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A non-uniform warping theory for beams

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Abstract

This Note proposes a non-uniform warping beam theory including the effects of torsion and shear forces. Based on a displacement model using three warping parameters associated to three St Venant warping functions corresponding to torsion and shear forces, this theory is free from the classical assumptions on the warpings or on the shears, and is valid for any kind of homogeneous elastic and isotropic cross-section. The result on the structural behavior of the beam specifies the effect of the non-symmetry of the cross-section, and the closed form results obtained for the stresses show the contribution of each internal force. Comparison with St Venant beam theory highlights the additional effects due to the non-uniformity of the warping. ***To cite this article: R. El Fatmi, C. R. Mecanique 335 (2007).***

Résumé

Une théorie de gauchissement non uniforme. Cette Note propose une théorie de gauchissement non uniforme prenant en compte les effets de la torsion et des efforts tranchants. Basée sur un modéle cinématique utilisant trois paramètres de gauchissement associés aux trois fonctions de gauchissement de torsion et d'efforts tranchants de St Venant, cette théorie, qui s'affranchit naturellement des hypothèses classiques sur les gauchissements ou les cisaillements, est valable pour toute section homogène élastique et isotrope. La comparaison aux résultats de St Venant permet aussitôt de dégager les effets induits par la non-uniformité du gauchissement. En particulier, cette théorie permet de préciser la contribution de chacun des efforts intérieurs ainsi que l'effet de la non-symétrie de la section sur le comportement global de la poutre. ***Pour citer cet article : R. El Fatmi, C. R. Mecanique 335 (2007).***

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Mots-clés : Mécanique des solides numérique ; Gauchissement ; Torsion ; Effort tranchant ; St Venant ; Couplage ; Contrainte axiale ; Cisaillement

1. Introduction

In the general case of loading and boundary conditions, warping is non-uniform along the axis of a beam. This leads to a beam mechanical behavior that may be sufficiently different from that predicted by the St Venant (SV) beam theory [1,2] or other theories which are restricted to uniform warping. To better describe the warping effects,

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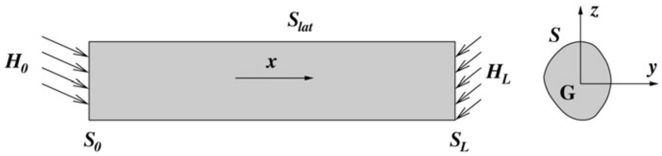

Fig. 1. St Venant problem.

high order beam theories have been proposed; they have generally been used to study, separately, the effects of torsional warping [3,4] and shear force warping [5,6]. These theories are based on displacement models ($\boldsymbol{\xi}$) including a warping (w) of the following shape:

$$\boldsymbol{\xi}(x, \boldsymbol{X}) = \boldsymbol{v}(x) + \boldsymbol{\theta}(x) \wedge \boldsymbol{X} + w(x, \boldsymbol{X})\boldsymbol{x} \quad \text{with} \quad w(x, \boldsymbol{X}) = \eta(x)\psi(\boldsymbol{X}) \tag{1}$$

where $\boldsymbol{x}$ is the unit vector along the beam axis, $\boldsymbol{X}$ the in-section vector position, $(\boldsymbol{v}, \boldsymbol{\theta})$ the cross-sectional displacements, η the warping parameter and ψ a warping mode associated to torsion or to one of the shear forces. In each case, ψ is supposed to represent the corresponding SV-warping-function, which is considered as the reference to describe the natural warping of a cross-section (CS). Further, η may be independent or linked to the cross-sectional strains, which can reduce the number of degrees of freedom (for the torsion, η is taken as the twisting rate (e.g. [3]), and for the shear-bending η is taken as the cross-sectional shear strain (e.g. [6])).

Non-uniform warping theories agree for the structural behavior of the beam in the case of bi-symmetrical-CS. There is also an agreement about the expression of the additional axial stresses due to warping. However, the situation is not so clear for shear stresses, because these are intimately associated to the choice of the warping mode and the warping parameter (independent or not). Also, there is no agreement concerning the effect of the non-symmetry of the CS. In most of the works, for non-symmetrical-CS, if the bending moments refer to the centroid while the torsional moment refers to the shear center, torsional and bending effects remain uncoupled for non-uniform warping theories as they were in classical beam theories. However, on an other hand, [4] has shown that a (new) flexural-torsional coupling is induced by the non-uniformity of the warping.

In order to obtain a beam theory valid for any CS and able to detect eventual elastic coupling between warpings, we propose, in this Note, a beam theory based on the following warping model:

$$w(x, \boldsymbol{X}) = \eta_x(x)\psi^x(\boldsymbol{X}) + \eta_y(x)\psi^y(\boldsymbol{X}) + \eta_z(x)\psi^z(\boldsymbol{X}) \tag{2}$$

using three independent warping parameters (η_x, η_y, η_z) associated to three warping functions (ψ^x, ψ^y, ψ^z) which are 'exactly' the SV-warping-functions corresponding to torsion and shear forces. This model, that could be considered as the most general one, leads to a non-uniform beam theory (denoted herein by NUW-BT) free from the classical assumptions on the warping functions or on the shear distributions (e.g. Vlasov assumptions for thin-walled profiles).

It should be noted that the theoretical development of this theory is completely based on the properties of 3D SV-solution of the original and complete SV-problem. Thus, it is necessary in the present Note, to first recall SV-problem and the detailed 3D SV-solution.

2. Three-dimensional solution of St Venant problem

In this section, we give, for an homogeneous elastic isotropic material, the SV-solution that refers to the shear center[1] of the cross-section, and wherein, for the sake of simplicity, the in-plane displacement related to Poisson's effects will be omitted (which is equivalent to assume that the Poisson's ratio ν is zero). Furthermore, several properties of SV-solution, needed for the theoretical development of the NUW-BT presented in the next section, are also specified.

The reference problem shown in Fig. 1 is a 3D equilibrium beam problem. The beam is of section S and length L. S_{lat} is the lateral surface and S_0 and S_L are the extremity sections. $\boldsymbol{y}$ and $\boldsymbol{z}$ are the inertia unit vectors of the CS. A point is marked $M = x\boldsymbol{x} + \boldsymbol{X}$, where $\boldsymbol{X}$ belongs to S. The material constituting the beam is characterized by the Young's modulus E and the shear modulus G. The beam is in equilibrium under surface force densities $\boldsymbol{H}_0$ and $\boldsymbol{H}_L$

[1] We refer to the shear center, because it is common, for non-symmetrical cross-sections (in order to uncouple torsional and bending effects) to express the bending moments referring to the centroid while the torsional moment is referred to the shear center.

acting on S_0 and S_L, respectively. SV-solution satisfies all the equations of the linearized equilibrium problem, except the boundary conditions on (S_0, S_L) which are satisfied only in terms of the resultant (force and moment). To give the expression of the SV-solution, it is convenient to first introduce these notations:

– Let (y_c, z_c) denote the components of the shear center C of the CS. We define by $\overline{\boldsymbol{X}} = (0, (y - y_c), (z - z_c))$ the in-section vector that refers to the C.
– $\boldsymbol{\sigma}$ denoting the stress tensor, we define the cross-sectional stresses $(\boldsymbol{R}, \boldsymbol{M})$, resultant and moment, by

$$\boldsymbol{R} = \boldsymbol{r}(\boldsymbol{\sigma} \cdot \boldsymbol{x}) = (N, T^y, T^z), \qquad \boldsymbol{M} = \overline{\boldsymbol{m}}(\boldsymbol{\sigma} \cdot \boldsymbol{x}) = (M^x, M^y, M^z) \tag{3}$$

where (N, T^y, T^z) are the axial and the shear forces, M^x is the torsional moment referring to the shear center C, whereas (M^y, M^z) are the bending moment referring to the centroid G.

The SV-solution is given by:

$$\boldsymbol{\xi}^{\text{sv}} = \boldsymbol{u}(x) + \omega_x(x)\boldsymbol{x} \wedge \overline{\boldsymbol{X}} + \big(\omega_y(x)\boldsymbol{y} + \omega_z(x)\boldsymbol{z}\big) \wedge \boldsymbol{X} + \big(M^x\phi^x(\boldsymbol{X}) + T^y\phi^y(\boldsymbol{X}) + T^z\phi^z(\boldsymbol{X})\big)\boldsymbol{x} \tag{4}$$

$$\boldsymbol{\sigma}^{\text{sv}} = \begin{bmatrix} \sigma_{xx}^{\text{sv}} & \sigma_{xy}^{\text{sv}} & \sigma_{xz}^{\text{sv}} \\ & 0 & 0 \\ & & 0 \end{bmatrix}_{\text{sym}}, \qquad \begin{cases} \sigma_{xx}^{\text{sv}} = \dfrac{N}{A} + \dfrac{M^y}{I_y}z - \dfrac{M^z}{I_z}y \\ \boldsymbol{\tau}^{\text{sv}} = \sigma_{xy}^{\text{sv}}\boldsymbol{y} + \sigma_{xz}^{\text{sv}}\boldsymbol{z} = M^x\boldsymbol{\tau}^x + T^y\boldsymbol{\tau}^y + T^z\boldsymbol{\tau}^z \end{cases} \tag{5}$$

$$\gamma_x = \frac{N}{EA}, \quad \gamma_y = \frac{T_y}{GA_y}, \quad \gamma_y = \frac{T_z}{GA_z}, \quad \chi_x = \frac{M^x}{GJ}, \quad \chi_y = \frac{M^y}{EI_y}, \quad \chi_y = \frac{M^z}{EI_z} \tag{6}$$

where $(\boldsymbol{u}, \boldsymbol{\omega})$ are the cross-sectional displacements, $(\boldsymbol{\gamma} = \boldsymbol{u}' + \boldsymbol{x} \wedge \boldsymbol{\omega}, \boldsymbol{\chi} = \boldsymbol{\omega}')$ the cross-sectional strains, $(A, A_y, A_z, I_y, I_z, J)$ are the area, the reduced areas, the moments of inertia and the torsional constant, respectively; and where $(\phi^i, \boldsymbol{\tau}^i)$ with $i \in \{x, y, z\}$ are the SV-warping-functions and the SV-shears corresponding to torsion and shear forces, respectively. In this solution, the cross-sectional stresses $(\boldsymbol{R}, \boldsymbol{M})$ are supposed to verify the one-dimensional (1D) equilibrium equations

$$\boldsymbol{R}' = \boldsymbol{0}, \qquad \boldsymbol{M}' + \boldsymbol{x} \wedge \boldsymbol{R} = \boldsymbol{0} \tag{7}$$

and are related to $(\boldsymbol{\gamma}, \boldsymbol{\chi})$ by the 1D structural behavior expressed by the uncoupled constitutive relations (Eq. (6)). $\boldsymbol{\sigma}^{\text{sv}}$ is unique ($[\ ]_{\text{sym}}$ indicates that the matrix is symmetric) and ξ^{sv} is given within an arbitrary rigid body displacement.

Eqs. (7), (6) and boundary conditions on S_0 and S_L, form the 1D problem that defines the SV beam theory (denoted herein by SV-BT) associated to the 3D SV-solution.

2.1. *Properties[2] of SV-warping-functions and SV-shears*

(a) The shears $\boldsymbol{\tau}^i$ verify the natural conditions:

$$\langle\boldsymbol{\tau}^y\rangle = \boldsymbol{y}, \quad \langle\boldsymbol{\tau}^z\rangle = \boldsymbol{z}, \quad \langle\boldsymbol{\tau}^x\rangle = \boldsymbol{0}, \qquad \langle\overline{\boldsymbol{X}} \wedge \boldsymbol{\tau}^y\rangle = \boldsymbol{0}, \quad \langle\overline{\boldsymbol{X}} \wedge \boldsymbol{\tau}^z\rangle = \boldsymbol{0}, \quad \langle\overline{\boldsymbol{X}} \wedge \boldsymbol{\tau}^x\rangle = \boldsymbol{x} \tag{8}$$

(b) Shear forces and torsion are not coupled (Eq. (6)), this implies that:

$$\langle\boldsymbol{\tau}^x \cdot \boldsymbol{\tau}^x\rangle = \frac{1}{J}, \quad \langle\boldsymbol{\tau}^y \cdot \boldsymbol{\tau}^y\rangle = \frac{1}{A_y}, \quad \langle\boldsymbol{\tau}^z \cdot \boldsymbol{\tau}^z\rangle = \frac{1}{A_z}, \qquad \langle\boldsymbol{\tau}^x \cdot \boldsymbol{\tau}^y\rangle = \langle\boldsymbol{\tau}^x \cdot \boldsymbol{\tau}^z\rangle = \langle\boldsymbol{\tau}^y \cdot \boldsymbol{\tau}^z\rangle = 0 \tag{9}$$

(c) The warping functions are chosen such that:

$$\langle\phi^i\rangle = \langle y\phi^i\rangle = \langle z\phi^i\rangle = 0, \quad i \in \{x, y, z\} \tag{10}$$

These conditions are always possible [1], since $\boldsymbol{\xi}^{\text{sv}}$ is unique within an arbitrary rigid body displacement.

[2] Within the framework of the *exact* beam theory [1], which constitutes our reference, SV-solution is expressed with *cross-sectional operators* that contain the SV-warping-functions, the SV-shears, the structural behavior, ... For the mathematical characterization that governs these operators see the details [2, Section-3] for a 2D-characterization (on the cross-section) and see [7, Section-3] for 3D-characterization (on a slice of the beam). For the numerical method that can be used to compute these operators see [8,2,7].

(d) Starting from Eqs. (4)–(6), one can derive the relations between the shears and the warping functions:

$$\boldsymbol{\tau}^x = \frac{1}{J}(\boldsymbol{x} \wedge \overline{\boldsymbol{X}} + GJ\nabla\phi^x), \qquad \boldsymbol{\tau}^y = \frac{1}{A_y}(\boldsymbol{y} + GA_y\nabla\phi^y), \qquad \boldsymbol{\tau}^z = \frac{1}{A_z}(\boldsymbol{z} + GA_z\nabla\phi^z) \tag{11}$$

(∇ denotes the gradient operator).

(e) When a CS presents y- or/and z-symmetry, we have the following properties:

$$\left.\begin{array}{l} y\text{-symmetry } z_c = 0 \ \phi^x \text{ is odd}/z \ \phi^y \text{ is even}/z \ \phi^z \text{ is odd}/z \\ z\text{-symmetry } y_c = 0 \ \phi^x \text{ is odd}/y \ \phi^y \text{ is odd}/y \ \phi^z \text{ is even}/y \end{array}\right\} \tag{12}$$

3. Non-uniform warping theory

For the sake of simplicity, we keep, as reference, the problem defined in Section 2. The theory is based on the following displacement model (similar to that of the St Venant equation (4)):

$$\boldsymbol{\xi}(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{\eta}) = \boldsymbol{v}(x) + \theta_x(x)\boldsymbol{x} \wedge \overline{\boldsymbol{X}} + (\theta_y(x)\boldsymbol{y} + \theta_z(x)\boldsymbol{z}) \wedge \boldsymbol{X} + \eta_i(x) \cdot \psi^i(\boldsymbol{X})\boldsymbol{x} \tag{13}$$

where $\eta_i\psi^i$ is a sum using the repeated indices convention with $i \in \{x, y, z\}$, $(\boldsymbol{v}, \boldsymbol{\theta})$ are the cross-sectional displacements, η_i ($\boldsymbol{\eta} = (\eta_x, \eta_y, \eta_z)$) the warping parameters, and ψ^i the warping functions corresponding to torsion and shear forces related to SV-warping-functions by: $\psi^x = GJ\phi^x$, $\psi^y = GA_y\phi^y$ and $\psi^z = GA_z\phi^z$. This choice assumes that the CS maintains its shape (no distortion). The beam theory associated with this displacement, parametrized by $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{\eta})$, will be derived hereafter by the principle of virtual work. Let us introduce first $\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}(\hat{\boldsymbol{v}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\eta}})$ denoting a virtual displacement and $\hat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}(\hat{\boldsymbol{\xi}})$ the corresponding strain tensor. With $\hat{\boldsymbol{\gamma}} = \hat{\boldsymbol{v}}' + \boldsymbol{x} \wedge \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\chi}} = \hat{\boldsymbol{\theta}}'$, the non-zero components of $\hat{\boldsymbol{\varepsilon}}$ can be written:

$$\hat{\varepsilon}_{xx} = \hat{\gamma}_x + z\hat{\chi}_y - y\hat{\chi}_z + \hat{\eta}_i'\psi^i, \quad \begin{bmatrix} 2\,\hat{\varepsilon}_{xy} \\ 2\,\hat{\varepsilon}_{xz} \end{bmatrix} = \hat{\gamma}_y\boldsymbol{y} + \hat{\gamma}_z\boldsymbol{z} + \hat{\chi}_x(\boldsymbol{x} \wedge \overline{\boldsymbol{X}}) + \hat{\boldsymbol{\eta}}_i\nabla\psi^i \tag{14}$$

The internal virtual work is $W_i = -\int_L \langle \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\hat{\boldsymbol{\xi}})\rangle\,\mathrm{d}x$. Using the expression (14) of the virtual deformations, W_i takes the form:

$$\begin{aligned} W_i &= -\int_L (\boldsymbol{R} \cdot \hat{\boldsymbol{\gamma}} + \boldsymbol{M} \cdot \hat{\boldsymbol{\chi}} + \boldsymbol{M}_\psi \cdot \hat{\boldsymbol{\eta}}' + \boldsymbol{M}_s \cdot \hat{\boldsymbol{\eta}})\,\mathrm{d}x \\ &= \int_L \big(\boldsymbol{R}' \cdot (\hat{\boldsymbol{v}} + \boldsymbol{x} \wedge \hat{\boldsymbol{\theta}}) + \boldsymbol{M}' \cdot \hat{\boldsymbol{\theta}} + (\boldsymbol{M}_\psi' - \boldsymbol{M}_s) \cdot \hat{\boldsymbol{\eta}}\big)\,\mathrm{d}x - [\boldsymbol{R} \cdot \hat{\boldsymbol{v}} + \boldsymbol{M} \cdot \hat{\boldsymbol{\theta}} + \boldsymbol{M}_\psi \cdot \hat{\boldsymbol{\eta}}]_0^L \end{aligned} \tag{15}$$

where

$$\boldsymbol{R} = \langle \boldsymbol{\sigma} \cdot \boldsymbol{x}\rangle, \quad \boldsymbol{M} = \boldsymbol{m}(\boldsymbol{\sigma} \cdot \boldsymbol{x}), \quad \boldsymbol{M}_\psi = \langle \sigma_{xx}\psi^i\rangle\boldsymbol{x}_i, \quad \boldsymbol{M}_s = \langle \sigma_{xy}\psi^i_{,y} + \sigma_{xz}\psi^i_{,z}\rangle\boldsymbol{x}_i \quad \big(\boldsymbol{x}_i \in \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}\big) \tag{16}$$

This defines the internal forces $\boldsymbol{R}, \boldsymbol{M}, \boldsymbol{M}_\psi$ and $\boldsymbol{M}_s$. The new ones denoted by $\boldsymbol{M}_\psi = (M_\psi^x, M_\psi^y, M_\psi^z)$ and $\boldsymbol{M}_s = (M_s^x, T_s^y, T_s^z)$ will be called[3] the *bimoment* vector and the *secondary internal force* vector, respectively.

The external virtual work is $W_e = \int_{S_0} \boldsymbol{H}_0 \cdot \hat{\boldsymbol{\xi}}\,\mathrm{d}S + \int_{S_L} \boldsymbol{H}_L \cdot \hat{\boldsymbol{\xi}}\,\mathrm{d}S$. Using (Eq. (13)), W_e takes the form:

$$W_e = \boldsymbol{P}_0 \cdot \hat{\boldsymbol{v}}_0 + \boldsymbol{C}_0 \cdot \hat{\boldsymbol{\theta}}_0 + \boldsymbol{Q}_0 \cdot \hat{\boldsymbol{\eta}}_0 + \boldsymbol{P}_L \cdot \hat{\boldsymbol{v}}_L + \boldsymbol{C}_L \cdot \hat{\boldsymbol{\theta}}_L + \boldsymbol{Q}_L \cdot \hat{\boldsymbol{\eta}}_L \tag{17}$$

where ($\boldsymbol{P}_I = \langle \boldsymbol{H}_I\rangle$; $\boldsymbol{C}_I = (\boldsymbol{H}_I)$; $\boldsymbol{Q}_I = \langle H_I^x\psi^x\rangle\boldsymbol{x} + \langle H_I^x\psi^y\rangle\boldsymbol{y} + \langle H_I^x\psi^z\rangle\boldsymbol{z}$) define, for the 1D theory, the external actions associated to the external surface force density $\boldsymbol{H}_I$, with $I \in \{0, L\}$.

[3] It is usual, in non-uniform torsional warping theories, to call M_ψ^x the *bimoment*, and M_s^x the *secondary torsional moment*. By analogy, we introduce the *bimoment vector* $\boldsymbol{M}_\psi$ and the *secondary internal force vector* $\boldsymbol{M}_s$. The components of $\boldsymbol{M}_s$ are denoted by (M_s^x, T_y^s, T_s^z) because M_s^x which is homogeneous to a moment, and the (T_s^y, T_s^z) are homogeneous to forces.

Thanks to the principle of virtual work, Eqs. (15)–(17) provide the equilibrium equations and the boundary conditions:

$$\left.\begin{aligned} &\boldsymbol{R}' = \boldsymbol{0}\\ &\boldsymbol{M}' + \boldsymbol{x}\wedge\boldsymbol{R} = \boldsymbol{0}\\ &\boldsymbol{M}'_\psi - \boldsymbol{M}_s = \boldsymbol{0}\end{aligned}\right\},\qquad \left.\begin{aligned} &x=0:\quad \{\boldsymbol{R},\boldsymbol{M},\boldsymbol{M}_\psi\}_0 = -\{\boldsymbol{P}_0,\boldsymbol{C}_0,\boldsymbol{Q}_0\}\\ &x=L:\quad \{\boldsymbol{R},\boldsymbol{M},\boldsymbol{M}_\psi\}_L = \{\boldsymbol{P}_L,\boldsymbol{C}_L,\boldsymbol{Q}_L\}\end{aligned}\right\} \tag{18}$$

Let $\boldsymbol{D} = (\boldsymbol{D}^\sigma, \boldsymbol{D}^\tau)$ denote the generalized strain vector and $\boldsymbol{T} = (\boldsymbol{T}^\sigma, \boldsymbol{T}^\tau)$ the corresponding generalized force vector defined by:

$$\left.\begin{aligned} &\boldsymbol{D}^\sigma = (\gamma_x, \chi_y, \chi_z, \eta'_x, \eta'_y, \eta'_z), &&\boldsymbol{D}^\tau = (\chi_x, \eta_x, \gamma_y, \eta_y, \gamma_z, \eta_z)\\ &\boldsymbol{T}^\sigma = (N, M^y, M^z, M^x_\psi, M^y_\psi, M^z_\psi), &&\boldsymbol{T}^\tau = (M^x, M^x_s, T^y, T^y_s, T^z, T^z_s)\end{aligned}\right\} \tag{19}$$

The 1D elastic constitutive relation can be written $\boldsymbol{T} = \boldsymbol{\Gamma}\boldsymbol{D}$ where $\boldsymbol{\Gamma}$ defines the structural rigidity operator. Using the matrix notation, the elastic strain energy for the 1D model of the beam is given by $W^{1D}_{el}(\boldsymbol{D},\boldsymbol{D}) = \frac{1}{2}\int_L [\boldsymbol{D}]^t[\boldsymbol{\Gamma}][\boldsymbol{D}]\,\mathrm{d}x$. Besides, for the 3D problem, $\boldsymbol{\varepsilon}$ denoting the strain tensor associated to the displacement $\boldsymbol{\xi}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{\eta})$ (Eq. (13)), and using Hooke's law, the beam elastic strain energy can be written as

$$W^{3D}_{el}(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}) = \frac{1}{2}\int_L \big\langle E\varepsilon^2_{xx} + 4G(\varepsilon^2_{xy} + \varepsilon^2_{xz})\big\rangle\,\mathrm{d}x \tag{20}$$

Identifying the strain energies W^{3D}_{el} and W^{1D}_{el} allows to derive the rigidity operator $\boldsymbol{\Gamma}$. This identification shows that $\boldsymbol{\Gamma}$ can be written $\boldsymbol{\Gamma} = \big[\begin{smallmatrix}\boldsymbol{\Gamma}^\sigma & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{\Gamma}^\tau\end{smallmatrix}\big]$ and leads to the uncoupled relations $\boldsymbol{T}^\sigma = \boldsymbol{\Gamma}^\sigma\boldsymbol{D}^\sigma$ and $\boldsymbol{T}^\tau = \boldsymbol{\Gamma}^\tau\boldsymbol{D}^\tau$. The rigidity operators $\boldsymbol{\Gamma}^\sigma$ and $\boldsymbol{\Gamma}^\tau$ are associated to the axial stress and the shear stresses, respectively. Using the properties Eqs. (8)–(11) for the SV-shears and the SV-warping functions, $\boldsymbol{\Gamma}^\sigma$ and $\boldsymbol{\Gamma}^\tau$ reduce to:

$$\boldsymbol{\Gamma}^\sigma = E\begin{bmatrix} A & 0 & 0 & 0 & 0 & 0\\ & I_y & 0 & 0 & 0 & 0\\ & & I_z & 0 & 0 & 0\\ & & & I^{xx}_\psi & I^{xy}_\psi & I^{xz}_\psi\\ & & & & I^{yy}_\psi & I^{yz}_\psi\\ & & & & & I^{zz}_\psi\end{bmatrix}_{\mathrm{sym}}$$

$$\boldsymbol{\Gamma}^\tau = G\begin{bmatrix} I_x & J-I_x & z_cA & -z_cA & -y_cA & y_cA\\ & I_x-J & -z_cA & z_cA & y_cA & -y_cA\\ & & A & A_y-A & 0 & 0\\ & & & A-A_y & 0 & 0\\ & & & & A & A_z-A\\ & & & & & A-A_z\end{bmatrix}_{\mathrm{sym}} \tag{21}$$

where the cross-sectional constants $I^{ij}_\psi = \langle\psi^i\psi^j\rangle$ define the *warping matrix* noted $\boldsymbol{I}_\psi$.

The expressions of the strain (Eq. (14)) and the inverse of the constitutive relations (Eq. (21)) allow one to express the normal and shear stresses with respect to the generalized stresses. Using the property Eq. (11), one can obtain the following explicit form

$$\begin{aligned}\sigma^{nuw}_{xx} = {}&\overbrace{\frac{N}{A} + \frac{M^y}{I_y}z - \frac{M^z}{I_z}y}^{\sigma^{sv}_{xx}}\\ &+ \frac{M^x_\psi}{\kappa_\sigma}\big[(I^{yy}_\psi I^{zz}_\psi - I^{yz2}_\psi)\psi^x + (-I^{xy}_\psi I^{zz}_\psi + I^{xz}_\psi I^{yz}_\psi)\psi^y + (I^{xy}_\psi I^{yz}_\psi - I^{xz}_\psi I^{yy}_\psi)\psi^z\big]\\ &+ \frac{M^y_\psi}{\kappa_\sigma}\big[(-I^{xy}_\psi I^{zz}_\psi + I^{xz}_\psi I^{yz}_\psi)\psi^x + (I^{xx}_\psi I^{zz}_\psi - I^{xz2}_\psi)\psi^y + (-I^{xx}_\psi I^{yz}_\psi + I^{xy}_\psi I^{xz}_\psi)\psi^z\big]\end{aligned}$$

$$+\frac{M_\psi^z}{\kappa_\sigma}\big[(I_\psi^{xy}I_\psi^{yz}-I_\psi^{xz}I_\psi^{yy})\psi^x+(-I_\psi^{xx}I_\psi^{yz}+I_\psi^{xy}I_\psi^{xz})\psi^y+(I_\psi^{xx}I_\psi^{yy}-I_\psi^{xy2})\psi^z\big]$$

with

$$\kappa_\sigma=I_\psi^{xx}I_\psi^{yy}I_\psi^{zz}-I_\psi^{xx}I_\psi^{yz2}-I_\psi^{xy2}I_\psi^{zz}+2I_\psi^{xy}I_\psi^{xz}I_\psi^{yz}-I_\psi^{xz2}I_\psi^{yy} \tag{22}$$

$$\begin{aligned}
\boldsymbol{\tau}^{nuw}={}&\overbrace{M^x\boldsymbol{\tau}^x+T^y\boldsymbol{\tau}^y+T^z\boldsymbol{\tau}^z}^{\boldsymbol{\tau}^{\mathrm{sv}}}\\
&+\frac{M_s^x}{\kappa_\tau}\big[(A-A_y)(A-A_z)(I_x\boldsymbol{\tau}^x-\boldsymbol{x}\wedge\overline{\boldsymbol{X}})-A^2\big(y_c^2(A-A_y)+z_c^2(A-A_z)\big)\boldsymbol{\tau}^x\\
&-z_cA(A-A_z)(A_y\boldsymbol{\tau}^y-\boldsymbol{y})+y_cA(A-A_y)(A_z\boldsymbol{\tau}^z-\boldsymbol{z})\big]\\
&+\frac{T_s^y}{\kappa_\tau}\big[-z_cA(A-A_z)(J\boldsymbol{\tau}^x-\boldsymbol{x}\wedge\overline{\boldsymbol{X}})\\
&+\big((I_x-J)(A-A_z)-y_c^2A^2\big)(A\boldsymbol{\tau}^y-\boldsymbol{y})-z_c^2A^2(A-A_z)\boldsymbol{\tau}^y-y_cz_cA^2(A_z\boldsymbol{\tau}^z-\boldsymbol{z})\big]\\
&+\frac{T_s^y}{\kappa_\tau}\big[y_cA(A-A_y)(J\boldsymbol{\tau}^x-\boldsymbol{x}\wedge\overline{\boldsymbol{X}})\\
&-y_cz_cA^2(A_y\boldsymbol{\tau}^y-\boldsymbol{y})+\big((I_x-J)(A-A_y)-z_c^2A^2\big)(A\boldsymbol{\tau}^z-\boldsymbol{z})-y_c^2A^2(A-A_y)\boldsymbol{\tau}^z\big]
\end{aligned}$$

with

$$\kappa_\tau=(I_x-J)(A-A_y)(A-A_z)-A^2\big(y_c^2(A-A_y)+z_c^2(A-A_z)\big) \tag{23}$$

This result makes clear the additional contribution of the new internal forces $\boldsymbol{M}_\psi$ and $\boldsymbol{M}_s$ induced by the non-uniformity of warping.

4. Comments

- Due to warping, torsional and bending effects are coupled in the present NUW-BT, even if the torsional moment refers to the shear center C whereas the bending moments refer to the centroid G. For an arbitrary-CS this coupling effect is related to the three coupling components $(I_\psi^{xy}, I_\psi^{xz}, I_\psi^{yz})$ of the warping matrix and to the coordinates (y_c, z_c) of C. This coupling appears clearly in the constitutive relations (Eq. (21)) and in the expressions of the stresses Eqs. (22), (23). For bi-symmetrical-CS, thanks to symmetry properties (Eq. (12)), $y_c=z_c=I_\psi^{xy}=I_\psi^{xz}=I_\psi^{yz}=0$ and the torsional-flexural coupling vanishes; in that case, the stresses reduce to:

$$\sigma_{xx}^{nuw}=\sigma_{xx}^{\mathrm{sv}}+\frac{M_\psi^x}{I_\psi^{xx}}\psi^x+\frac{M_\psi^y}{I_\psi^{yy}}\psi^y+\frac{M_\psi^z}{I_\psi^{zz}}\psi^z \tag{24}$$

$$\boldsymbol{\tau}^{nuw}=\boldsymbol{\tau}^{\mathrm{sv}}+\frac{M_s^x}{(I_x-J)}(I_x\boldsymbol{\tau}^x-\boldsymbol{x}\wedge\boldsymbol{X})+\frac{T_s^y}{(A-A_y)}(A\boldsymbol{\tau}^y-\boldsymbol{y})+\frac{T_s^z}{(A-A_z)}(A\boldsymbol{\tau}^z-\boldsymbol{z}) \tag{25}$$

- Equilibrium equations (18), constitutive relation $\boldsymbol{T}=\boldsymbol{\Gamma}\boldsymbol{D}$ (Eq. (21)) and boundary conditions on S_0 and S_L form the 1D problem that defines the NUW-BT. For a cantilever beam submitted to shear-bending or torsion, the warping is restrained ($\boldsymbol{\eta}=\mathbf{0}$) for the built-in section, and free ($\boldsymbol{M}_\psi=\mathbf{0}$) in the other extremity; the warping is then non-uniform along the span.
- The lateral surface S_{lat} of the beam is free of loading. In Eq. (23), the contribution of $\boldsymbol{M}_s$ to the shear is expressed with the SV-shears $(\boldsymbol{\tau}^x, \boldsymbol{\tau}^y, \boldsymbol{\tau}^z)$ and with the supplementary terms $(\boldsymbol{x}\wedge\overline{\boldsymbol{X}})$, $(\boldsymbol{y})$ and $(\boldsymbol{z})$. The SV-shears naturally vanish at the free edge of the section but the supplementary terms violate the 'no shear' boundary conditions at the edge. Thus, for this theory founded on the displacement model (13), the result on the shear distribution over the section is not quite satisfying.[4]

[4] To better evaluate the shear, an alternative consists of considering the equilibrium of an elementary slice $(\mathrm{d}x)$ of the beam and of calculating the shear that equilibrates the variation of the normal stresses σ_{xx}^{nuw} given by Eq. (22).

	TP-SV	Vlasov	Kim-BT	NUW-BT
$\theta_x(L)$	43.330	4.119	4.236	4.203
$v_z(L)$	–	–	2.163	2.369

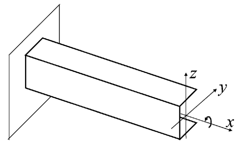

Fig. 2. Torsion of a cantilever beam with a channel-CS (aspect ratio 5): coupling effect.

Table 1
Torsion. Axial and shear stresses

	σ_{xx}^{3D}	σ_{xx}^{1D}	σ_{xx}^{sv}
CS_1	29.104	20.690	0
CS_2	1499.8	1234.0	0
	τ^{3D}	τ^{1D}	τ^{sv}
CS_1	11.168	9.600	16.28
CS_2	50.735	32.873	620.64

Table 2
Shear-bending of a short beam (aspect ratio 2.5). Comparison of the axial stresses due to warping and flexure

3D-FEM	σ_{xx}^{3D}	σ_{xx}^{sv}	$(\sigma_{xx}^{3D})^w$	$\frac{(\sigma_{xx}^{3D})^w}{\sigma_{xx}^{sv}}\%$
CS_1	34.306	30.000	4.306	14.35
CS_2	122.720	87.234	35.486	40.68
NUW-BT	σ_{xx}^{1D}	σ_{xx}^{sv}	$(\sigma_{xx}^{1D})^w$	$\frac{(\sigma_{xx}^{1D})^w}{\sigma_{xx}^{sv}}\%$
CS_1	33.346	30.000	3.346	11.15
CS_2	135.909	87.234	48.675	55.77

	$\left\lvert\frac{\sigma_{xx}^{1D}-\sigma_{xx}^{3D}}{\sigma_{xx}^{3D}}\right\rvert\%$
CS_1	2.80
CS_2	10.75

$(\cdot)^{3D}$, $(\cdot)^{1D}$ and $(\cdot)^{sv}$ denote quantities related to 3D-FEM, NUW-BT, and SV-BT, respectively.

– It should be noted that for the application of the NUW-BT, one needs to previously know, for any given CS, all its constants (A, A_y, A_z, I_y, I_z, J, y_c, z_c) and in particular its SV-warping-functions (ψ^x, ψ^y, ψ^z) and shears ($\boldsymbol{\tau}^x$, $\boldsymbol{\tau}^y$, $\boldsymbol{\tau}^z$). This can be achieved by using one of the numerical methods proposed by [8,2,7] for the computation of the 3D SV-solution. In such conditions, it is worthwhile to note that we have just to compute the six scalars of the warping matrix $\boldsymbol{I}_\psi$, and for the stresses, the closed form results (Eqs. (22), (23)) can be directly used without any additional computation.

This Note has been limited to the key points of the theory; however, more details can be found in [9, Part-I]. Further, in [9, Part-II], this theory is used to analyze, for a representative set of cross-sections (CS) (solid-CS and thin-walled open/closed-CS, bi-symmetric or not), the elastic behavior of cantilever beams subjected to torsion or shear-bending; numerical results are given for the 1D-structural behavior and also for the 3D-stress distributions close to the built-in section: the stress predictions of the NUW-BT are compared to those obtained by three-dimensional finite elements computations 3D-FEM. However, for the present Note, some significant results due to warping effects may be given:

– the first result is given to illustrate the flexural-torsional coupling that occurs for non-symmetrical-CS. Fig. 2 concerns the channel-CS studied by [4]: for $\theta_x(L)$, the results are similar for the three theories (Vlasov-BT, Kim-BT,[5] NUW-BT) and its magnitude is 90% lower than for uniform theory (SV-BT). The transversal displacement due to the flexural-torsional coupling computed by the present theory and that of Kim are very close;
– the second result concerns the stresses in the built-in section of cantilever beams subjected to torsion or shear-bending and made of two kinds of CS: a solid-rectangular-CS (CS_1) and an open thin-walled-I-CS (CS_2). For the torsion (Table 1), the results indicate that the axial stresses (due to the restrained warping) may be much larger than the shears. For the shear bending of a short beam (Table 2), the results indicate that the axial stresses σ_{xx}^w due to (the restrained warping) may reach 50% of the axial stresses σ_{xx}^{sv} due to flexure.

References

[1] P. Ladevèze, J. Simmonds, New concepts for linear beam theory with arbitrary geometry and loading, Eur. J. Mech. 17 (3) (1998) 377–402.
[2] R. El Fatmi, H. Zenzri, On the structural behavior and the Saint Venant solution in the exact beam theory. Application to laminated composite beams, Comput. Struct. 80 (2002) 1441–1456.

[5] Kim-BT, written for open/closed thin-walled cross-section, is built on a mixed approach using the Hellinger–Reissner principle. This approach considers a kinematic model similar to that of NUW-BT, but where only the torsional warping function is considered, and introduces (thin-walled) Vlasov assumptions for the shear.

[3] V. Vlasov, Thin Walled Elastic Beams, Israel Program for Scientific Transactions, 1961.
[4] N. Kim, M. Kim, Exact dynamic/static stiffness matrices of non symmetric thin-walled beams considering coupled shear deformation effects, Thin-Walled Structures 43 (2005) 701–734.
[5] C.M. Wang, J.N. Reddy, K.H. Lee, Shear Deformable Beams and Plates, Elsevier, New York, 2000.
[6] L. Dufort, S. Drapier, M. Grédiac, Closed-form solution for the cross-section warping in short beams under three-point bending, Composite Structures 52 (2001) 233–246.
[7] R. El Fatmi, H. Zenzri, A numerical method for the exact elastic beam theory. Applications to homogeneous and composite beams, Int. J. Solids Struct. 41 (2004) 2521–2537.
[8] P. Ladevèze, P. Sanchez, J.G. Simmonds, On application of the exact theory of elastic beams, in: D. Durban, D. Givoli, J.G. Simmonds (Eds.), Advances in the Mechanics of Plates and Shells, Kluwer Academic Publishers, 2001.
[9] R. El Fatmi, Non-uniform warping including the effects of torsion and shear forces. Part-I: A general beam theory and Part-II: Analytical and numerical applications, Int. J. Solids Struct., submitted for publication.