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Modeling of linearly electromagneto-elastic thin plates

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Abstract

We extend to linearized electromagneto-elasticity the thin plate modeling previously derived by the authors in the framework of linearized piezoelectricity. According to the type of electromagnetic boundary conditions, four different models appear. This leads to new mixed 'senso-actuator' and 'actuato-sensor' behaviors. Moreover, it is pointed out that the method can be extended to other multi-physical couplings. ***To cite this article: T. Weller, C. Licht, C. R. Mecanique 335 (2007).***
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Résumé

Modélisation de plaques minces linéairement électromagnéto-élastiques. On étend à l' électromagnéto-élasticité linéarisée la modélisation de plaques minces déjà obtenue par les auteurs dans le cadre de la piézoélectricité linéarisée. En fonction du type de chargement électromagnétique considéré, on montre que quatre modèles différents apparaissent. Deux de ces modèles présentent des comportements métissés originaux de type « capto-actionneur » ou « actionno-capteur ». De plus, la méthode utilisée est suffisamment robuste pour être étendue à d'autres couplages multi-physiques. ***Pour citer cet article : T. Weller, C. Licht, C. R. Mecanique 335 (2007).***
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Mots-clés : Solides et structures ; Plaques magnéto-électro-élastiques ; Analyse asymptotique

1. Introduction

It is a classical technique that thin plates models can be justified in a rigorous way by giving to the thickness of the plate a role of *parameter* whose aim is to tend to zero. In [1], the authors have extended the method introduced in [2] to anisotropic heterogeneous linearly piezoelectric thin plates (also see [3–8], etc.). In particular, it has been shown that the electrical boundary conditions lead to two different models which have been fully described in all admissible piezoelectric crystal classes. Here, the method is extended to linearized electromagneto-elasticity whose field is currently a subject of intensive research (see [9]). We show in this Note that *this physical phenomenon leads to*

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four different thin plate models. In two of them, the plate can be used *at the same time* as a sensor *and* as an actuator. We also show that the method can be easily extended to other multi-physical couplings.

2. Setting the problem

Greek indexes (except ε) take their value in $\{1,2\}$ and Latin ones (except p and q) in $\{1,2,3\}$. The reference configuration of a linearly electromagneto-elastic thin plate is the closure in $\mathbb{R}^3$ of the set $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, where ω is a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\omega$ and ε a small positive number. Let $\Gamma^\varepsilon_{\text{lat}} := \partial\omega \times (-\varepsilon, \varepsilon)$, $\Gamma^\varepsilon_\pm := \omega \times \{\pm\varepsilon\}$ and let there be three suitable partitions of $\partial\Omega^\varepsilon$: $(\Gamma^\varepsilon_{mD}, \Gamma^\varepsilon_{mN})$, $(\Gamma^\varepsilon_{eD}, \Gamma^\varepsilon_{eN})$ and $(\Gamma^\varepsilon_{gD}, \Gamma^\varepsilon_{gN})$ with Γ^ε_{mD}, Γ^ε_{eD} and Γ^ε_{gD} of strictly positive surface measures. The plate is clamped along Γ^ε_{mD}, is at an electrical potential φ^ε_0 on Γ^ε_{eD} and at a magnetic potential ϕ^ε_0 on Γ^ε_{gD}. It is subjected to body forces f^ε in Ω^ε and to surface forces g^ε in Γ^ε_{mN}. Furthermore, we will consider an electrical loading d^ε on Γ^ε_{eN} and a magnetic one b^ε on Γ^ε_{gN}. We note n^ε the outward unit normal to $\partial\Omega^\varepsilon$ and assume that $\Gamma^\varepsilon_{mD} = \gamma_0 \times (-\varepsilon, \varepsilon)$, with $\gamma_0 \subset \partial\omega$. In the absence of body electric currents and charges, the equations determining the electromagneto-elastic state $s^\varepsilon := (u^\varepsilon, \varphi^\varepsilon, \phi^\varepsilon)$ at equilibrium are:

$$\mathcal{P}(\Omega^\varepsilon)\begin{cases} \operatorname{div}\sigma^\varepsilon + f^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \quad \sigma^\varepsilon n^\varepsilon = g^\varepsilon \text{ on } \Gamma^\varepsilon_{mN}, \quad u^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_{mD} \\ \operatorname{div} D^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \quad D^\varepsilon \cdot n^\varepsilon = d^\varepsilon \text{ on } \Gamma^\varepsilon_{eN}, \quad \varphi^\varepsilon = \varphi^\varepsilon_0 \text{ on } \Gamma^\varepsilon_{eD} \\ \operatorname{div} B^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \quad B^\varepsilon \cdot n^\varepsilon = b^\varepsilon \text{ on } \Gamma^\varepsilon_{gN}, \quad \phi^\varepsilon = \phi^\varepsilon_0 \text{ on } \Gamma^\varepsilon_{gD} \\ (\sigma^\varepsilon, D^\varepsilon, B^\varepsilon) = M^\varepsilon(x)(e(u^\varepsilon), \nabla\varphi^\varepsilon, \nabla\phi^\varepsilon) \text{ in } \Omega^\varepsilon \end{cases}$$

where u^ε, φ^ε, ϕ^ε, σ^ε, $e(u^\varepsilon)$, D^ε and B^ε respectively stand for the displacement, the electric and the magnetic potential fields, the stress tensor, the tensor of small strains, the electric and the magnetic inductions. If we denote the set of all linear mappings from a space V into a space W by $\mathcal{L}(V, W)$, the set of all 3×3 symmetric matrices by S^3 and define $\mathcal{H} := S^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, the operator M^ε is an element of $\mathcal{L}(\mathcal{H}, \mathcal{H})$ such that:

$$\begin{aligned} \sigma^\varepsilon &= \mathsf{a}^\varepsilon e(u^\varepsilon) - \mathsf{b}^\varepsilon \nabla\varphi^\varepsilon - \mathsf{c}^\varepsilon \nabla\phi^\varepsilon \\ D^\varepsilon &= \mathsf{b}^{\varepsilon^T} e(u^\varepsilon) + \mathsf{d}^\varepsilon \nabla\varphi^\varepsilon + \mathsf{e}^\varepsilon \nabla\phi^\varepsilon \\ B^\varepsilon &= \mathsf{c}^{\varepsilon^T} e(u^\varepsilon) + \mathsf{e}^{\varepsilon^T} \nabla\varphi^\varepsilon + \mathsf{f}^\varepsilon \nabla\phi^\varepsilon \end{aligned} \tag{1}$$

In these constitutive equations, a^ε, b^ε, c^ε, d^ε, e^ε and f^ε respectively stand for the elastic, piezoelectric, piezomagnetic, dielectric, electromagnetic coupling and magnetic permeability tensors while the superscript T denotes the transpose operation. Of course, M^ε is not symmetric but under realistic assumption of boundedness of a^ε, $\mathsf{b}^\varepsilon, \ldots, \mathsf{f}^\varepsilon$ and of uniform ellipticity of a^ε, d^ε and f^ε, the physical problem $\mathcal{P}(\Omega^\varepsilon)$ has a unique weak solution. To get plates models, the question is to study its behavior when $\varepsilon \to 0$.

3. A convergence result

We will show that four different limit behaviors appear according to the type of boundary conditions in $\mathcal{P}(\Omega^\varepsilon)$. These limit behaviors will be indexed by a couple (p, q) with $1 \leqslant p, q \leqslant 2$. Let us recall the main steps of the method.

First we come down to a fixed open set $\Omega := \omega \times (-1, 1)$ through the bijection

$$x = (x_1, x_2, x_3) \in \overline{\Omega} \mapsto x^\varepsilon = \pi^\varepsilon(x) = (x^\varepsilon_1, x^\varepsilon_2, \varepsilon x^\varepsilon_3) \in \overline{\Omega^\varepsilon}$$

At the same time, we drop the index ε for the images by $(\pi^\varepsilon)^{-1}$ of the geometric sets defined at the beginning of Section 2. We also assume that the electromagneto-elastic coefficients are such that:

$$M^\varepsilon(\pi^\varepsilon x) =: M(x), \quad M \in L^\infty(\Omega, \mathcal{L}(\mathcal{H}, \mathcal{H})), \quad \exists\kappa > 0\text{:} \quad M(x)h \cdot h \geqslant \kappa |h|^2_{\mathcal{H}}, \quad \forall h \in \mathcal{H}, \text{ a.e. } x \in \Omega \tag{2}$$

while:

$$\begin{cases} f^\varepsilon_\alpha(\pi^\varepsilon x) = \varepsilon f_\alpha(x), \quad f^\varepsilon_3(\pi^\varepsilon x) = \varepsilon^2 f_3(x), \quad \forall x \in \Omega \\ g^\varepsilon_\alpha(\pi^\varepsilon x) = \varepsilon^2 g_\alpha(x), \quad g^\varepsilon_3(\pi^\varepsilon x) = \varepsilon^3 g_3(x), \quad \forall x \in \Gamma_{mN} \cap \Gamma_\pm \\ g^\varepsilon_\alpha(\pi^\varepsilon x) = \varepsilon g_\alpha(x), \quad g^\varepsilon_3(\pi^\varepsilon x) = \varepsilon^2 g_3(x), \quad \forall x \in \Gamma_{mN} \cap \Gamma_{\text{lat}} \\ d^\varepsilon(\pi^\varepsilon x) = \varepsilon^{3-p} d(x), \quad \forall x \in \Gamma_{eN} \cap \Gamma_\pm, \qquad b^\varepsilon(\pi^\varepsilon x) = \varepsilon^{3-q} d(x), \quad \forall x \in \Gamma_{gN} \cap \Gamma_\pm \\ d^\varepsilon(\pi^\varepsilon x) = \varepsilon^{2-p} d(x), \quad \forall x \in \Gamma_{eN} \cap \Gamma_{\text{lat}}, \qquad b^\varepsilon(\pi^\varepsilon x) = \varepsilon^{2-q} d(x), \quad \forall x \in \Gamma_{gN} \cap \Gamma_{\text{lat}} \\ \varphi^\varepsilon_0(\pi^\varepsilon x) = \varepsilon^p \varphi_0(x), \quad \forall x \in \Gamma_{eD}, \qquad \phi^\varepsilon_0(\pi^\varepsilon x) = \varepsilon^q \phi_0(x), \quad \forall x \in \Gamma_{gD} \end{cases} \tag{3}$$

where (f, g, d, b) is an element (independent of ε) of $L^2(\Omega)^3 \times L^2(\Gamma_{mN})^3 \times L^2(\Gamma_{eN}) \times L^2(\Gamma_{gN})$. We also suppose that φ_0 and ϕ_0 have $H^1(\Omega)$ extensions into Ω still denoted by φ_0 and ϕ_0. Furthermore, to bound the 'work of the exterior loading', we assume that:

$$\begin{cases} \text{if } p \text{ (resp. } q) = 1\text{: the extensions of } \varphi_0 \text{ (resp. } \phi_0) \text{ into } \Omega \text{ do not depend on } x_3 \\ \text{if } p \text{ (resp. } q) = 2\text{: the closure of the projection of } \Gamma_{eD} \text{ (resp. } \Gamma_{gD}) \text{ on } \omega \text{ coincides with } \overline{\omega} \\ \qquad\qquad\qquad \text{moreover, either } d \text{ (resp. } b) = 0 \text{ on } \Gamma_{eN} \cap \Gamma_{\text{lat}} \text{ (resp. } \Gamma_{gN} \cap \Gamma_{\text{lat}}) \\ \qquad\qquad\qquad \text{or } \Gamma_{eN} \cap \Gamma_{\text{lat}} \text{ (resp. } \Gamma_{gN} \cap \Gamma_{\text{lat}}) = \emptyset \end{cases} \tag{4}$$

Next, with the true physical state $s^\varepsilon = (u^\varepsilon, \varphi^\varepsilon, \phi^\varepsilon)$ defined on Ω^ε, we associate a *scaled* electromagneto-mechanical state $s_{p,q}(\varepsilon) := (u_{p,q}(\varepsilon), \varphi_{p,q}(\varepsilon), \phi_{p,q}(\varepsilon))$ defined by:

$$\begin{aligned} u^\varepsilon_\alpha(x^\varepsilon) &= \varepsilon\big(u_{p,q}(\varepsilon)\big)_\alpha(x) \\ u^\varepsilon_3(x^\varepsilon) &= \big(u_{p,q}(\varepsilon)\big)_3(x) \\ \varphi^\varepsilon(x^\varepsilon) &= \varepsilon^p \varphi_{p,q}(\varepsilon)(x) \\ \phi^\varepsilon(x^\varepsilon) &= \varepsilon^q \phi_{p,q}(\varepsilon)(x) \end{aligned} \tag{5}$$

for all $x = \pi^\varepsilon(x)$ in Ω and for $1 \leqslant p, q \leqslant 2$, so that $s_{p,q}(\varepsilon)$ is the solution of the following mathematical problem, equivalent to the genuine physical one:

$$\mathcal{P}(\varepsilon, \Omega)_{p,q} \begin{cases} \text{Find } s_{p,q}(\varepsilon) \in (0, \varphi_0, \phi_0) + \mathbf{V} \text{ such that } m_{p,q}(\varepsilon)(s_{p,q}(\varepsilon), r) = L(r), \quad \forall r \in \mathbf{V} \\ \mathbf{V} := \{r = (v, \psi, \Psi) \in H^1_{\Gamma_{mD}}(\Omega)^3 \times H^1_{\Gamma_{eD}}(\Omega) \times H^1_{\Gamma_{gD}}(\Omega)\} \end{cases}$$

where $H^1_\Gamma(\Omega)$ denotes the subset of $H^1(\Omega)$ whose elements vanish on $\Gamma \subset \partial\Omega$ and with

$$\begin{cases} m_{p,q}(\varepsilon)(s, r) := \int_\Omega M(x) k_{p,q}(\varepsilon, s) \cdot k_{p,q}(\varepsilon, r)\, \mathrm{d}x \\ k_{p,q}(\varepsilon, r) = k_{p,q}(\varepsilon, (v, \psi, \Psi)) = (e(\varepsilon, v), \nabla_p(\varepsilon, \psi), \nabla_q(\varepsilon, \Psi)) \\ e(\varepsilon, v)_{\alpha\beta} = e(v)_{\alpha\beta}, e(\varepsilon, v)_{\alpha 3} = \varepsilon^{-1} e(v)_{\alpha 3}, e(\varepsilon, v)_{33} = \varepsilon^{-2} e(v)_{33} \\ 2e_{ij} = \partial_i v_j + \partial_j v_i, \quad \nabla_n(\varepsilon, \phi)_\alpha = \varepsilon^{n-1}\partial_\alpha \phi, \nabla_n(\varepsilon, \phi)_3 = \varepsilon^{n-2}\partial_3\phi, \quad \text{with } n = p, q \text{ and } \phi = \psi, \Psi \\ L(r) = L(v, \psi, \Psi) = \int_\Omega f \cdot v\, \mathrm{d}x + \int_{\Gamma_{mN}} g \cdot v\, \mathrm{d}x + \int_{\Gamma_{eN}} d\psi\, \mathrm{d}x + \int_{\Gamma_{gN}} b\Psi\, \mathrm{d}x \end{cases} \tag{6}$$

The signs of the various powers of ε in the components of $k_{p,q}(\varepsilon, r)$ induce an orthogonal decomposition of $\mathcal{H}$ in subspaces $\mathcal{H}^\star_{p,q}$, with $\star \in \{-, 0, +\}$, which is crucial to fully describe plates models in all admissible crystal classes. We denote by $h^\star_{p,q}$ the projection on $\mathcal{H}^\star_{p,q}$ of any element h of $\mathcal{H}$. For example, we have

$$\begin{aligned} \mathcal{H}^-_{1,2} &:= \big\{(e, E, H) \in \mathcal{H};\ e_{\alpha\beta} = 0,\ E_\alpha = 0,\ H_i = 0\big\} \\ \mathcal{H}^0_{1,2} &:= \big\{(e, E, H) \in \mathcal{H};\ e_{i3} = 0,\ E_3 = 0,\ H_\alpha = 0\big\} \\ \mathcal{H}^+_{1,2} &:= \big\{(e, E, H) \in \mathcal{H};\ e_{ij} = 0,\ E_i = 0,\ H_3 = 0\big\} \end{aligned}$$

For given (p, q), M can then be decomposed in nine elements $M^{\star\diamond}_{p,q} \in \mathcal{L}(\mathcal{H}^\diamond_{p,q}, \mathcal{H}^\star_{p,q})$, with $\star, \diamond \in \{-, 0, +\}$. Because $M^{00}_{p,q}$ and $M^{--}_{p,q}$ are positive operators on $\mathcal{H}^0_{p,q}$ and $\mathcal{H}^-_{p,q}$, the Schur complement $\widetilde{M}_{p,q} := M^{00}_{p,q} - M^{0-}_{p,q}(M^{--}_{p,q})^{-1} M^{-0}_{p,q}$ is an element of $\mathcal{L}(\mathcal{H}^0_{p,q}, \mathcal{H}^0_{p,q})$. The key point of the asymptotic study is to show that if $\bar{k}_{p,q}$ is the limit (in a suitable topology) of $k_{p,q}(\varepsilon, s_{p,q}(\varepsilon))$, then $(M\bar{k}_{p,q})^-_{p,q} = (\bar{k}_{p,q})^+_{p,q} = 0$. This will enable us to exhibit $\widetilde{M}_{p,q}$ *as the operator governing the limit constitutive equations due to the fundamental relation*:

$$(Mh)^-_{p,q} = h^+_{p,q} = 0 \quad \Rightarrow \quad \widetilde{M}_{p,q} h^0_{p,q} = (Mh)^0_{p,q} \text{ and } \widetilde{M}_{p,q} h^0_{p,q} \cdot h^0_{p,q} = Mh \cdot h \tag{7}$$

The limit space of displacements will be the space of Kirchhoff–Love displacements defined by $\mathbf{V}_{\text{KL}} := \{v \in H^1_{\Gamma_{mD}}(\Omega)^3;\ e_{i3}(v) = 0\}$ while the limit electrical spaces will be $\Phi_{e,1} := \{\psi \in H^1_{\Gamma_{eD}}(\Omega);\ \partial_3\psi = 0\}$ and $\Phi_{e,2} := \{\psi \in H^1_{\partial_3}(\Omega);\ \psi|_{\Gamma_{eD} \cap \Gamma^\pm} = 0\}$, where $H^1_{\partial_3}(\Omega) := \{\psi \in L^2(\Omega);\ \partial_3\psi \in L^2(\Omega)\}$. Of course, the limit magnetic spaces $\Phi_{g,p}$ are defined as $\Phi_{e,p}$ by replacing Γ_{eD} by Γ_{gD}.

Finally, we have the following convergence result:

Theorem 3.1. *Let* $\mathbf{K}_1 := H^1(\Omega)$ *and* $\mathbf{K}_2 := H^1_{\partial_3}(\Omega)$. *When* $\varepsilon \to 0$, *the family* $(s_{p,q}(\varepsilon))_{\varepsilon > 0}$ *of the unique solutions of* $\mathcal{P}(\varepsilon, \Omega)_{p,q}$ *strongly converges in* $\mathbf{X}_{p,q} := H^1_{\Gamma_{mD}}(\Omega)^3 \times \mathbf{K}_p \times \mathbf{K}_q$ *to the unique solution* $\bar{s}_{p,q}$ *of*

$$\overline{\mathcal{P}}(\Omega)_{p,q}\begin{cases}\textit{Find } s\in(0,\varphi_0,\phi_0)+\mathbf{S}_{p,q} \textit{ such that}\\ \widetilde{m}_{p,q}(s,r):=\int_\Omega \widetilde{M}_{p,q}k(s)^0_{p,q}\cdot k(r)^0_{p,q}\,\mathrm{d}x=L(r),\quad \forall r\in\mathbf{S}_{p,q}:=\mathbf{V}_{\mathrm{KL}}\times\Phi_{e,p}\times\Phi_{g,q}\end{cases}$$

Proof. For any $r\in\mathbf{V}$, we define $k(r)=k(v,\psi,\Psi):=(e(v),\nabla\psi,\nabla\Psi)$. Assumptions (2) and (4) together with Korn and Poincaré inequalities imply that $(s_{p,q}(\varepsilon),k(\varepsilon,s_{p,q}(\varepsilon)))$ is bounded in $\mathbf{X}_{p,q}\times L^2(\Omega,\mathcal{H})$ so that there exists a subsequence, still indexed by ε, such that $(s_{p,q}(\varepsilon),k(\varepsilon,s_{p,q}(\varepsilon)))\rightharpoonup(\bar{s}_{p,q},\bar{k}_{p,q})$ in $\mathbf{X}_{p,q}\times L^2(\Omega;\mathcal{H})$ and $k(s_{p,q}(\varepsilon))^-_{p,q}\to 0$ in $L^2(\Omega;\mathcal{H})$ with $k(\bar{s}_{p,q})^0_{p,q}=(\bar{k}_{p,q})^0_{p,q}$, where $\rightharpoonup$ and $\to$ respectively stand for weak and strong convergences. A suitable choice of test functions in $\mathcal{P}(\varepsilon,\Omega)_{p,q}$, as in [2], yields $(M\bar{k}_{p,q})^-_{p,q}=(\bar{k}_{p,q})^+_{p,q}=0$. Taking r arbitrary in $\mathbf{S}_{p,q}\cap C^\infty(\overline{\Omega},\mathbb{R}^5)$ gives that $\bar{s}_{p,q}$ is *the* unique solution of $\overline{\mathcal{P}}(\Omega)_{p,q}$, hence the whole family $(s_{p,q}(\varepsilon))_{\varepsilon>0}$ weakly converges to $\bar{s}_{p,q}$. The strong convergence follows by taking $h=k(\varepsilon,s_{p,q}(\varepsilon))-\bar{k}_{p,q}$ in (2). □

4. The four models

In Theorem 3.1, the limit behavior is pointed out by a problem posed over Ω, namely $\overline{\mathcal{P}}(\Omega)_{p,q}$. To get physically meaningful results, we define an electromagneto-mechanical state $\bar{s}^\varepsilon_{p,q}$ over the real plate Ω^ε by the descaling $\bar{s}^\varepsilon_{p,q}(\pi^\varepsilon x):=\bar{s}_{p,q}(x),\ \forall x\in\Omega$. This electromagneto-mechanical state is the unique solution of a problem posed over Ω^ε which is the transportation by π^ε of the (limit scaled) problem $\overline{\mathcal{P}}(\Omega)_{p,q}$:

$$\overline{\mathcal{P}}(\Omega^\varepsilon)_{p,q}\begin{cases}\text{Find } s^\varepsilon\in(0,\varphi_0^\varepsilon,\phi_0^\varepsilon)+\mathbf{S}^\varepsilon_{p,q} \text{ such that}\\ \widetilde{m}^\varepsilon_{p,q}(s,r):=\int_{\Omega^\varepsilon}\widetilde{M}^\varepsilon_{p,q}k(s)^0_{p,q}\cdot k(r)^0_{p,q}\,\mathrm{d}x=L^\varepsilon(r),\quad\forall r\in\mathbf{S}^\varepsilon_{p,q}\end{cases}$$

The superscript ε in $\mathbf{S}^\varepsilon_{p,q}$ and L^ε means that Ω is replaced by Ω^ε in their definitions. This transported problem is our proposal to model thin linearly electromagneto-elastic plates of thickness 2ε. In these models, constitutive laws are given by $\widetilde{M}^\varepsilon_{p,q}$ where $\widetilde{M}^\varepsilon_{p,q}(\pi^\varepsilon x):=\widetilde{M}_{p,q}(x),\ \forall x\in\Omega$. This modeling with a simplified kinematic exhibits decoupling properties that are analogous to those highlighted in [10]. It is also accurate in the sense that the convergence result on the scaled states implies that s^ε is asymptotically equivalent to $\bar{s}^\varepsilon_{p,q}$.

Remark 4.1. This modeling of electromagneto-elastic thin plates can be extended without difficulties to other multiphysical couplings as soon as the constitutive equations (or their derivatives) involve state variables which implicate space gradients.

5. The limit constitutive equations

Introducing mechanical (m), electrical (e) and magnetic (g) components of $k^0_{p,q}$, we associate to $\widetilde{M}^\varepsilon_{p,q}$ the sub-operators $\widetilde{M}^\varepsilon_{p,q_{mm}}$, $\widetilde{M}^\varepsilon_{p,q_{me}}$, $\widetilde{M}^\varepsilon_{p,q_{mg}}$, $\widetilde{M}^\varepsilon_{p,q_{ee}}$, $\widetilde{M}^\varepsilon_{p,q_{eg}}$, $\widetilde{M}^\varepsilon_{p,q_{gg}}$. These operators are obtained by suitable algebraic operations involving M^ε (see (1) for its definition). We can then show, and it is to be pointed out, that $\widetilde{M}^\varepsilon_{p,q}$ keeps the same structure than M^ε in spite of the dimension reduction process. That means that $\widetilde{M}^\varepsilon_{p,q_{mm}}$, $\widetilde{M}^\varepsilon_{p,q_{ee}}$ and $\widetilde{M}^\varepsilon_{p,q_{gg}}$ are symmetric while $\widetilde{M}^\varepsilon_{p,q_{me}}=-\widetilde{M}^{\varepsilon^T}_{p,q_{em}}$, $\widetilde{M}^\varepsilon_{p,q_{mg}}=-\widetilde{M}^{\varepsilon^T}_{p,q_{gm}}$ and $\widetilde{M}^\varepsilon_{p,q_{eg}}=\widetilde{M}^{\varepsilon^T}_{p,q_{ge}}$. It is important to note that a^ε, b^ε, d^ε, and f^ε are polar tensors while c^ε and e^ε are axial ones. Moreover, magnetic effects involve symmetry with respect to time. Recall that a^ε, b^ε, d^ε, and f^ε are even while c^ε and e^ε are odd with respect to time reversal. Of course, we restrict our study to materials that are at the same time piezoelectric and piezomagnetic. From the symmetry point of view, there are forty five such crystal classes (see [11] for example). In the following, we will use the prime to denote time reversal. In the case of a polarization normal to the plate, we have the following properties:

- When $(p,q)=(1,1)$, $\widetilde{M}^\varepsilon_{1,1}$ may be represented through Voigt notations by a 7×7 matrix. There is a piezoelectric *and* a piezomagnetic decoupling (i.e. $\widetilde{M}^\varepsilon_{1,1_{me}}=\widetilde{M}^\varepsilon_{1,1_{mg}}=0$) for all the crystal classes of the orthorhombic, tetragonal and cubic systems. This also occurs in the monoclinic system for the class 2 and in the hexagonal system for the classes 6, 622, $62'2'$, $6mm$ and $6m'm'$. There are only ten crystal classes for which piezoelectric and piezomagnetic couplings simultaneously occur (i.e. $\widetilde{M}^\varepsilon_{1,1_{me}}$ *and* $\widetilde{M}^\varepsilon_{1,1_{mg}}\neq 0$): 1, m', 3, $\bar{6}'$, 32, $32'$, $3m$, $3m'$, $\bar{6}'m2'$ and $\bar{6}'m'2$.

– When $(p,q)=(1,2)$, $\widetilde{M}^{\varepsilon}_{1,2}$ may be represented by a 6×6 matrix. The piezoelectric and piezomagnetic decouplings simultaneously occur with the classes $2'$, $6'$, 422, $4mm$, $\bar{4}m2$, 622, $6'22'$, $6mm$ and $6'm'm$. There are only seven classes for which piezoelectric and piezomagnetic couplings simultaneously occur: 1, m, 3, $\bar{6}$, $32'$, $3m'$ and $\bar{6}m'2'$. Moreover, the electromagnetic coupling always vanishes (i.e. $\widetilde{M}^{\varepsilon}_{12_{eg}}=0$), except for the classes 1, $2'$ and m.
– When $(p,q)=(2,1)$, $\widetilde{M}^{\varepsilon}_{2,1}$ may be represented by a 6×6 matrix. The piezoelectric and piezomagnetic decouplings simultaneously occur with the classes m, $\bar{6}$, 422, $42'2'$, $4'22'$, 622, $62'2'$, $\bar{6}m2$ and $\bar{6}m'2'$. There are only seven classes for which piezoelectric and piezomagnetic couplings simultaneously occur: 1, $2'$, 3, $6'$, $3m$, $3m'$ and $6'm'm$. Moreover, the electromagnetic coupling always vanishes, except for the classes 1, $2'$ and m.
– When $(p,q)=(2,2)$, $\widetilde{M}^{\varepsilon}_{2,2}$ may be represented by a 5×5 matrix and $\widetilde{M}^{\varepsilon}_{2,2_{mm}}$ involves only mechanical terms. The crystal classes for which the piezoelectric and the piezomagnetic couplings simultaneously occur are: 1, 2, 3, 4, $4'$, $\bar{4}$, $\bar{4}'$, 6, 222, $2'2'2$, $2mm$, $2m'm'$, $2'mm'$, $3m'$, $4m'm'$, $4'mm'$, $\bar{4}m'2'$, $\bar{4}'m2'$, $\bar{4}'m'2$, $6m'm'$, 23 and $\bar{4}'3m'$. Moreover, the piezoelectric *and* the piezomagnetic decouplings occur with the classes m', $\bar{6}'$, 32, 422, 622, $6'22'$, $\bar{6}m2$, $\bar{6}'m2'$ and $\bar{6}'m'2$.
– In the later cases, that is for the classes m', $\bar{6}'$, 32, 422, 622, $6'22'$, $\bar{6}m2$, $\bar{6}'m2'$ and $\bar{6}'m'2$ (and only these), *all the* $\widetilde{M}^{\varepsilon}_{p,q_{mm}}$ *are identical.*
– For the classes $\bar{6}'$, $\bar{6}'m2'$ and $\bar{6}'m'2$ when $p=q=1$ and for the classes 222, $\bar{4}'$, $\bar{4}'2'm$, $\bar{4}'2m'$, 23 and $\bar{4}'3m'$ when $p=q=2$, the operators $\widetilde{M}^{\varepsilon}_{p,q_{mm}}$, $\widetilde{M}^{\varepsilon}_{p,q_{me}}$, $\widetilde{M}^{\varepsilon}_{p,q_{mg}}$, $\widetilde{M}^{\varepsilon}_{p,q_{ee}}$, $\widetilde{M}^{\varepsilon}_{p,q_{eg}}$ and $\widetilde{M}^{\varepsilon}_{p,q_{gg}}$ involve only mechanical, piezoelectric, piezomagnetic, dielectric, electromagnetic and magnetic permeability coefficients respectively, i.e. there is no mixing even if coupling always appears. In *all* other situations, these operators involve a mixture of coefficients of different types.

When $p=q=2$, the constitutive equations for an electromagneto-elastic plate designed with a 222 symmetry class material are very simple and read as:

$$\begin{pmatrix}\sigma^{\varepsilon}_{11}\\ \sigma^{\varepsilon}_{22}\\ \sqrt{2}\,\sigma^{\varepsilon}_{12}\\ D^{\varepsilon}_{3}\\ B^{\varepsilon}_{3}\end{pmatrix}=\underbrace{\begin{pmatrix}(\mathsf{a}^{\varepsilon}_{11}\mathsf{a}^{\varepsilon}_{33}-\mathsf{a}^{\varepsilon^2}_{13})/\mathsf{a}^{\varepsilon}_{33} & (\mathsf{a}^{\varepsilon}_{12}\mathsf{a}^{\varepsilon}_{33}-\mathsf{a}^{\varepsilon}_{13}\mathsf{a}^{\varepsilon}_{23})/\mathsf{a}^{\varepsilon}_{33} & 0 & 0 & 0\\ (\mathsf{a}^{\varepsilon}_{12}\mathsf{a}^{\varepsilon}_{33}-\mathsf{a}^{\varepsilon}_{13}\mathsf{a}^{\varepsilon}_{23})/\mathsf{a}^{\varepsilon}_{33} & (\mathsf{a}^{\varepsilon}_{22}\mathsf{a}^{\varepsilon}_{33}-\mathsf{a}^{\varepsilon^2}_{23})/\mathsf{a}^{\varepsilon}_{33} & 0 & 0 & 0\\ 0 & 0 & \mathsf{a}^{\varepsilon}_{66} & -\mathsf{b}^{\varepsilon}_{63} & -\mathsf{c}^{\varepsilon}_{63}\\ 0 & 0 & \mathsf{b}^{\varepsilon}_{63} & \mathsf{d}^{\varepsilon}_{33} & \mathsf{e}^{\varepsilon}_{33}\\ 0 & 0 & \mathsf{c}^{\varepsilon}_{63} & \mathsf{e}^{\varepsilon}_{33} & \mathsf{f}^{\varepsilon}_{33}\end{pmatrix}}_{\widetilde{M}^{\varepsilon}_{2,2}}\cdot\begin{pmatrix}e^{\varepsilon}_{11}\\ e^{\varepsilon}_{22}\\ \sqrt{2}\,e^{\varepsilon}_{12}\\ \partial^{\varepsilon}_{3}\varphi^{\varepsilon}\\ \partial^{\varepsilon}_{3}\phi^{\varepsilon}\end{pmatrix} \tag{8}$$

which is an example of a coupled but not mixed operator. Of course $\widetilde{M}^{\varepsilon}_{p,q}$ is symmetric as soon as piezoelectric and piezomagnetic decouplings simultaneously occur.

Remark 5.1. The physical situation when the thin plate is used as an electrical (resp. magnetic) sensor corresponds to $p=1$ (resp. $q=1$) while the actuator case corresponds to $p,q=2$. It therefore appears two original mixed behaviors when $p\neq q$. In these situations, the plate is *at the same time* a sensor *and* an actuator excepted for the classes listed *supra* for which the plate is no more electromagneto-elastic (i.e. $\widetilde{M}^{\varepsilon}_{p,q_{me}}=\widetilde{M}^{\varepsilon}_{p,q_{mg}}=0$). The two cases $p\neq q$ allow the modeling of electrically commanded magnetic devices and of magnetically commanded electric ones, which is of considerable interest in the development of non-volatile magnetic random access memories. We emphasize on the point that this behavior is here fully describe for any admissible crystal class (for a detailed discussion and a classification of piezoelectric and piezomagnetic symmetry classes, see [10]).

References

[1] T. Weller, C. Licht, Analyse asymptotique de plaques minces linéairement piézoélectriques, C. R. Acad. Sci. Paris, Sér. I 335 (2002) 309–314.
[2] P.G. Ciarlet, Mathematical Elasticity, vol. II, North-Holland, Amsterdam, 1997.
[3] M. Rahmoune, A. Benjeddou, R. Ohayon, New thin piezoelectric plates models, J. Intelligent Material Systems and Structures 9 (1998) 1017–1029.
[4] M. Lenczner, Modèle d'assemblage de plaques piézoélectriques, Rapport d'activité du Laboratoire de Calcul Scientifique, Equipe de Mathématiques, Université de Franche-Comté, Besançon.

[5] A. Sène, Modélisation asymptotique de plaques : contrôlabilité exacte frontière, piezoélectricité, Thèse de l'Université Joseph Fourrier – Grenoble I, 2000.
[6] G.A. Maugin, D. Attou, An asymptotic theory of thin piezoelectric plates, Quart. J. Mech. Appl. Math. 43 (1990) 347–362.
[7] P. Bisegna, F. Maceri, A consistent theory of thin piezoelectric plates, J. Intelligent Material Systems and Structures 7 (1996) 372–389.
[8] I.N. Figueiredo, C.F. Leal, A piezoelectric anisotropic plate model, Asymptot. Anal. 44 (2005) 327–346.
[9] M. Fiebig, Revival of the magnetoelectric effect, J. Phys. D: Appl. Phys. 38 (2005) R123–R152.
[10] T. Weller, Étude des symétries et modèles de plaques en piézoélectrictité linéarisée, Thèse de l'Université Montpellier II, 2004.
[11] J. Sivardière, La symétrie en mathématiques, physique et chimie, Presses Universitaires de Grenoble, 1995.