

# Nonlinear sequential laminates reproducing hollow sphere assemblages

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## Abstract

A special class of nonlinear porous materials with isotropic ‘sequentially laminated’ microstructures is found to reproduce exactly the hydrostatic behavior of ‘hollow sphere assemblages’. It is then argued that this result supports the conjecture that Gurson’s approximate criterion for plastic porous materials, and its viscoplastic extension of Leblond et al. (1994), may actually yield rigorous upper bounds for the hydrostatic flow stress of porous materials containing an isotropic, but otherwise arbitrary, distribution of porosity. *To cite this article: M.I. Idiart, C. R. Mecanique 335 (2007).*

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## Résumé

**Matériaux séquentiellement stratifiés non-linéaires reproduisant des ‘assemblages des sphères composites’.** On montre qu’une classe spéciale des matériaux poreux non-linéaires avec des microstructures ‘séquentiellement stratifiées’ isotropes reproduit exactement le comportement hydrostatique des ‘assemblage des sphères composites’ de Hashin. On argumente que ce résultat conforte la conjecture suivant laquelle le critère de Gurson pour les matériaux poreux plastiques, et son extension viscoplastique due à Leblond et al. (1994), peuvent constituer des bornes supérieures rigoureuses pour le seuil plastique hydrostatique des matériaux poreux contenant une distribution de porosité arbitraire mais macroscopiquement isotrope. *Pour citer cet article : M.I. Idiart, C. R. Mecanique 335 (2007).*

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## 1. Introduction

In this Note, a special class of *nonlinear* porous materials with isotropic ‘sequentially laminated’ microstructures is found to reproduce exactly the hydrostatic behavior of ‘hollow sphere assemblages’ (HSAs). The interest in this result

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stems from the fact that, as will be argued below, it supports the conjecture that Gurson’s approximate criterion [1] for plastic porous materials, and its viscoplastic extension [2], may actually yield rigorous upper *bounds* for the hydrostatic flow stress of porous materials containing an isotropic, but otherwise *arbitrary*, distribution of porosity [3–5]. Given that Gurson’s criterion is the most widely used estimate for plastic porous materials, and the fact that for sufficiently high triaxialities it can be much more restrictive than the best upper bounds for porous materials known to date [6–8]—especially at small porosities—, the validity of this conjecture would have important implications, both, practical and theoretical.

The subsequent analysis is set in the theoretical framework of nonlinear composites. Composites refer to heterogeneous materials made of  $N$  different homogeneous constituents, or *phases*, which are *randomly* distributed in a volume  $\Omega$ , at a length scale that is much smaller than the size of  $\Omega$  and the scale of variation of the loading conditions. The constitutive behavior of each phase is taken to be characterized by convex strain potentials  $w^{(r)}$  ( $r = 1, \dots, N$ ) such that

$$\boldsymbol{\sigma} = \partial_{\boldsymbol{\varepsilon}} w^{(r)}(\boldsymbol{\varepsilon}) \tag{1}$$

where  $\partial_{\boldsymbol{\varepsilon}}$  denotes differentiation with respect to  $\boldsymbol{\varepsilon}$ . Within the context of the deformation theory of plasticity,  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\sigma}$  represent the infinitesimal strain and stress, while in the context of viscoplasticity, they represent the Eulerian strain rate and Cauchy stress, respectively. Then, the effective behavior of the composite, which is defined as the relation between the volume averages of the stress  $\bar{\boldsymbol{\sigma}} = \langle \boldsymbol{\sigma} \rangle$  and the strain  $\bar{\boldsymbol{\varepsilon}} = \langle \boldsymbol{\varepsilon} \rangle$ , can be characterized by an effective strain potential  $\tilde{W}$ , such that (e.g., [9])

$$\bar{\boldsymbol{\sigma}} = \partial_{\bar{\boldsymbol{\varepsilon}}} \tilde{W}(\bar{\boldsymbol{\varepsilon}}), \quad \tilde{W}(\bar{\boldsymbol{\varepsilon}}) = \min_{\boldsymbol{\varepsilon} \in \mathcal{K}(\bar{\boldsymbol{\varepsilon}})} \sum_{r=1}^N c^{(r)} \langle w^{(r)}(\boldsymbol{\varepsilon}) \rangle^{(r)} \tag{2}$$

Here,  $\langle \cdot \rangle$  and  $\langle \cdot \rangle^{(r)}$  denote the volume averages over the composite ( $\Omega$ ) and over phase  $r$  ( $\Omega^{(r)}$ ), respectively,  $c^{(r)}$  is the volume fraction of phase  $r$ , and  $\mathcal{K}$  is the set of kinematically admissible strain fields with a prescribed volume average  $\bar{\boldsymbol{\varepsilon}}$ . In general, the function  $\tilde{W}$  is extremely difficult to compute, since it amounts to solving a set of nonlinear partial differential equations with randomly oscillating coefficients. However, for the special class of sequentially laminated composites considered in Section 2, expression (2) simplifies considerably, becoming amenable of analytical treatment.

**2. Two-phase composites with sequentially laminated microstructures**

A sequential laminate is an iterative construction obtained by layering laminated materials (which in turn have been obtained from lower-order lamination procedures) with other laminated materials, or directly with the homogeneous phases that make up the composite, in such a way as to produce hierarchical microstructures of increasing complexity (e.g., [10]). The *rank* of the laminate refers to the number of layering operations required to reach the final sequential laminate. Of the many possible types of sequential laminates, we restrict attention to two-phase, sequential laminates formed by layering at every step a laminate with one of the original phases, identified with  $r = 1$ . Thus, a rank-1 laminate corresponds to a simple laminate with a given layering direction  $\mathbf{n}^{(1)}$ , with phases 1 and 2 in proportions

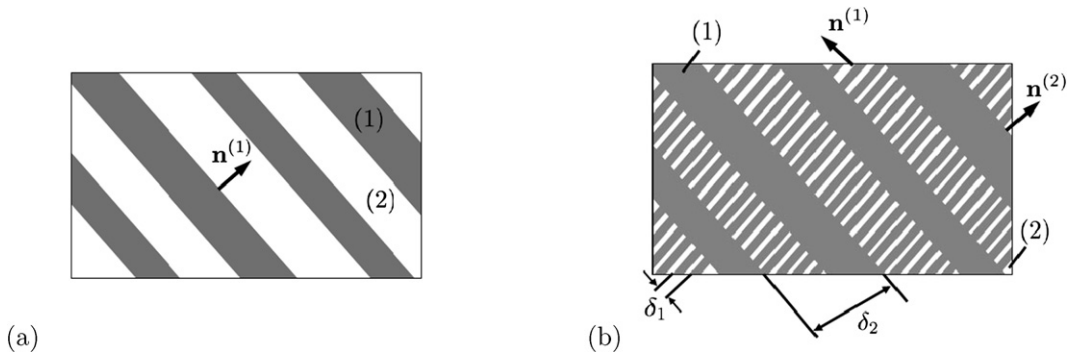


Fig. 1. Two-phase laminates: (a) simple or rank 1 laminate, (b) rank 2 laminate ( $\delta_2 \gg \delta_1$ ).

$1 - f_1$  and  $f_1$  (see Fig. 1(a)). In turn, a rank-2 laminate is constructed by layering the rank-1 laminate with phase 1, in a different layering direction  $\mathbf{n}^{(2)}$ , in proportions  $f_2$  and  $1 - f_2$ , respectively (see Fig. 1(b)). Rank- $M$  laminates are obtained by iterating this procedure  $M$  times, layering the rank- $(M - 1)$  laminate with phase 1 in direction  $\mathbf{n}^{(M)}$ , in proportions  $f_M$  and  $1 - f_M$ , respectively. In this procedure, it is assumed that the length scale of the embedded laminate is much smaller than the length scale of the embedding laminate (i.e.,  $\delta_1 \ll \delta_2$  in Fig. 1(b)). This assumption allows one to regard the rank- $(M - 1)$  laminate in the rank- $M$  laminate as a homogeneous phase, so that available expressions for the effective potential of simple laminates (e.g., [11]) can be used at each step of the process to obtain an exact expression for the effective potential of the rank- $M$  sequential laminate (e.g., [12,13]).

It is important to note that the effective behavior of the sequential laminates considered here, unlike that of typical nonlinear composites, does not depend on all the details of the microstructure, but only on partial information of it in the form of the volume fractions  $f_i$  and lamination directions  $\mathbf{n}^{(i)}$ . Furthermore, from their construction process described above, it follows that the microstructure of these sequential laminates can be regarded as *random* and *particulate*, with phase 2 playing the role of the *inclusion* (discontinuous) phase embedded in a (continuous) *matrix* made up of phase 1 (see Fig. 1(b)). A distinctive feature of this very special class of ‘particulate’ composites is that the local fields are *uniform* in the inclusion phase, but not in the matrix phase.

In general, sequentially laminated microstructures are *anisotropic*. However, deBotton [13] has recently proposed a lamination sequence, namely a set of  $f_i$  and  $\mathbf{n}^{(i)}$  ( $i = 1, \dots, M$ ), such that *isotropic* microstructures result in the limit of *infinite* rank ( $M \rightarrow \infty$ ). In this sequence, the  $M$  unit vectors  $\mathbf{n}^{(i)}$  are chosen so that their endpoints are uniformly distributed over the unit sphere  $S$ , and the corresponding  $f_i$  are all taken to be  $f_i = 1 - c^{(1)}/M$ , where the volume fraction of the matrix phase  $c^{(1)}$  in the rank- $M$  laminate is assumed *infinitesimally* small (i.e.,  $c^{(1)} \ll 1$ ). Thus, expanding the effective potential of such rank- $M$  laminate (e.g., [13]) to *first order* in  $c^{(1)}$ , and then letting  $M \rightarrow \infty$  in the resulting expression, the following result is obtained for the effective potential of the infinite-rank laminate [14]:

$$\tilde{W}(\bar{\boldsymbol{\varepsilon}}) = \min_{\mathbf{a}(\mathbf{n})} \left\{ (1 - c^{(1)})w^{(2)}(\bar{\boldsymbol{\varepsilon}}) + c^{(1)} \frac{1}{|S|} \int_S [w^{(1)}(\bar{\boldsymbol{\varepsilon}} + \mathbf{a}(\mathbf{n}) \otimes_s \mathbf{n}) - \mathbf{n} \cdot \partial_{\boldsymbol{\varepsilon}} w^{(2)}(\bar{\boldsymbol{\varepsilon}})\mathbf{a}(\mathbf{n})] dS(\mathbf{n}) \right\} \quad (3)$$

Here,  $|S|$  and  $dS(\mathbf{n})$  denote, respectively, the total and differential surface area over the unit sphere  $S$ ,  $\mathbf{a}(\mathbf{n})$  is the (three-dimensional) amplitude vector associated with the strain jump between laminae, and the symbol  $\otimes_s$  denotes the symmetric part of the outer product.

It is emphasized that expression (3) constitutes an *exact* result for nonlinear composites with the class of random and isotropic, ‘particulate’ microstructures described above. Even though this result, being valid only for composites with a *dilute* matrix concentration, may not seem very useful at first, in fact, it can be used in an iterative process to obtain exact results for composites with *non-dilute* matrix concentrations, as will be seen in the next section.

Finally, it is useful to note that the optimization problem in (3) simplifies when the *matrix* phase is *incompressible*. Indeed, the argument of  $w^{(1)}$  in (3) must be traceless in this case, and therefore, the minimization must be carried out over functions  $\mathbf{a}(\mathbf{n})$  of the form

$$\mathbf{a}(\mathbf{n}) = -\text{tr}(\bar{\boldsymbol{\varepsilon}})\mathbf{n} + a_1(\mathbf{n})\mathbf{m}_1 + a_2(\mathbf{n})\mathbf{m}_2 \quad (4)$$

where  $\mathbf{m}_1$  and  $\mathbf{m}_2$  are two mutually orthogonal, unit vectors lying on the plane with normal  $\mathbf{n}$ . Thus, incompressibility reduces by one the dimension of the optimization problem in (3).

### 3. Porous, power-law materials under hydrostatic loadings

In this section, the result provided above is used to generate an exact expression for the effective potential of a special class of sequential laminates made of a power-law matrix containing an *arbitrary* concentration of pores. Thus, the inclusion phase is taken to be *vacuous*, so that  $w^{(2)} = 0$ , and the matrix phase is taken to be characterized by an isotropic, *incompressible* potential of the form

$$w^{(1)}(\boldsymbol{\varepsilon}) = \frac{\sigma_0 \varepsilon_0}{1 + m} \left( \frac{\varepsilon_e}{\varepsilon_0} \right)^{1+m} \quad (5)$$

where  $m$  is the strain-rate sensitivity, such that  $0 \leq m \leq 1$ ,  $\sigma_0$  is a flow stress,  $\varepsilon_0$  is a reference strain, and the von Mises equivalent strain is given in terms of the deviatoric part of the strain tensor by  $\varepsilon_e = \sqrt{(2/3)\boldsymbol{\varepsilon}_d \cdot \boldsymbol{\varepsilon}_d}$ .

Following deBotton [13], porous laminates with *non-dilute* matrix concentrations are constructed by means of an iterative process based on a differential scheme. The idea is to take an infinite-rank porous laminate with a dilute matrix concentration, as the inclusion phase of an infinite-rank laminate with a dilute matrix concentration, which in turn is taken as the inclusion phase of an infinite-rank laminate with a dilute matrix concentration, and so on. Thus, each time, an infinitesimal amount of matrix material is added to the porous material, in such a way that, after repeating the process an infinite number of times, a non-dilute concentration of the matrix material is reached. This process allows repeated use of the exact result (3), so that, at the end, an exact expression for the effective potential of the non-dilute porous laminate is obtained in the form of a differential equation. It is noted that this process is contrary to those followed in common differential schemes, where the starting point is a composite with a dilute concentration of the *inclusion* phase (see Ref. [10] and references therein).

Thus, the first step in this process is to consider an infinite-rank porous laminate with a *dilute* concentration  $c^{[1]}$  of the power-law matrix (5), subjected to *hydrostatic* loadings. A superscript  $[k]$  is used here to denote quantities corresponding to the  $k$ th iteration. According to (3) and (4), the corresponding effective potential is given by

$$\tilde{W}^{[1]}(\bar{\boldsymbol{\varepsilon}}) = \min_{a_1^{[1]}(\mathbf{n}), a_2^{[1]}(\mathbf{n})} c^{[1]} \frac{1}{|S|} \int_S \frac{\sigma_0 \varepsilon_0}{1+m} \left[ (2\bar{\varepsilon}_m)^2 + \frac{(a_1^{[1]}(\mathbf{n}))^2 + (a_2^{[1]}(\mathbf{n}))^2}{3} \right]^{(1+m)/2} dS(\mathbf{n}) \tag{6}$$

where  $\bar{\varepsilon}_m = (1/3) \text{tr}(\bar{\boldsymbol{\varepsilon}})$  is the macroscopic mean strain. Clearly, the minimum in this expression is attained at  $a_1^{[1]}(\mathbf{n}) = a_2^{[1]}(\mathbf{n}) = 0$ , the integrand in (6) becomes independent of  $\mathbf{n}$ , and so the effective potential  $\tilde{W}^{[1]}$  can be written explicitly as

$$\tilde{W}^{[1]}(\bar{\boldsymbol{\varepsilon}}) = \frac{\tilde{\sigma}_h^{[1]} \varepsilon_0}{1+m} \left| 2 \frac{\bar{\varepsilon}_m}{\varepsilon_0} \right|^{1+m}, \quad \frac{\tilde{\sigma}_h^{[1]}}{\sigma_0} = c^{[1]} \tag{7}$$

where  $\tilde{\sigma}_h^{[1]}$  denotes the hydrostatic effective flow stress of the porous laminate in the first iteration. Note that, because of the isotropy of  $\tilde{W}^{[1]}$ , the derivative  $\partial_{\bar{\boldsymbol{\varepsilon}}} \tilde{W}^{[1]}(\bar{\boldsymbol{\varepsilon}})$  evaluated at hydrostatic loadings must be hydrostatic, the resulting mean stress being  $\bar{\sigma}_m = (2/3) \tilde{\sigma}_h^{[1]} |\bar{\varepsilon}_m / \varepsilon_0|^m \text{sgn}(\bar{\varepsilon}_m)$ .

Next, let the potential (7) be that of the inclusion phase in an infinite-rank laminate with a dilute concentration  $c^{[2]}$  of a power-law matrix phase. It then follows from the specialized version of (3), together with (4), that the effective potential  $\tilde{W}^{[2]}$  of this laminate subjected to hydrostatic loadings is of the same form (7)<sub>1</sub>, but with an effective flow stress  $\tilde{\sigma}_h^{[2]}$  given by

$$\frac{\tilde{\sigma}_h^{[2]}}{\sigma_0} = \frac{\tilde{\sigma}_h^{[1]}}{\sigma_0} + c^{[2]} \left( 1 + m \frac{\tilde{\sigma}_h^{[1]}}{\sigma_0} \right) \tag{8}$$

where  $\tilde{\sigma}_h^{[1]}$  is given by (7)<sub>2</sub>. In deriving this expression, use has been made of the facts that the derivative  $\partial_{\bar{\boldsymbol{\varepsilon}}} \tilde{W}^{[1]}(\bar{\boldsymbol{\varepsilon}})$  evaluated at hydrostatic loadings is hydrostatic, and the minimum in (3) is, once again, attained at  $a_1^{[2]}(\mathbf{n}) = a_2^{[2]}(\mathbf{n}) = 0$ . Note that  $\tilde{\sigma}_h^{[2]}$  corresponds to an infinite-rank, porous laminate with a *total* porosity  $(1 - c^{[2]})(1 - c^{[1]})$ .

By following exactly similar arguments, it can be shown that the effective potential at the  $k + 1$  iteration is of the same form (7)<sub>1</sub>, but with an effective flow stress given by

$$\frac{\tilde{\sigma}_h^{[k+1]}}{\sigma_0} = \frac{\tilde{\sigma}_h^{[k]}}{\sigma_0} + c^{[k+1]} \left( 1 + m \frac{\tilde{\sigma}_h^{[k]}}{\sigma_0} \right) \tag{9}$$

where  $\tilde{\sigma}_h^{[k]}$  is the effective flow stress of the porous laminate in the previous iteration, with  $\tilde{\sigma}_h^{[0]} = 0$ , and  $c^{[k+1]}$  denotes the (infinitesimal) concentration of matrix material *added* to the porous laminate at the  $k + 1$  iteration. The *total* concentration  $c_{k+1}^{(1)}$  of matrix material in the porous laminate at the  $k + 1$  iteration is given by

$$c_{k+1}^{(1)} = 1 - \prod_{j=1}^{k+1} (1 - c^{[j]}) \tag{10}$$

and the *increment* of total matrix concentration at each iteration is thus given by

$$c_{k+1}^{(1)} - c_k^{(1)} = \prod_{j=1}^k (1 - c^{[j]}) - \prod_{j=1}^{k+1} (1 - c^{[j]}) = c^{[k+1]} \prod_{j=1}^k (1 - c^{[j]}) = c^{[k+1]} (1 - c_k^{(1)}) \quad (11)$$

Then, from relations (9) and (11), it follows that

$$\left(1 + m \frac{\tilde{\sigma}_h^{[k]}}{\sigma_0}\right)^{-1} \frac{(\tilde{\sigma}_h^{[k+1]}/\sigma_0^{(1)}) - (\tilde{\sigma}_h^{[k]}/\sigma_0^{(1)})}{c_{k+1}^{(1)} - c_k^{(1)}} = \frac{1}{1 - c_k^{(1)}} \quad (12)$$

Since the  $c^{[k]}$  are infinitesimally small, relation (12) leads to the following differential equation for the effective flow stress  $\tilde{\sigma}_h$  of a porous laminate with a total, non-dilute matrix concentration  $c^{(1)}$ :

$$\left(1 + m \frac{\tilde{\sigma}_h}{\sigma_0}\right)^{-1} \frac{d(\tilde{\sigma}_h/\sigma_0)}{dc^{(1)}} = \frac{1}{1 - c^{(1)}}, \quad \left. \frac{\tilde{\sigma}_h}{\sigma_0} \right|_{c^{(1)}=0} = 0 \quad (13)$$

Finally, integrating this equation gives

$$\frac{\tilde{\sigma}_h}{\sigma_0} = \frac{1}{m} (f^{-m} - 1) \quad (14)$$

where  $f = 1 - c^{(1)}$  denotes the total porosity in the final infinite-rank, porous laminate. In the limiting case of an ideally plastic matrix ( $m \rightarrow 0$ ), this expression becomes

$$\frac{\tilde{\sigma}_h}{\sigma_0} = -\ln f \quad (15)$$

Note that the corresponding hydrostatic yield stress is  $|\bar{\sigma}_m| = (2/3)\tilde{\sigma}_h = -(2/3)\sigma_0 \ln f$ .

Expressions (14) and (15) agree exactly with those corresponding to HSAs (e.g., [2]). Finally, it is worth noting that the two-dimensional analogue of this result, i.e., transversely isotropic laminates reproducing the in-plane hydrostatic behavior of ‘hollow cylinder assemblages’, follows from exactly similar arguments.

#### 4. Discussion

Thus, isotropic porous materials with HSAs and sequentially laminated microstructures exhibit identical hydrostatic behaviors. In turn, this behavior is reproduced exactly by the hydrostatic limit of Gurson’s approximate criterion [1] and its viscoplastic extension [2]. Furthermore, since expression (14) with  $m = 1$  reduces to the linear Hashin–Shtrikman upper bound for the class of porous materials exhibiting overall isotropy (e.g., [10]), it follows that HSAs and sequentially laminated microstructures are ‘optimal’ within that class. Even though this ‘optimality’ is only valid in the linear case, the fact that porous materials with these two very different types of microstructures exhibit identical hydrostatic behaviors for any degree of matrix nonlinearity suggests that these microstructures may still be ‘optimal’ in the nonlinear domain. In turn, this supports the conjecture that the hydrostatic limit of Gurson’s criterion and its viscoplastic extension may constitute rigorous upper bounds for the hydrostatic flow stress of porous materials containing an isotropic, but otherwise arbitrary, distribution of porosity [3–5]. In this connection, it is noted that recent numerical simulations [15] of porous, ideally plastic materials with random distributions of porosity different than HSAs, indeed exhibit a weaker hydrostatic stress than that given by Gurson’s criterion.

A distinctive feature of porous materials with the above-mentioned HSAs and sequentially laminated microstructures is that the fields in the ‘inclusion’ phase (i.e., the pores) are uniform, what may hint at proving Gurson’s bounding character. A proof of this conjecture, however, remains elusive.

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