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Homogenization of thermo-viscoelastic Kelvin–Voigt model

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Abstract

We consider an ε-periodic composite material, $\varepsilon \ll 1$, constituted of periodic fibres surrounded by a polymer matrix, solidifying under a heating process. The mechanical behaviour of the material is described by the Kelvin–Voigt visco-elasticity equation with rapidly oscillating space and time dependent coefficients. This time dependence is caused by the dependence of the state of the material on the temperature, that is a solution of a thermo-chemical model studied earlier. The existence and uniqueness of a solution of the Kelvin–Voigt visco-elasticity model are proved, the homogenized model is obtained and the existence and uniqueness of its solution are studied. The estimates for the difference between the solution of the original problem and the homogenized one are obtained. ***To cite this article: Z. Abdessamad et al., C. R. Mecanique 335 (2007).***

Résumé

Homogénéisation du modèle thermo-visco-élastique Kelvin–Voigt. On considère un matériau composite de structure périodique de période $\varepsilon \ll 1$, constitué d'un tissu de fibres noyé dans une résine qui se soldifie sous l'effet de la chaleur. Les propriétés mécaniques du matériau sont décrites par l'équation de viscoélasticité de Kelvin–Voigt avec des coefficients oscillants dépendant des variables spatiale et temporaire x et t. Cette dépendance de temps est engendrée par la dépendance de l'état déformé du matériau de la temperature, une solution du problème thermo-chimique étudié précedement. On établit un résultat d'existence et d'unicité de la solution, puis à l'aide de la méthode du développement asymptotique on détermine le problème homogénéisé. On prouve l'existence et l'unicité de la solution du problème homogénéisé, puis on obtient une estimation pour la différence entre la solution du problème de départ et la solution du problème homogénéisé lorsque ε tend vers zéro. ***Pour citer cet article : Z. Abdessamad et al., C. R. Mecanique 335 (2007).***

Keywords: Computational solid mechanics; Homogenization; Visco-elasticity; Time dependent coefficients; Memory effect

Mots-clés : Mécanique des solides numérique ; Homogénéisation ; Visco-élasticité ; Coefficients dépendents de temps ; Effet de mémoire

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The present Note continues the topic [1] where the following thermo-chemical model of formation of a composite material has been considered. (See [1] for the precise physical meaning of all the entities.)

$$\begin{cases} C\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial T_\varepsilon}{\partial t} - \operatorname{div}\left(K\left(\dfrac{x}{\varepsilon}\right)\nabla T_\varepsilon\right) = f_\varepsilon, & (x,t)\in\Omega\times(0,\tau)\\ \dfrac{\partial\alpha_\varepsilon}{\partial t} = \hat{f}(\alpha_\varepsilon, T_\varepsilon), & (x,t)\in\Omega\times(0,\tau)\\ T_\varepsilon(x,t)=0, & (x,t)\in\partial\Omega\times(0,\tau)\\ T_\varepsilon(x,0)=T^0(x), \quad \alpha_\varepsilon(x,0)=\alpha^0(x), & x\in\Omega \end{cases} \tag{1}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, $n\geqslant 2$; its boundary $\partial\Omega$ is assumed to be C^4-smooth; $f_\varepsilon = q(\frac{x}{\varepsilon})\hat{f}(\alpha_\varepsilon, T_\varepsilon) + f_0(x,t)$; $C(\xi)$, $K(\xi)$ and $q(\xi)$ are 1-periodic measurable bounded functions of $\xi\in\mathbb{R}^n$, piecewise smooth in the sense of [2] §4.1, such that $\exists k_1, q_1 > 0$: $\forall\xi\in\mathbb{R}^n$, $C(\xi), K(\xi)\geqslant k_1$; $|q(\xi)|\leqslant q_1$; $\hat{f}\in C^4(\mathbb{R}^2)$, λ-Lipschitz (i.e. $|\hat{f}(\alpha_1,T_1)-\hat{f}(\alpha_2,T_2)|\leqslant\lambda(|\alpha_1-\alpha_2|^2+|T_1-T_2|^2)^{1/2}$) with some

$$\lambda < \frac{k_1}{\max\{1,\frac{\operatorname{diam}\Omega}{\sqrt{2}}\}(k_1\tau + (1+\frac{1}{\sqrt{2}})q_1\operatorname{diam}\Omega)}$$

$T^0\in C^4(\overline{\Omega})$, $f_0\in C^2(\overline{\Omega}\times[0,\tau])$, $\alpha^0\in C^2(\overline{\Omega})$.

It was shown in [1] that if (α_0, T_0) is a solution of the homogenized problem

$$\begin{cases} \langle C\rangle_\xi\dfrac{\partial T_0}{\partial t} - \operatorname{div}\big(\widehat{K}\nabla T_0\big) = \langle q\rangle_\xi\hat{f}(\alpha_0,T_0)+f_0(x,t), & (x,t)\in\Omega\times(0,\tau)\\ \dfrac{\partial\alpha_0}{\partial t} = \hat{f}(\alpha_0,T_0), & (x,t)\in\Omega\times(0,\tau)\\ T_0(x,t)=0, & (x,t)\in\partial\Omega\times(0,\tau)\\ T_0(x,0)=T^0(x), \quad \alpha_0(x,0)=\alpha^0(x), & x\in\Omega \end{cases} \tag{2}$$

such that $(\alpha_0,T_0)\in C^2(\overline{\Omega}\times[0,\tau])\times C^3(\overline{\Omega}\times[0,\tau])$, then we have the following estimate:

$$\|T_\varepsilon - T_0\|_{L^2(\Omega\times(0,\tau))} + \|\alpha_\varepsilon-\alpha_0\|_{L^2(\Omega\times(0,\tau))} = \mathrm{O}\big(\sqrt{\varepsilon}\big)$$

Henceforth the angular brackets with subscript ξ denote the mean value of the appropriate function with respect to ξ: $\langle u(\xi)\rangle_\xi := \int_Q u(\xi)\,\mathrm{d}\xi$, where $Q=(0,1)^n$ is the reference periodicity cell. The symbol $\widehat{K}$ denotes the homogenized conductivity matrix (see [2]).

In this Note, we refer to a particular case of problem (1) where we take $\alpha^0 = T^0 = 0$ and $\hat{f}(0,0)=0$. In this case, the above results can be improved as follows:

Lemma 1. *Assume that $m\in\mathbb{N}^*$, $f_0\in H^m(0,\tau;H^{2m}(\Omega))$ and that there exists a constant $\tau^*<\tau$ such that $f_0(x,t)=0$, for all $t\leqslant\tau^*$, $\hat{f}\in C^{2m}(\mathbb{R}^2)$ (and all the derivatives of $\hat{f}$ of order up to $2m$ are bounded), $\lambda^{(m)}$-Lipschitz in the sense*

$$\lambda^{(m)} := \sup_{u_1,u_2,v_1,v_2\in H^m(0,\tau;H^{2m}(\Omega))}\frac{\|\hat{f}(u_1,v_1)-\hat{f}(u_2,v_2)\|_{H^m(0,\tau;H^{2m}(\Omega))}}{\|u_1-u_2\|_{H^m(0,\tau;H^{2m}(\Omega))}+\|v_1-v_2\|_{H^m(0,\tau;H^{2m}(\Omega))}} < \infty$$

Then there exists $\mu^{(m)}>0$ depending only on m, k_1, q_1, τ and Ω, such that as long as $\lambda^{(m)}\leqslant\mu^{(m)}$ a solution (α_0,T_0) of problem (2) exists, it is unique and belongs to $C^{m-[n/2]-1}(\overline{\Omega}\times[0,\tau])\times C^{m-[n/2]}(\overline{\Omega}\times[0,\tau])$; it is equal to zero for all $t\leqslant\tau^*$.

Lemma 2. *Assume that there exists a finite number p of open 1-periodic sets $(\mathcal{D}_i)_{1\leqslant i\leqslant p}$ with $C^{1,\beta}$-smooth boundary $(0<\beta\leqslant 1)$ such that $D_i\cap D_j=\emptyset$, $i\neq j$, and $\mathbb{R}^n=\bigcup_{i=1}^p\overline{\mathcal{D}}_i$, and assume that $K\in C^{0,\mu}(\overline{\mathcal{D}}_i)$ for all $i=1,\ldots,p$. Let the assumptions of Lemma 1 hold for $m=3+[n/2]$ and let $(\alpha_\varepsilon,T_\varepsilon)$, (α_0,T_0) be the solutions of problems (1) and (2) respectively. Then there exists a constant C independent of ε such that*

$$\|T_\varepsilon-T_0\|_{L^\infty(\overline{\Omega}\times[0,\tau])}+\|\alpha_\varepsilon-\alpha_0\|_{L^\infty(\overline{\Omega}\times[0,\tau])}\leqslant C\varepsilon \quad \textit{and} \quad \left\|\frac{\partial T_\varepsilon}{\partial t}\right\|_{L^\infty(\overline{\Omega}\times[0,\tau])}\leqslant C$$

Consider now the Kelvin–Voigt visco-elasticity equation

$$\begin{cases} \rho\left(\dfrac{x}{\varepsilon}\right)\ddot{u}^{\varepsilon}_{T_\varepsilon} - \dfrac{\partial}{\partial x_i}\left(B^{\varepsilon,T_\varepsilon}_{ij}\dfrac{\partial \dot{u}^{\varepsilon}_{T_\varepsilon}}{\partial x_j}\right) - \dfrac{\partial}{\partial x_i}\left(A^{\varepsilon,T_\varepsilon}_{ij}\dfrac{\partial u^{\varepsilon}_{T_\varepsilon}}{\partial x_j}\right) = f(x,t), \quad (x,t)\in\Omega\times(0,\tau) \\ u^{\varepsilon}_{T_\varepsilon}(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,\tau) \\ u^{\varepsilon}_{T_\varepsilon}(x,0)=\dot{u}^{\varepsilon}_{T_\varepsilon}(x,0)=0, \quad x\in\Omega \end{cases} \tag{3}$$

Here $u^{\varepsilon}_{T_\varepsilon}(x,t)$ is the unknown displacement vector field, $u^{\varepsilon}_{T_\varepsilon}(x,t)=(u^{\varepsilon}_{T_\varepsilon,k}(x,t))_{1\leqslant k\leqslant n}$; $\dot{u}^{\varepsilon}_{T_\varepsilon}$ and $\ddot{u}^{\varepsilon}_{T_\varepsilon}$ denote the first and the second time derivatives of $u^{\varepsilon}_{T_\varepsilon}$, respectively. The summation over the repeated indices is assumed. The volume density $\rho(\xi)$ is a scalar function which is also 1-periodic. The smooth vector function $f(x,t)$ is given, and describes forces due to the thermal effects. The linear elastic tensor $A^{\varepsilon,T_\varepsilon}_{ij}$ and the viscosity tensor $B^{\varepsilon,T_\varepsilon}_{ij}$ are matrix valued entities:

$$A^{\varepsilon,T_\varepsilon}_{ij}=\left(A^{kl}_{ij}\right)_{1\leqslant k,l\leqslant n}, \quad B^{\varepsilon,T_\varepsilon}_{ij}=\left(B^{kl}_{ij}\right)_{1\leqslant k,l\leqslant n}, \quad i,j=1,\ldots,n$$

These tensors are functions of T_ε, the solution of problem (1) and they are assumed to depend periodically on $\xi := x/\varepsilon$. This dependence means that the visco-elastic properties depend on the temperature. For example, for low temperatures the elastic tensor A plays a more significant role than the viscosity tensor B while for high temperatures it may be other way round. So, this temperature depending Kelvin–Voigt model simulates the thermo-chemico-visco-elastic process for a composite material. In this model the thermo-chemical and visco-elastic problems are decoupled: first the thermo-chemical problem (1) has to be solved, and then its solution is used in the visco-elastic equation (3). There is a high interest to the combined models of this type, see e.g. a recent paper [3] where a visco-plastic model taking into account the non-linear hardening effect is considered. Our model (1), (3) takes into account the dependence of the visco-elastic properties on the temperature.

Assume that $A_{ij}(\xi,T)$ and $B_{ij}(\xi,T)$ are Lipschitz functions of T and that their derivatives in T are also Lipschitz functions of T. We assume also that:

(H1) For all $T\in\mathbb{R}$, $\xi\in\mathbb{R}^n$ and $i,j=1,\ldots,n$, $A_{ij}(\xi,T)$, $B_{ij}(\xi,T)$ and their first derivative in T belong to $L^\infty(\mathbb{R}^{n+1};\mathcal{M}_{n,n}(\mathbb{R}))$. Additionally, $B^{kl}_{ij}=B^{il}_{kj}=B^{lk}_{ji}$ and there exists a positive constant ν independent of ξ and T, such that for all symmetric matrices $\eta=(\eta^l_j)\in\mathbb{R}^{n\times n}$,

$$\nu\eta^k_i\eta^k_i \leqslant B^{kl}_{ij}(\xi,T)\eta^k_i\eta^l_j \leqslant \nu^{-1}\eta^k_i\eta^k_i, \quad \forall\xi\in\mathbb{R}^n,\ \forall T\in\mathbb{R} \tag{4}$$

(H2) The 1-periodic function ρ belongs to $L^\infty(Q;\mathbb{R})$ and is uniformly positive.

We will adopt the following notational conventions throughout the Note. $H^1_\#(Q)$ denotes the closure of $C^\infty_\#(Q)$ in the norm of the standard space $H^1(Q)$ where $C^\infty_\#(Q)$ stands for the subspace of infinitely smooth functions $C^\infty(\mathbb{R}^n)$ whose elements are periodic with respect to Q. For any space X, we denote the space X^n by $\mathbf{X}$. The space $\mathcal{H}$ is the Hilbert space consisting of all elements u of $\mathbf{H}^1_\#(Q)$ that have a zero mean value with respect to ξ. It is equipped with the norm $\|u\|_{\mathcal{H}} := \|\nabla_\xi u\|_{(\mathbf{L}^2(Q))^n}$. The space $\mathbf{H}^{-1}_\#(Q)$ is the 'dual' space of $\mathbf{H}^1_\#(Q)$ with respect to the dual product denoted $\langle\cdot,\cdot\rangle_Q$.

The last result of Lemma 2 concerning the estimate of the first time derivative of the function T_ε will be employed with the assumption (H1) to prove that the eslasticity and the viscosity functions $A^{\varepsilon,T_\varepsilon}_{ij}$ and $B^{\varepsilon,T_\varepsilon}_{ij}$ have uniformly bounded first time derivatives.

Theorem 1. *Let $f\in L^2(0,\tau;\mathbf{H}^{-1}(\Omega))$ and let* (H1)–(H2) *and the assumptions of Lemma* 2 *hold. Then for all $\varepsilon>0$, problem* (3) *admits a unique solution $u^{\varepsilon}_{T_\varepsilon}$ in $H^1(0,\tau;\mathbf{H}^1_0(\Omega))$ and there exists a constant C_1 independent of ε such that*

$$\left|\left|\left|u^{\varepsilon}_{T_\varepsilon}\right|\right|\right|_{\Omega_\tau} \equiv \left\|u^{\varepsilon}_{T_\varepsilon}\right\|_{L^\infty(0,\tau;\mathbf{H}^1_0(\Omega))} + \left\|\dot{u}^{\varepsilon}_{T_\varepsilon}\right\|_{L^2(0,\tau;\mathbf{H}^1_0(\Omega))} + \left\|\dot{u}^{\varepsilon}_{T_\varepsilon}\right\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))} \leqslant C_1\|f\|_{L^2(0,\tau;\mathbf{H}^{-1}(\Omega))}$$

Let us replace T_ε by T_0 in the coefficients A_{ij}, B_{ij} of problem (3). Then, applying the Lipschitz property for the coefficients A_{ij} and B_{ij} in T, we can prove that the solution $u^{\varepsilon}_{T_0}$ of this new modified problem is close to the solution u^ε of problem (3):

$$\left|\left|\left|u^{\varepsilon}_{T_\varepsilon} - u^{\varepsilon}_{T_0}\right|\right|\right|_{\Omega_\tau} = \mathrm{O}(\varepsilon) \tag{5}$$

This new problem for $u^\varepsilon_{T_0}$ is a particular case of the time dependent problem for the Kelvin–Voigt equation. Taking into account estimate (5) we will further consider the following auxiliary visco-elasticity equation:

$$\rho\left(\frac{x}{\varepsilon}\right)\ddot{u}^\varepsilon - \frac{\partial}{\partial x_i}\left(B_{ij}\left(x,t,\frac{x}{\varepsilon}\right)\frac{\partial \dot{u}^\varepsilon}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(A_{ij}\left(x,t,\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x_j}\right) = f(x,t), \quad \text{in } \Omega\times(0,\tau) \tag{6}$$

$$u^\varepsilon = 0, \quad \text{on } \partial\Omega \tag{7}$$

$$u^\varepsilon_{|t=0} = \dot{u}^\varepsilon_{|t=0} = 0 \tag{8}$$

where, the tensors $A_{ij}(x,t,\xi)$ and $B_{ij}(x,t,\xi)$ are matrix-valued entities:

$$A_{ij}(x,t,\xi) = \left(A^{kl}_{ij}(x,t,\xi)\right)_{1\leqslant k,l\leqslant n}, \qquad B_{ij}(x,t,\xi) = \left(B^{kl}_{ij}(x,t,\xi)\right)_{1\leqslant k,l\leqslant n}$$

which are 1-periodic with respect to ξ. The properties of A_{ij} and B_{ij} with respect to x and t result from those of T_0, which are given by Lemma 1.

Let $\Omega_\tau := \Omega\times(0,\tau)$, and henceforth we assume that:

(H3) For all $i,j = 1,\ldots,n$, $A_{ij}(x,t,\xi)$, $B_{ij}(x,t,\xi)$ and their first time derivative $\dot{A}_{ij}$ and $\dot{B}_{ij}$ belong to $L^\infty(\Omega\times(0,\tau)\times\mathbb{R}^n;\mathcal{M}_{n,n}(\mathbb{R}))$. Additionally, $B^{kl}_{ij} = B^{il}_{kj} = B^{lk}_{ji}$ and there exists a positive constant ν independent of x, ξ and t, such that for all symmetric matrices $\eta = (\eta^l_j)\in\mathbb{R}^{n\times n}$ and for all $(x,t,\xi)\in\Omega\times(0,\tau)\times\mathbb{R}^n$, the tensor B_{ij} satisfies (4).

Thus, from Theorem 1, we deduce that for a given $f\in L^2(0,\tau;\mathbf{H}^{-1}(\Omega))$, and for all $\varepsilon>0$, problem (6)–(8) admits a unique solution u^ε in $H^1(0,\tau;\mathbf{H}^1_0(\Omega))$. In order to derive the homogenized problem, we use the traditional asymptotic expansion method (see, e.g., [2,4]) which suggests that the solution to the problem (6)–(8) be sought in the following two-scale form:

$$u^\varepsilon(x,t) \sim v(x,t) + \varepsilon N\left(x,t,\frac{x}{\varepsilon}\right) + \varepsilon^2 u^{(2)}\left(x,t,\frac{x}{\varepsilon}\right) + \cdots \tag{9}$$

where N and $u^{(2)}$ are 1-periodic functions with respect to the third argument. These functions are found via solving appropriate versions of the 'cell' problem in $\xi = x/\varepsilon$ and t, formulated in its generality as follows:
For any $x\in\overline{\Omega}$, find $u(x,\cdot,\cdot)\in H^1(0,\tau;\mathcal{H})$ such that:

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,t,\xi)\frac{\partial \dot{u}}{\partial \xi_j}\right) - \frac{\partial}{\partial \xi_i}\left(A_{ij}(x,t,\xi)\frac{\partial u}{\partial \xi_j}\right) = g(x,t,\xi), \quad u(x,0,\xi) = \phi(x,\xi) \tag{10}$$

with some given g and ϕ.

Theorem 2. *Assume that for all $x\in\overline{\Omega}$, ϕ and g belong respectively to $\mathcal{H}$ and $H^1(0,\tau;\mathbf{H}^{-1}_\#(Q))$. Let $g(x,t,\cdot)$ satisfy $\langle g\rangle_\xi = 0$. Let also for any $x\in\Omega$ the straightforward modifications of assumption* (H3) *hold, namely with x replaced by ξ, Ω by Q and Ω_τ by $Q_\tau := Q\times(0,\tau)$. Then the problem* (10) *admits for any x a unique solution $u(x,\cdot,\cdot)$ in $H^1(0,\tau;\mathcal{H})$. Moreover, there exists a constant C_2 such that*:

$$\|u\|_{L^\infty(0,\tau;\mathcal{H})} + \|\dot{u}\|_{L^2(0,\tau;\mathcal{H})} \leqslant C_2\left(\|g\|_{L^2(0,\tau;\mathbf{H}^{-1}(Q))} + \|\phi\|_{\mathcal{H}}\right), \quad \text{for any } x\in\overline{\Omega}$$

Substituting the ansatz (9) into (6) and collecting formally the terms with equal powers of ε, one can identify the terms corresponding to order ε^{-1}. Equation generated by these terms and the first initial condition (8) give the main 'unit cell' problem satisfied by N, the second term of the ansatz (9):

$$-\frac{\partial}{\partial \xi_i}\left(B_{ij}(x,t,\xi)\frac{\partial^2 N}{\partial \xi_j\partial t}(x,t,\xi)\right) - \frac{\partial}{\partial \xi_i}\left(A_{ij}(x,t,\xi)\frac{\partial N}{\partial \xi_j}(x,t,\xi)\right) = F(x,t,\xi), \quad \xi\in\mathbb{R}^n,\ t>0 \tag{11}$$

$$N(x,0,\xi) = 0, \quad \xi\in\mathbb{R}^n \tag{12}$$

where

$$F(x,t,\xi) := \frac{\partial}{\partial \xi_i}B_{ik}(x,t,\xi)\frac{\partial \dot{v}}{\partial x_k}(x,t) + \frac{\partial}{\partial \xi_i}A_{ik}(x,t,\xi)\frac{\partial v}{\partial x_k}(x,t)$$

By Theorem 2, the periodic function $N(x,\xi,t)$ satisfying the 'unit cell' problem exists for any given $x\in\overline{\Omega}$ and $v(\cdot,t)$ such that $v(\cdot,t)\in H^1(0,\tau)$. For uniqueness of the solution we require that N has zero mean value with respect to ξ. The following lemma establishes the structure of function N:

Lemma 3. *The following representation holds*:

$$N(x,t,\xi)=\int_0^t \frac{\partial\dot{v}}{\partial x_k}(x,t')\mathcal{N}_k^B(x,t-t',t',\xi)\,\mathrm{d}t'+\int_0^t \frac{\partial v}{\partial x_k}(x,t')\mathcal{N}_k^A(x,t-t',t',\xi)\,\mathrm{d}t'$$

Here $\mathcal{N}_k^A(x,t,s,\xi)$ and $\mathcal{N}_k^B(x,t,s,\xi)$ are periodic with respect to ξ solutions of the following initial boundary value problems:

$$-\frac{\partial}{\partial\xi_i}\left(B_{ij}(x,s+t,\xi)\frac{\partial\dot{\mathcal{N}}_k^A}{\partial\xi_j}(x,t,s,\xi)\right)-\frac{\partial}{\partial\xi_i}\left(A_{ij}(x,s+t,\xi)\frac{\partial\mathcal{N}_k^A}{\partial\xi_j}(x,t,s,\xi)\right)=0 \tag{13}$$

$$-\frac{\partial}{\partial\xi_i}\left(B_{ij}(x,s+t,\xi)\frac{\partial\dot{\mathcal{N}}_k^B}{\partial\xi_j}(x,t,s,\xi)\right)-\frac{\partial}{\partial\xi_i}\left(A_{ij}(x,s+t,\xi)\frac{\partial\mathcal{N}_k^B}{\partial\xi_j}(x,t,s,\xi)\right)=0 \tag{14}$$

$$\mathcal{N}_k^A(x,0,s,\xi)=g_k^A(x,s,\xi),\qquad \mathcal{N}_k^B(x,0,s,\xi)=g_k^B(x,s,\xi) \tag{15}$$

In turn, $g_k^A(x,s,\xi)$, $g_k^B(x,s,\xi)$ solve the following cell problems:

$$-\frac{\partial}{\partial\xi_i}\left(B_{ij}(x,s,\xi)\frac{\partial}{\partial\xi_j}g_k^A(x,s,\xi)\right)=\frac{\partial}{\partial\xi_i}A_{ik}(x,s,\xi) \tag{16}$$

$$-\frac{\partial}{\partial\xi_i}\left(B_{ij}(x,s,\xi)\frac{\partial}{\partial\xi_j}g_k^B(x,s,\xi)\right)=\frac{\partial}{\partial\xi_i}B_{ik}(x,s,\xi) \tag{17}$$

In the same way, the identification of the terms corresponding to order ε^0 after the substitution of (9) into (6) and in (8) implies that the function $u^{(2)}(x,t,\xi)$ solves the problem (10) with $\phi=\phi_{u^{(2)}}=0$ and g equal to some function $\mathcal{F}$. According to Theorem 2, $u^{(2)}(x,t,\xi)$ exists if and only if the function $\mathcal{F}(x,t,\xi)$ has zero mean value with respect to ξ over Q: $\langle\mathcal{F}(x,t,\xi)\rangle_\xi=0$. This equation implies the homogenized equation:

$$\hat{\rho}\ddot{v}(x,t)-\frac{\partial}{\partial x_i}\sigma_i(x,t)=f(x,t) \tag{18}$$

where $\hat{\rho}=\langle\rho\rangle_\xi$ and

$$\sigma_i(x,t)=\widehat{B}_{ij}(x,t)\frac{\partial\dot{v}}{\partial x_j}+\widehat{A}_{ij}(x,t)\frac{\partial v}{\partial x_j}+\int_0^t\left(\widehat{E}_{ij}(x,t,t')\frac{\partial\dot{v}}{\partial x_j}(x,t')+\widehat{D}_{ij}(x,t,t')\frac{\partial v}{\partial x_j}(x,t')\right)\mathrm{d}t' \tag{19}$$

Let us note that the homogenized relations (18)–(19) display the 'memory effect' due to the integral terms in (19). Expression (19) uses the notation

$$\widehat{A}_{ij}(x,t)=\left\langle A_{ij}(x,t,\xi)+B_{ik}(x,t,\xi)\frac{\partial g_j^A}{\partial\xi_k}\right\rangle_\xi,\qquad \widehat{B}_{ij}(x,t)=\left\langle B_{ik}(x,t,\xi)\left(\delta_{kj}I+\frac{\partial g_j^B}{\partial\xi_k}\right)\right\rangle_\xi \tag{20}$$

$$\widehat{E}_{ij}(x,t,t')=\left\langle A_{ik}(x,t,\xi)\frac{\partial\mathcal{N}_j^B}{\partial\xi_k}(x,t-t',t',\xi)\right\rangle_\xi+\left\langle B_{ik}(x,t,\xi)\frac{\partial^2\mathcal{N}_j^B}{\partial\xi_k\partial t}(x,t-t',t',\xi)\right\rangle_\xi \tag{21}$$

$$\widehat{D}_{ij}(x,t,t')=\left\langle A_{ik}(x,t,\xi)\frac{\partial\mathcal{N}_j^A}{\partial\xi_k}(x,t-t',t',\xi)\right\rangle_\xi+\left\langle B_{ik}(x,t,\xi)\frac{\partial^2\mathcal{N}_j^A}{\partial\xi_k\partial t}(x,t-t',t',\xi)\right\rangle_\xi \tag{22}$$

Now, we define the operator $\mathcal{L}: H^1(0,\tau;\mathbf{H}_0^1(\Omega))\to H^1(0,\tau;\mathbf{H}_0^1(\Omega))$, such that if $x\in\Omega$, $v\in H^1(0,\tau;\mathbf{H}_0^1(\Omega))$ then $\tilde{v}=\mathcal{L}v$ is defined as a solution of the following equation:

$$\hat{\rho}\ddot{\tilde{v}}-\frac{\partial}{\partial x_i}\left(\widehat{B}_{ij}\frac{\partial\dot{\tilde{v}}}{\partial x_j}+\widehat{A}_{ij}\frac{\partial\tilde{v}}{\partial x_j}\right)=f+\frac{\partial}{\partial x_i}\int_0^t\left(\widehat{E}_{ij}(x,t,t')\frac{\partial\dot{v}}{\partial x_j}(x,t')+\widehat{D}_{ij}(x,t,t')\frac{\partial v}{\partial x_j}(x,t')\right)\mathrm{d}t'$$

such that, $\tilde{v}(x,0)=\dot{\tilde{v}}(x,0)=0$ and $\tilde{v}(x,t)|_{\partial\Omega\times(0,\tau)}=0$. Thus, by using the Banach fixed point theorem we establish the following results:

Theorem 3. *Let $f\in L^2(0,\tau;\mathbf{H}^{-1}(\Omega))$ such that there exists a constant $\tau^*<\tau$, such that $f(x,t)=0$ for all $t\leqslant\tau^*$, and assume that* (H2)–(H3) *hold. Then there exists a unique $v\in H^1(0,\tau;\mathbf{H}^1_0(\Omega))$, such that*

$$\hat{\rho}\ddot{v}(x,t)-\frac{\partial\sigma_i}{\partial x_i}(x,t)=f(x,t),\quad v|_{\partial\Omega\times(0,\tau)}=0,\ v|_{t=0}=\dot{v}|_{t=0}=0$$

In order to justify the asymptotic expansion of u^ε we will impose additional regularity assumptions on the viscoelastic coefficients. Namely, we will assume that A_{ij} and B_{ij} are both smooth with respect to x and t, and periodic and piecewise smooth with respect to ξ. More precisely, we will assume (cf. Lemma 2) that there exist a finite number p of disjoint periodic subdomains $\mathcal{D}_i\subset\mathbb{R}^n$, $i=1,\dots,p$, such that $\mathbb{R}^n=\bigcup_{i=1}^p\overline{\mathcal{D}}_i$ and that each A_{lj}, B_{lj} is in Hölder class $C^{1,\zeta}(\overline{\mathcal{D}}_i)$ of periodic functions, with $0<\zeta\leqslant 1$. We hence require the physical characteristics of the composite media to be smooth in ξ in each subdomain $\overline{\mathcal{D}}_i$ assumed itself having a sufficiently smooth boundary, but possibly discontinuous across their boundaries. Henceforth we denote for a fixed $0<\zeta<1$ by $\mathcal{K}$ the space consisting of all elements u of $C^{1,\zeta}(\overline{\mathcal{D}}_i)$ for all $i=1,\dots,p$.

Theorem 4. *Assume that*: *v, $\dot{v}\in C^3(\overline{\Omega}_\tau)$; A_{ij}, B_{ij}, N and $\dot{N}\in C^2(\overline{\Omega}_\tau;\mathcal{K})$ and $u^{(2)}$, $\dot{u}^{(2)}\in C^1(\overline{\Omega}_\tau;\mathcal{K})$. Then there exists a constant C independent of ε such that*

$$\left\|u^\varepsilon-(v+\varepsilon N)\right\|_{L^\infty(0,\tau;\mathbf{H}^1_0(\Omega))}+\left\|\dot{u}^\varepsilon-(\dot{v}+\varepsilon\dot{N})\right\|_{L^2(0,\tau;\mathbf{H}^1_0(\Omega))}\leqslant C\varepsilon^{1/2}$$
$$\|u^\varepsilon-v\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))}+\|\dot{u}^\varepsilon-\dot{v}\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))}\leqslant C\varepsilon^{1/2}$$

Now we describe sufficient conditions which ensure the validity of the assumptions of Theorem 4. Namely, we consider the case when the elastic and viscous characteristics are proportional, i.e. there exists a constant $\kappa>0$ such that for all $i,j=1,\dots,n$, $A_{ij}=\kappa B_{ij}$. In this case we check that $\widehat{A}_{ij}=\kappa\widehat{B}_{ij}$, $\widehat{E}_{ij}=\widehat{D}_{ij}=0$ and from (18)–(19), we conclude that the homogenized problem takes the form

$$\hat{\rho}\ddot{v}(x,t)-\frac{\partial}{\partial x_i}\left(\widehat{B}_{ij}(x,t)\frac{\partial}{\partial x_j}(\dot{v}+\kappa v)\right)=f(x,t),\quad v|_{t=0}=\dot{v}|_{t=0}=0$$

Then by using the Faedo–Galerkin method we prove that it is enough to have A_{ij}, $B_{ij}\in C^{[n/2]+4}(\overline{\Omega}_\tau;L^\infty(Q))$, and for all $k=0,1,2,3$, $f^{(k)}\in L^2(0,\tau;\mathbf{H}^{6-2k}(\Omega))$, such that there exists a constant $\tau^*<\tau$, such that $f(x,t)=0$ for all $t\leqslant\tau^*$ to ensure the inclusions v, $\dot{v}\in C^3(\overline{\Omega}_\tau)$ which are assumed in Theorem 4. Additionally, using the results of [5] (or of [6]) and [7] we prove:

Lemma 4. *Let $\mathcal{D}_i\subset\mathbb{R}^n$, $i=1,\dots,p$, be as in Lemma 2. We also assume that $A_{lj}(x,t,\xi)$ and $B_{lj}(x,t,\xi)$ are 1-periodic with respect to ξ, such that A_{lj}, B_{lj} belong to $C^2(\overline{\Omega}_\tau;C^{0,\lambda}(\overline{\mathcal{D}}_i))$ for all $i=1,\dots,p$. Then*

$$N,\dot{N}\in C^2(\overline{\Omega}_\tau;\mathcal{K}),\quad u^{(2)},\dot{u}^{(2)}\in C^1(\overline{\Omega}_\tau;\mathcal{K})$$

Thereby we conclude that under the assumptions of Lemma 4 and those immediately preceding it, the proportionality condition of A_{ij} and B_{ij} is sufficient to ensure the validity of the assumptions made in Theorem 4 and consequently it ensures the results of Theorem 4 which proves the estimate for the difference between the exact solution u^ε and v when ε tends to zero.

Remark. Let us recall that for (α_0,T_0), the solution of problem (2), the function T_0 depends only on variables x and t which belong to $\overline{\Omega}_\tau$, thus we will consider A_{ij}^{ε,T_0} and B_{ij}^{ε,T_0} as functions of (x,t,ξ). Then, by using Lemma 1 and Lemma 4, we can find sufficient conditions for A_{ij}^{ε,T_0} and B_{ij}^{ε,T_0} to be regular enough, e.g. for them to belong to $C^{4+[n/2]}(\overline{\Omega}_\tau;L^\infty(Q))\cap C^2(\overline{\Omega}_\tau;\mathcal{K})$. Thus, we can apply the main result of the present paper Theorem 4 to a viscoelasticity equation coupled with problem (2). Indeed, the associated homogenized problem has the same structure as the problem (18)–(19). It suffices to replace A_{ij} and B_{ij} by A_{ij}^{ε,T_0} and B_{ij}^{ε,T_0} in definitions of the homogenized coefficients given by (20)–(22) and in problems (13)–(17). In particular, if A_{ij}^{ε,T_0} and B_{ij}^{ε,T_0} are proportional, then the result of Theorem 4 is also valid for the difference between $u_{T_0}^{\varepsilon}$ and the solution of this homogenized problem, also denoted by v. Finally, we use (5) to deduce the following estimate:

$$\|u_{T_\varepsilon}^{\varepsilon} - v\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))} + \|\dot{u}_{T_\varepsilon}^{\varepsilon} - \dot{v}\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))} \leqslant C\varepsilon^{1/2}$$

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