $x \in \overline{\Omega}$, $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$. *It has continuous partial derivatives* G^0_u, $\mathsf{G}^0_{\xi_i}$ *and satisfies the condition* (A.2) *with the function* p_0.

A typical example of the integrand G^ε is given in Section 5.

We consider the following variational problem:

$$J^\varepsilon[u^\varepsilon] \equiv \int_{\Omega^\varepsilon} \mathsf{G}^\varepsilon\big(x,u^\varepsilon,\nabla u^\varepsilon\big)\,\mathrm{d}x \to \inf, \quad u^\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}\big(\Omega^\varepsilon\big) \tag{3}$$

The functional $J^\varepsilon[u^\varepsilon]$ is assumed to be bounded from below that is

$$J^\varepsilon[u^\varepsilon] \geqslant \Phi\big(\|u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)}\big) - \mu \tag{4}$$

where $\Phi(t)$ is a continuous function such that $\Phi(t) \to +\infty$ as $t \to +\infty$ and $\mu \in \mathbb{R}$.

It is known from [16–18] that, for each $\varepsilon > 0$, there exists a solution $u^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)$ of (3).

Let us extend u^ε in $\mathcal{F}^\varepsilon$ by zero (keeping for it the same notation). Then we obtain the sequence $\{u^\varepsilon\} \subset W^{1,p_\varepsilon(\cdot)}(\Omega)$. We study the asymptotic behavior of u^ε as $\varepsilon \to 0$.

Instead of the classical periodicity assumption on the microstructure of the perforated domain Ω^ε, we impose certain conditions on the local energy characteristic of the set $\mathcal{F}^\varepsilon$. To this end we introduce K_h^z an open cube centered at $z \in \Omega$ with length equal to h ($0 < \varepsilon \ll h < 1$) and we define a function of $b \in \mathbb{R}$:

$$c^{\varepsilon,h}(z,b) = \inf_{v^\varepsilon} \int_{K_h^z} \left\{\mathsf{G}^\varepsilon\big(x,0,\nabla u^\varepsilon\big) + h^{-\mathsf{p}^{(+)}-\gamma}\big(|v^\varepsilon - b|^{p_\varepsilon(x)} + |v^\varepsilon - b|^{p_0(x)}\big)\right\} \mathrm{d}x \tag{5}$$

where $\gamma > 0$, and the infimum is taken over $v^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(K_h^z)$ that equal zero in $\mathcal{F}^\varepsilon$. We make the following further assumptions:

(C.1) *there exist a continuous function* $c(x,b)$ *such that for any* $x\in\Omega$, $b\in\mathbb{R}$, *and a certain* $\gamma=\gamma_0>0$,

$$\lim_{h\to0}\overline{\lim_{\varepsilon\to0}}\,h^{-n}c^{\varepsilon,h}(z,b)=\lim_{h\to0}\underline{\lim_{\varepsilon\to0}}\,h^{-n}c^{\varepsilon,h}(z,b)=c(x,b)$$

(C.2) *there exists a constant* A *independent of* ε,γ *such that, for any* $x\in\Omega$,

$$\lim_{h\to0}\overline{\lim_{\varepsilon\to0}}\,h^{-n}c^{\varepsilon,h}(z,b)\leqslant A\left(1+|b|^{p_0(x)}\right)$$

The main result of the Note is the following:

Theorem 3.1. *Let assumptions* (i)–(iv), (A.1)–(A.5) *and* (C.1)–(C.2) *hold. Then for any sequence of solutions of the variational problem* (3) *there is a subsequence which converges weakly in* $W^{1,p_0(\cdot)}(\Omega)$ *to a solution of the following variational problem*:

$$\int_\Omega\left\{\mathsf{G}^0(x,u,\nabla u)+c(x,u)\right\}\mathrm{d}x\to\inf,\quad u\in W_0^{1,p_0(\cdot)}(\Omega)\tag{6}$$

4. Sketch of the proof of Theorem 3.1

It follows from (A.4) and (4) that the solution of (3) satisfies the estimate $\|u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)}\leqslant C$ with a constant C independent on ε. We extend u^ε by zero to $\mathcal{F}^\varepsilon$ and consider $\{u^\varepsilon\}$ as a sequence in $W^{1,p_\varepsilon(\cdot)}(\Omega)$. It is clear that $\|u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega)}\leqslant C$. Now the condition (iv) imply that $\|u^\varepsilon\|_{W^{1,p_0(\cdot)}(\Omega)}\leqslant C$. This means that $\{u^\varepsilon\}$ is a weakly compact set in $W^{1,p_0(\cdot)}(\Omega)$. Hence, one can extract a subsequence that converges weakly to $u\in W^{1,p_0(\cdot)}(\Omega)$. We will show that u is a solution of (6). The proof will be done in two steps.

Step 1. Upper bound. Let $\{x^\alpha\}$ be a periodic grid in Ω with a period $h'=h-h^{1+\gamma/\mathsf{p}^{(+)}}$ $(0<\varepsilon\ll h\ll1)$. Cover the domain Ω by the cubes K_h^α of length $h>0$ centered at the points x^α. We associate with this covering a partition of unity $\{\varphi_\alpha\}$: $0\leqslant\varphi_\alpha(x)\leqslant1$; $\varphi_\alpha(x)=0$ for $x\notin K_h^\alpha$; $\varphi_\alpha(x)=1$ for $x\in K_h^\alpha\setminus\bigcup_{\beta\neq\alpha}K_h^\beta$; $\sum_\alpha\varphi_\alpha(x)=1$ for $x\in\Omega$; $|\nabla\varphi_\alpha(x)|\leqslant Ch^{-1-\gamma/p^{(+)}}$.

Let w be a smooth function in Ω such that $w(x)=0$ on $\partial\Omega$ and let $\mathcal{K}_\theta$ denotes a subset of the cubes K_h^α such that $|w(x)|>\theta>0$ for any $x\in K_h^\alpha$. We set $b_\alpha=w(x^\alpha)$ for $K_h^\alpha\in\mathcal{K}_\theta$ and $b_\alpha=1$ for $K_h^\alpha\notin\mathcal{K}_\theta$. For any K_h^α, we define the set $\mathcal{B}^\alpha(\varepsilon,h;\vartheta)=\{x\in K_h^\alpha\colon v_\alpha^\varepsilon(x)\,\mathrm{sign}\,b_\alpha\leqslant|b_\alpha|-\vartheta\}$ and the function V_α^ε, $V_\alpha^\varepsilon=v_\alpha^\varepsilon(x)$ in $\mathcal{B}^\alpha(\varepsilon,h;\vartheta)$ and $V_\alpha^\varepsilon=b_\alpha^\vartheta\equiv(|b_\alpha|-\vartheta)\,\mathrm{sign}\,b_\alpha$ in $K_h^\alpha\setminus\mathcal{B}^\alpha(\varepsilon,h;\vartheta)$, where v_α^ε is a function minimizing (5) with $b=b_\alpha$ and $z=x^\alpha$, $0<\vartheta\ll\theta/2\ll1$. We set $\vartheta=h$ and introduce the function $w_h^\varepsilon(x)=w(x)+\sum_\alpha\frac{w(x)}{b_\alpha^\vartheta}(V_\alpha^\varepsilon(x)-b_\alpha^\vartheta)\varphi_\alpha(x)$. Using the condition (C.1) one can show that $w_h^\varepsilon\in W_0^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)$. Therefore, we have that $J^\varepsilon[u^\varepsilon]\leqslant J^\varepsilon[w_h^\varepsilon]$. Estimating the right hand side of this inequality we get:

$$\overline{\lim_{\varepsilon\to0}}\,J^\varepsilon\left[u^\varepsilon\right]\leqslant J_{\mathrm{hom}}[w]\equiv\int_\Omega\left\{\mathsf{G}^0(x,w,\nabla w)+c(x,w)\right\}\mathrm{d}x\tag{7}$$

This inequality is obtained for $w\in C_0^\infty(\Omega)$. It holds true for any $w\in W_0^{1,p_0(\cdot)}(\Omega)$. This follows from the density of $C_0^\infty(\Omega)$ in $W_0^{1,p_0(\cdot)}(\Omega)$ (see Section 2) and the continuity of the functional J_{hom} in $W_0^{1,p_0(\cdot)}(\Omega)$.

Step 2. Lower bound. Let $u\in W_0^{1,p_0(\cdot)}(\Omega)$ be a weak limit in $W^{1,p_0(\cdot)}(\Omega)$ of the sequence $\{u^\varepsilon\}\subset W_0^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)\cap W_0^{1,p_0(\cdot)}(\Omega^\varepsilon)$ (extended by zero in $\mathcal{F}^\varepsilon$) by a subsequence $\varepsilon=\varepsilon_k$. For any $\delta>0$, we introduce a function $u_\delta\in C_0^1(\Omega)$ such that $\|u-u_\delta\|_{W^{1,p_0(\cdot)}(\Omega)}<\delta$. One can show that there is a sequence $\{w_\delta^\varepsilon\}\subset W_0^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)\cap W_0^{1,p_0(\cdot)}(\Omega^\varepsilon)$ and $w_\delta^\varepsilon=0$ in $\mathcal{F}^\varepsilon$ which converges weakly in $W^{1,p_0(\cdot)}(\Omega)$ to the function $(u-u_\delta)$. We set $u_\delta^\varepsilon=u^\varepsilon+w_\delta^\varepsilon$ and show that $\lim_{\delta\to0}\overline{\lim}_{\varepsilon=\varepsilon_k\to0}\|u_\delta^\varepsilon-u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)}=0$. Using this equality, the definition of u_δ, and the continuity of J_{hom} in $W^{1,p_0(\cdot)}(\Omega)$ we see that $\lim_{\delta\to0}\overline{\lim}_{\varepsilon=\varepsilon_k\to0}|J^\varepsilon[u_\delta^\varepsilon]-J^\varepsilon[u^\varepsilon]|=0$ and $\lim_{\delta\to0}J_{\mathrm{hom}}[u_\delta]=J_{\mathrm{hom}}[u]$. These relations imply that if we prove that $\underline{\lim}_{\varepsilon=\varepsilon_k\to0}J^\varepsilon[u_\delta^\varepsilon]\geqslant J_{\mathrm{hom}}[u_\delta]$ (δ-*lower bound*) we immediately obtain the lower bound:

$$\underline{\lim_{\varepsilon=\varepsilon_k\to0}}\,J^\varepsilon\left[u^\varepsilon\right]\geqslant J_{\mathrm{hom}}[u]\tag{8}$$

Let us prove the δ-*lower bound*. To this end we cover $\mathbb{R}^n$ by cubes K_h^α with nonintersecting interiors centered at x^α forming a periodic, with the period h, grid in $\mathbb{R}^n$ and denote: $\Omega_\theta^\pm = \{x \in \Omega \mid \pm u_\delta > \theta > 0\}$; $\Omega_{\theta,h}^\pm = \{\bigcup_\alpha K_h^\alpha \mid K_h^\alpha \subset \Omega_\theta^\pm\}$; $\Omega_\theta = \Omega_\theta^+ \cup \Omega_\theta^-$; $\Omega_{\theta,h} = \Omega_{\theta,h}^+ \cup \Omega_{\theta,h}^-$; $\mathcal{O}_\theta = \Omega \setminus \Omega_\theta$; $\Omega_\theta^\varepsilon = \Omega_\theta \cap \Omega^\varepsilon$; $\Omega_{\theta,h}^\varepsilon = \Omega_{\theta,h} \cap \Omega^\varepsilon$; $\mathcal{O}_\theta^\varepsilon = \mathcal{O}_\theta \cap \Omega^\varepsilon$. We rewrite $J^\varepsilon[u_\delta^\varepsilon]$ as follows:

$$J^\varepsilon\big[u_\delta^\varepsilon\big] = \int_{\Omega_{\theta,h}^\varepsilon} \mathsf{G}^\varepsilon\big(x, u_\delta^\varepsilon, \nabla u_\delta^\varepsilon\big)\,\mathrm{d}x + \int_{\Omega_\theta^\varepsilon \setminus \Omega_{\theta,h}^\varepsilon} \mathsf{G}^\varepsilon\big(x, u_\delta^\varepsilon, \nabla u_\delta^\varepsilon\big)\,\mathrm{d}x + \int_{\mathcal{O}_\theta^\varepsilon} \mathsf{G}^\varepsilon\big(x, u_\delta^\varepsilon, \nabla u_\delta^\varepsilon\big)\,\mathrm{d}x \tag{9}$$

Conditions (i), (iv), (A.1)–(A.5) and the fact that $\lim_{h\to 0} \operatorname{meas}[\Omega_\theta \setminus \Omega_{\theta,h}] = 0$ imply the inequalities

$$\lim_{h\to 0} \varliminf_{\varepsilon=\varepsilon_k\to 0} \int_{\Omega_\theta^\varepsilon \setminus \Omega_{\theta,h}^\varepsilon} \mathsf{G}^\varepsilon\big(x, u_\delta^\varepsilon, \nabla u_\delta^\varepsilon\big)\,\mathrm{d}x \geqslant 0; \qquad \lim_{h\to 0} \varliminf_{\varepsilon=\varepsilon_k\to 0} \int_{\mathcal{O}_\theta^\varepsilon} \mathsf{G}^\varepsilon\big(x, u_\delta^\varepsilon, \nabla u_\delta^\varepsilon\big)\,\mathrm{d}x \geqslant \int_{\mathcal{O}_\theta} \mathsf{G}^0(x, u_\delta, \nabla u_\delta)\,\mathrm{d}x \tag{10}$$

Finally, following the lines of [10] and using the definition (5), for any $K_h^\alpha \subset \Omega_\theta^+$ ($K_h^\alpha \subset \Omega_\theta^-$), we get

$$\varliminf_{\varepsilon=\varepsilon_k\to 0} \int_{K_h^\alpha \cap \Omega^\varepsilon} \mathsf{G}^\varepsilon\big(x, u_\delta^\varepsilon, \nabla u_\delta^\varepsilon\big)\,\mathrm{d}x \geqslant \int_{K_h^\alpha} \mathsf{G}^0(x, u_\delta, \nabla u_\delta)\,\mathrm{d}x + \varliminf_{\varepsilon=\varepsilon_k\to 0} c^{\varepsilon,h}\big(x^\alpha, b_\alpha\big) + \mathrm{o}\big(h^n\big) \quad \text{as } h \to 0 \tag{11}$$

Now we take the union in (11) over all cubes $K_h^\alpha \subset \Omega_{\theta,h}$ and pass to the limit as $h \to 0$. Then by (9), (10), and condition (C.1) we obtain the δ-lower bound and, therefore, the lower bound (8).

Now it follows from (7) and (8) that $J_{\mathrm{hom}}[u] \leqslant J_{\mathrm{hom}}[w]$ for any $w \in W_0^{1,p_0(\cdot)}(\Omega)$. Thus, any weak limit of the solution of (3), extended by zero to the set $\mathcal{F}^\varepsilon$, is a solution of (6). This completes the proof of Theorem 3.1.

5. A periodic example

As an application of the previous general result, we now give an example of a perforated medium, where the distribution of the perforated domain and the growth function are specified.

Theorem 3.1 provides sufficient conditions for the existence of the homogenized problem (6). The goal of this section is to prove that, for an appropriate example, all the conditions of Theorem 3.1 are satisfied and to compute explicitly the functions G^0 and c in the homogenized problem (6).

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary. Let $\mathcal{F}^\varepsilon$ be a union of balls $\mathcal{F}_i^\varepsilon$ $(i = 1, 2, \ldots, N_\varepsilon)$ periodically, with a period ε, distributed in the domain Ω. We assume that $\mathcal{F}_i^\varepsilon$ is centered at the point $x^{i,\varepsilon}$ and of radius $r_\varepsilon = r\,\varepsilon^3$, where $r > 0$.

We define a function $p_\varepsilon \in C^1(\Omega)$ as follows. Let $\mathcal{B}_{\varepsilon/8}^i$ and $\mathcal{B}_{\varepsilon/4}^i$ be the balls centered at the point $x^{i,\varepsilon}$ and of radii $\varepsilon/8$ and $\varepsilon/4$, respectively. The function p_ε is a smooth ε-periodic function in Ω such that $p_\varepsilon(x) = 2$ in $\mathcal{B}_{\varepsilon/8}^i$ and $p_\varepsilon(x) = 2 + \varepsilon$ in $\Omega \setminus \bigcup_i \mathcal{B}_{\varepsilon/4}^i$, where $i = 1, 2, \ldots, N_\varepsilon$ and $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$. It is clear that p_ε satisfies the conditions (i)–(iv), and it converges uniformly in Ω to the function $p_0 \equiv 2$.

The following result holds:

Theorem 5.1. *Let u^ε be the solution of* (3) *with* $\mathsf{G}^\varepsilon(x, u, \xi) = \frac{1}{p_\varepsilon(x)}|\xi|^{p_\varepsilon(x)} + \frac{1}{p_\varepsilon(x)}|u|^{p_\varepsilon(x)} - f(x)u$, *where* $f \in C^1(\Omega)$. *Then u^ε converges weakly in $W^{1,2}(\Omega)$ to a solution of* (6) *with*

$$\mathsf{G}^0(x, u, \xi) = \frac{1}{2}|\xi|^2 + \frac{1}{2}|u|^2 - f(x)u \quad \text{and} \quad c(x, u) = 4\pi r|u|^2$$

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