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C. R. Mecanique 335 (2007) 757–762
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Asymptotics of eigenfrequencies of an elastic body with a heavy and hard peak-shaped inclusion ☆

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Received 4 June 2007; accepted 12 June 2007
Available online 19 November 2007
Presented by Évariste Sanchez-Palencia

Abstract

A heavy and hard peak-shaped inclusion in an elastic body provokes to concentration of eigenvalues in the low-frequency range of the spectrum and localization of the corresponding eigenmodes near the peak tip. ***To cite this article: S.A. Nazarov, C. R. Mecanique 335 (2007).***

Résumé

Développement asymptotique des fréquences propres d'un corps élastique avec une inclusion lourde et de forme très pointue. Une inclusion lourde et dure de forme pointue dans un corps élastique provoque la concentration des valeurs propres dans le domaine basses fréquences du spectre et la localisation des modes propres correspondants près de l'extrémité du pic. ***Pour citer cet article : S.A. Nazarov, C. R. Mecanique 335 (2007).***

Keywords: Computational solid mechanics; Elasticity system; Heavy and hard peak-shaped inclusion; Localization of eigenvectors

Mots-clés : Mécanique des solides numérique ; Système de l'elasticité ; Inclusion lourde et de forme très pointue ; Localisation des vecteurs propres

1. Eigenoscillations of contrast composite solids

Let Ω^0 and $\Omega^1 = \Omega \setminus \overline{\Omega^0}$ be two-dimensional anisotropic heterogeneous solids while the exterior boundary $\Gamma = \partial\Omega$ is smooth and free of traction, and the contact contour $\Gamma^0 = \partial\Omega^0$ is smooth everywhere except at the point $\mathcal{O}$. In the vicinity of $\mathcal{O}$ the inclusion Ω^0 is determined by the relations

$$x_1 > 0, \quad -b_- x_1^{1+\gamma} < x_2 < b_+ x_1^{1+\gamma} \tag{1}$$

where $x = (x_1, x_2)$ are dimensionless Cartesian coordinates centered at $\mathcal{O}$ while $b = b_+ + b_-$ and γ are positive constants. Because of (1), the inclusion is peak-shaped (see Fig. 1).

☆ The author gratefully acknowledges the support by N.W.O., the Netherlands Organization for Scientific Research.
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1631-0721/$ – see front matter
doi:10.1016/j.crme.2007.06.004

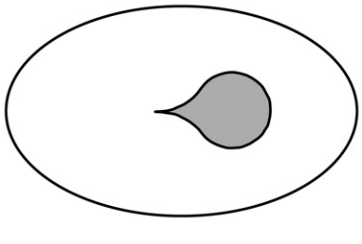

Fig. 1. The peak-shaped inclusion.

The eigenoscillations of the composite body Ω are described by the problem

$$D(-\nabla_x)^\top A^i(x)D(\nabla_x)u^i(x) = \Lambda\rho^i(x)u^i(x), \quad x \in \Omega^i,\ i = 0, 1 \tag{2}$$

$$u^0(x) = u^1(x), \quad D\big(n(x)\big)^\top A^0(x)D(\nabla_x)u^0(x) = D\big(n(x)\big)^\top A^1(x)D(\nabla_x)u^1(x), \quad x \in \Gamma^0 \tag{3}$$

$$D\big(n(x)\big)^\top A^1(x)D(\nabla_x)u^1(x) = 0, \quad x \in \Gamma \tag{4}$$

Here we employ a matrix notation in the elasticity theory, i.e., $u = (u_1, u_2)^\top$ is the displacement column, $\top$ stands for transposition, u^i for the restriction of the field u on Ω^i, and

$$\varepsilon(u) = (\varepsilon_{11}, \sqrt{2}\varepsilon_{12}, \varepsilon_{22})^\top = D(\nabla_x)u, \quad D(\nabla_x)^\top = \begin{pmatrix} \partial_1 & 2^{-1/2}\partial_2 & 0 \\ 0 & 2^{-1/2}\partial_1 & \partial_2 \end{pmatrix}, \quad \partial_j = \frac{\partial}{\partial x_j}, \quad \nabla_x = \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix}$$

while ε_{jk} are Cartesian components of the strain tensor. The strain column $\varepsilon(u)$ and the analogous stress column $\sigma^i(u)$ are related by the Hooke law $\sigma^i(u) = A^i\varepsilon(u)$ where A^i is 3×3-matrix of elastic moduli, symmetric, positive definite, and smooth in a neighborhood $\mathcal{V}^i$ of the set $\overline{\Omega^i} = \Omega^i \cup \partial\Omega^i$. Furthermore Λ denotes an eigenvalue (square of an eigenfrequency), ρ^i the material density which is positive and smooth is $\mathcal{V}^i$, and $n = (n_1, n_2)^\top$ the outward normal on the contours Γ and Γ^0. We assume that

$$A^1(x) = \tau A^{1\bullet}(x), \quad \rho^1(x) = \tau^{1+\beta}\rho^{1\bullet}(x) \tag{5}$$

where $\tau > 0$ is a small parameter, $\beta \geqslant 0$, and the characteristics $A^{1\bullet}$ and $\rho^{1\bullet}$ are of the same order as A^0 and ρ^0, respectively, i.e., the inclusion Ω^0 is much more hard and heavy than the body Ω^1.

Problem (2)–(4) is among the so-called *stiffness* problems which have been under consideration in many publications (cf. [1–4]). In particular, it was proved in [3,4] that, for $\beta > 0$ and a *smooth* contour Γ^0 (the Lipschitz property of Γ^0 is sufficient), the eigenvalues of problem (2)–(4) listed in the sequence

$$0 = \Lambda_1^\tau = \Lambda_2^\tau = \Lambda_3^\tau < \Lambda_4^\tau \leqslant \Lambda_5^\tau \leqslant \cdots \leqslant \Lambda_k^\tau \leqslant \cdots \to +\infty \tag{6}$$

according to their multiplicities, converge as $\tau \to +0$ to the corresponding eigenvalues

$$0 = \Lambda_1^0 = \Lambda_2^0 = \Lambda_3^0 < \Lambda_4^0 \leqslant \Lambda_5^0 \leqslant \cdots \leqslant \Lambda_k^0 \leqslant \cdots \to +\infty \tag{7}$$

of the spectral problem on the isolated inclusion

$$\begin{aligned} &D(-\nabla_x)^\top A^0(x)D(\nabla_x)v^0(x) = \Lambda^0\rho^0(x)v^0(x), \quad x \in \Omega^0 \\ &D\big(n(x)\big)^\top A^0(x)D(\nabla_x)v^0(x) = 0, \quad x \in \Gamma^0. \end{aligned} \tag{8}$$

Moreover, papers [3,4] present certain information on the eigenvectors of problem (2)–(4) and the asymptotic structure of the spectrum in the middle-frequency range together with estimates of asymptotic remainders where majoring constants are independent of the eigenvalue number k, i.e., an explicit dependence of the bounds on attributes of the limit spectrum (7) is established.

For a sharp peak (1) with $\gamma \geqslant 1$, the boundary Γ^0 is not Lipschitz and the energy space $E(\Omega^0)$ equipped with the norm $(\|D(\nabla_x)v; L^2(\Omega^0)\|^2 + \|v; L^2(\Omega^0 \setminus \mathbb{B}_d)\|^2)^{1/2}$, where $\mathbb{B}_d = \{x\colon\ |x| < d\}$ and the radius $d > 0$ is small, is not compactly embedded into the Lebesgue space $L^2(\Omega^0)$ (cf. [5, § 3.1] and [6]) and, therefore, the spectrum of problem (8) cannot be discrete so that even the convergence $\Lambda_k^\tau \to \Lambda_k^0$ cannot be construed. Note that the inclusion $\mathcal{H} \subset H^1(\Omega^0)$ is wrong for any $\gamma > 0$ (see [5, § 3.1]).

The aim of this Note is to prove another type of convergence

$$\tau^{-2\beta_-}\Lambda_k^\tau \to \lambda_{k-3}, \quad k=4,5,\dots, \quad \beta_\pm = 3^{-1}(1\pm\gamma^{-1})>0 \tag{9}$$

in the case $\gamma>1$ and to indicate the resultant spectral problem with the eigenvalues

$$0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_j\leqslant\cdots\to+\infty. \tag{10}$$

By (9), the eigenvalue Λ_k^τ of problem (2)–(4) with the sharp ($\gamma>1$) peak (1) vanishes as $\tau\to+0$.

2. Asymptotic ansatz

Introducing the stretched coordinates

$$\xi=(\xi_1,\xi_2)=\tau^{-\beta_0}x, \quad \beta_0=(3\gamma)^{-1} \tag{11}$$

turns inequalities (1) into the following ones:

$$\eta:=\xi_1>0, \quad -b_-\xi_1^{1+\gamma}<\tau^{-1/3}\xi_2<b_+\xi_1^{1+\gamma}. \tag{12}$$

Besides, the peak-shaped inclusion becomes thin so that it is characterized by the new small parameter $h=\tau^{1/3}$. Defining the ultra-rapid variable

$$\zeta=h^{-1}\xi_2=\tau^{-\beta_+}x_2\in\Upsilon(\eta)=\left(-b_-\eta^{1+\gamma}, b_+\eta^{1+\gamma}\right) \tag{13}$$

we employ the standard asymptotic ansatz (cf. [7], [5, Ch. 7]) for elasticity problems in thin domains

$$\Lambda^\tau=\tau^{-2\beta_0}h^2\lambda+\cdots=\tau^{2\beta_-}\lambda+\cdots \tag{14}$$

$$u^0(x)=w_2(\eta)e_2+hU^1(\eta,\zeta)+h^2U^2(\eta,\zeta)+h^3U^3(\eta,\zeta)+h^4U^4(\eta,\zeta)+\cdots \tag{15}$$

where $e_j=(\delta_{j,1},\delta_{j,2})^\top$, $\delta_{j,k}$ denotes the Kronecker symbol and other entries of the ansatz are to be found. The asymptotic ansatz in the framing body Ω^1

$$u^1(x)=v(\xi)+\cdots \tag{16}$$

is adjusted with (15). We emphasize that the limit passage $h\to+0$ shrinks set (12) into the semi-infinite slit $\Sigma=\{\xi\colon\ \xi_1>0,\ \xi_2=0\}$ and, thus, the vector function v in (16) is defined on the set $\mathbb{R}\setminus\overline{\Sigma}$.

Taking formulas (14), (16) and (5), (11) into account, we derive from (2) and (3) the relations

$$D(\nabla_x)^\top A^{1\bullet}(\mathcal{O})D(\nabla_\xi)v(\xi)=0, \quad \xi\in\mathbb{R}^2\setminus\Sigma, \quad v_1(\xi_1,\pm0)=0, \quad \xi_1>0 \tag{17}$$

and the transmission condition

$$v_2(\xi_1,\pm0)=w_2(\xi_1), \quad \xi_1>0 \tag{18}$$

To complete the problem for v, we need to detect the second transmission condition on Σ for the vector function v. To this end, we analyze ansatz (15). According to (11)–(13) the change of variables $x\mapsto(\eta,\zeta)$ requires freezing of coefficients at the point $\mathcal{O}$ and provides the following decompositions:

$$\begin{gathered}L^0=D(-\nabla_x)^\top A^0(\mathcal{O})D(\nabla_x)=\tau^{-2\beta_0}h^{-2}(L^{00}+hL^{01}+h^2L^{02})\\ \mathbf{n}_\pm^{1/2}N^{0\pm}=\mathbf{n}_\pm^{1/2}D(n^\pm)^\top A^0(\mathcal{O})D(\nabla_x)=\tau^{-\beta_0}h^{-1}(N^{00\pm}+hN^{01\pm}+h^2N^{02\pm})\end{gathered} \tag{19}$$

where $n^\pm(x_1)=\mathbf{n}_\pm(x_1)^{-1/2}(\pm1, b_\pm(1+\gamma)x_1^\gamma)$ are unit normal vectors on the arcs $\Gamma^\pm$ forming the peak (1), and $\mathbf{n}_\pm(x_1)=1+b_\pm(1+\gamma)^2x_1^{2\gamma}$. Using the notation $A^0_{(jk)}=D(e_j)^\top A^0(\mathcal{O})D(e_k)$, we get

$$L^{00}(\partial_\zeta)=-A^0_{(22)}\partial_\zeta^2, \quad L^{01}(\partial_\eta,\partial_\zeta)=-\left(A^0_{(21)}+A^0_{(12)}\right)\partial_\eta,\partial_\zeta, \quad L^{02}(\partial_\eta)=-A^0_{(11)}\partial_\eta^2$$

$$N^{00\pm}(\partial_\zeta)=\pm A^0_{(22)}\partial_\zeta, \quad N^{01\pm}(\eta,\partial_\eta,\partial_\zeta)=\pm A^0_{(21)}\partial_\eta+b_\pm(1+\gamma)\eta^\gamma A^0_{(12)}\partial_\zeta$$

$$N^{02\pm}(\eta,\partial_\eta)=b_\pm(1+\gamma)\eta^\gamma A^0_{(11)}\partial_\eta$$

Furthermore, the right-hand side of the second transmission condition (3) on $\Gamma^\pm$ takes the form

$$\tau^{1-\beta_0}D(\pm e_2)^\top A^{1\bullet}(\mathcal{O})D(\nabla_\xi)v(\xi_1,\pm0)+\cdots=:\tau^{1-\beta_0}G^{v\pm}(\xi_1)+\cdots \tag{20}$$

Inserting formulas (14), (15) and (19) into (2), (3) and gathering coefficients on similar powers of τ, we arrive at the following recursive sequence of problems on the segment $\Upsilon(\eta)$ with the parameter $\eta > 0$:

$$\begin{aligned} &L^{00}U^q = F^q := -L^{01}U^{q-1} - L^{02}U^{q-2} + \delta_{q,4}\lambda\rho(\mathcal{O})U^0 \quad \text{on } \Upsilon(\eta) \\ &N^{00\pm}U^q = G^{q\pm} := -N^{01\pm}U^{q-1} - N^{02\pm}U^{q-2} + \delta_{q,4}G^{v\pm} \quad \text{at } \zeta = \pm b_\pm \eta^{1+\gamma} \end{aligned} \tag{21}$$

where $U^0 = e_2 w_2$ and $U^q = 0$ for $q < 0$. The dilation factor in (11) and the exponent β_- in (14) were chosen in such a way that the terms $\tau^{-2\beta_-}\Lambda\rho(\mathcal{O})u^0 = \lambda\rho(\mathcal{O})U^0 + \cdots$ in (2) and $G^{v\pm}$ in (20) come to the problem (21) with the index $q = 4$. The problems (21) with $q = 1$ and $q = 2$ get the solutions

$$U^1(\eta,\zeta) = e_1\big(w_1(\eta) - \zeta w_2(\eta)\big), \quad U^2(\eta,\zeta) = -\big(A^0_{(22)}\big)^{-1}A^0_{(21)}\big(\zeta\partial_\eta w_1(\eta) - 2^{-1}\zeta^2\partial^2\eta w_2(\eta)\big) \tag{22}$$

The compatibility condition

$$\int_{\Upsilon(\eta)} F^q_p(\eta,\zeta)\,\mathrm{d}\zeta + G^{q+}_p(\eta) + G^{q-}_p(\eta) = 0 \tag{23}$$

with $q = 3$, $p = 2$ is satisfied while the one with $q = 3$, $p = 1$ provides the relation

$$\partial_\eta w_1(\eta) = 2^{-1}(b_+ - b_-)\eta^{1+\gamma}\partial^2_\eta w_2(\eta) \tag{24}$$

In view of (22), (24) and (20), the compatibility condition (23) with $q = 4$, $p = 2$ reads

$$\begin{aligned} &e_2^\top D(-e_2)^\top A^{1\bullet}(\mathcal{O})\big(D(\nabla_\xi)v(\xi_1,+0) - D(\nabla_\xi)v(\xi_1,-0)\big) \\ &\quad = \lambda\rho^0(\mathcal{O})b\xi_1^{1+\gamma}w_2(\xi_1) - \frac{a}{12}b^3\frac{\partial^2}{\partial\xi_1^2}\xi_1^{3(1+\gamma)}\frac{\partial^2 w_2}{\partial\xi_1^2}(\xi_1), \quad \xi_1 > 0 \end{aligned} \tag{25}$$

and delivers the necessary transmission condition in the problem for v. The coefficient

$$a = e_1^\top D(e_1)^\top\big(A^0(\mathcal{O}) - A^0(\mathcal{O})D(e_2)\big(A^0_{(22)}\big)^{-1}D(e_2)^\top A^0(\mathcal{O})\big)D(e_1)e_1$$

in (25) is known to be positive (see, e.g., [8], [5, Ch. 1.4]) and $a = 4\mu(1-2\nu)^{-1}$ for an isotropic material with the Poisson ratio $\nu < 1/2$ and the shear modulus $2\mu > 0$.

3. The resultant spectral problem

Let $\mathfrak{C}$ denote a linear space of smooth (up to the boundary) vector functions on $\mathbb{R}^2 \setminus \overline{\Sigma}$ which satisfy the stable conditions (17), (18) and have compact supports in $\mathbb{R}^2 \setminus 0$. With any test function $V \in \mathfrak{C}$, we derive from (17), (18), (25) the integral identity

$$\big(A^{10}(\mathcal{O})D(\nabla_\xi)v, D(\nabla_\xi)V\big)_{\mathbb{R}^2} + \frac{a}{12}b^3\left(\xi_1^{3(1+\gamma)}\frac{\partial^2 v_2}{\partial\xi_1^2}, \frac{\partial^2 V_2}{\partial\xi_1^2}\right)_\Sigma = \lambda\rho^0(\mathcal{O})b(\xi_1^{1+\gamma}v_2, V_2)_\Sigma \tag{26}$$

where $(\,,\,)_\Xi$ stands for the inner product in the Lebesgue space $L^2(\Xi)$.

Let $\mathfrak{H}$ denote the completion of $\mathfrak{C}$ with respect to the norm generated by the scalar product $\langle v, V\rangle$ on the left of (26). By Korn's and Hardy's inequalities, any vector function $v \in \mathfrak{C}$ meets the estimates

$$\begin{aligned} &\big\|\xi_1^{(3\gamma-1)/2}v_2; L_2(\Sigma)\big\| \leqslant c\big\|\xi_1^{3(1+\gamma)/2}\partial^2 v_2/\partial\xi_1^2; L_2(\Sigma)\big\| \\ &\big\|(1+\xi_1)^{-1/2}v_2; L_2(\Sigma)\big\| \leqslant c\big(\big\|(1+\xi)^{-1}v; L_2(\mathbb{R}^2)\big\| + \big\|\nabla_\xi v; L_2(\mathbb{R}^2)\big\|\big) \\ &\qquad \leqslant c\big(\big\|D(\nabla_\xi)v; L_2(\mathbb{R}^2)\big\| + \big\|\xi_1^{(3\gamma-1)/2}v_2; L_2(\Sigma)\big\|\big) \end{aligned}$$

Since $3\gamma - 1 > 1 + \gamma$ due to the condition $\gamma > 1$, the operator $\mathfrak{K}$ in $\mathfrak{H}$ determined by $\langle\mathfrak{K}v, V\rangle = \rho^0(\mathcal{O})b(\xi_1^{1+\gamma}v_2, V_2)_\Sigma$, $v, V \in \mathfrak{H}$, is compact. Obviously, it is continuous, symmetric, and non-negative while its kernel coincides with the subspace $\mathfrak{H}_0 = \{v \in \mathfrak{H}\colon\ v_2 = 0 \text{ on } \Sigma\}$. Thus, a general result of the operator theory (see, e.g., [9]) delivers the following assertion:

Fig. 2. The imperfect coating.

Lemma 3.1. *The variational formulation* (26) *of problem* (17), (18), (25) *possesses the eigenvalue sequence* (10). *The corresponding eigenvectors* $v^{(1)}, v^{(2)}, \ldots, v^{(j)}, \ldots$ *in* $\mathfrak{H}$ *can be subject to the orthogonality and normalization condition* $\rho^0(\mathcal{O})b(\xi_1^{1+\gamma}v_2^{(j)}, v_2^{(k)})_\Sigma = \delta_{j,k}$, $j,k = 1,2,\ldots$.

4. The result and open questions

A procedure of inverse and direct reductions, developed in [5, Ch. 7] and applied in [10,3,4], allows to derive estimates of asymptotic remainders with majoring constants independent of the eigenvalue number. Since even a formulation of such results needs a cumbersome notation, we restrict ourselves to present here only two facts. First, an appropriate approximation to the eigenvector $u^{(j+3)}$ of problem (2)–(4) looks as follows:

$$\mathcal{U}^{(j+3)0}(x) = \chi(x)\tau^{-\beta_1}\big(e_2 v_2^{(j)}(\xi_1,0) + hU^{(j)1}(\xi_1,h^{-1}\xi_2) + h^2U^{(j)2}(\xi_1,h^{-1}\xi_2)\big)$$
$$\mathcal{U}^{(j+3)1}(x) = \chi(x)\tau^{-\beta_1}v^{(j)}(\xi), \quad \beta_1 = 6^{-1}(1+2\gamma^{-1}) \tag{27}$$

Here ξ is the stretched coordinate system (11), $v^{(j)} \in \mathfrak{H}$ is an eigenvector of problem (26), $U^{(j)1}$ and $U^{(j)2}$ are constructed from $v^{(j)}$ according to (22), χ is a cut-off function with a small support and $\chi = 1$ in the vicinity of the peak tip $\mathcal{O}$. The factor $\tau^{-\beta_1}$ is put into (27) in order to satisfy approximately the natural normalization condition for eigenvectors of problem (2)–(4), namely,

$$\big\|\rho^{1/2}\mathcal{U}^{(j+3)}; L_2(\Omega)\big\|^2 = \rho^0(\mathcal{O})b\big\|\xi_1^{(1+\gamma)/2}v_2^{(j)}; L_2(\Sigma)\big\|^2 + \cdots = 1 + \cdots$$

The asymptotic formula (27) shows that the eigenmode $u^{(j+3)0}$ is mainly realized as transversal oscillations of the peak end which stimulate localized oscillations of the framing medium.

Second, the following simplified assertion on the normalized eigenvalues (6) is valid:

Theorem 4.1. *For any* $k > 3$, *convergence* (9) *holds true where* (10) *implies the eigenvalue sequence of problem* (26).

This Theorem needs minor changes in the case when a part Γ_D of the exterior boundary Γ is clamped, namely, $\Lambda_j^\tau > 0$ and $\tau^{-\beta_\Lambda}\Lambda_j^\tau \to \lambda_j$ for $j = 1,2,\ldots$. This observation does agree with the localization of eigenvectors of problem (2)–(4) in a neighborhood of the peak end; indeed, an approach developed in [11] proves the decay properties $|v^{(j)}(\xi)| = O(|\xi|^{-1/2})$ and $|v_2^{(j)}(\xi_1,0)| = O(\xi_1^{-3\gamma-1/2})$ as $|\xi| \to +\infty$. This also explains why the point $\Lambda = 0$ cannot belong to the spectrum of problem (26): The Dirichlet conditions on Γ_D removes the eigenvalues $\Lambda_1^\tau = \Lambda_2^\tau = \Lambda_3^\tau = 0$ from sequence (6), but at the same time the limit problem (17), (18), (25) is not influenced by a boundary condition imposed at a distance from the peak tip.

The same asymptotics are attributed to elastic field in a body with hard and heavy (cf. (5)) imperfect coating (see Fig. 2): If the arc Γ^0 is a straight segment near the point $\mathcal{O}$, the coating Ω^0 is defined by the formulas $0 < x_2 < H(x_1)$, $H(x_1) = |x_1|^{1+\gamma}(b_0 + O(|x_1|))$, $b_0 > 0$. Investigation of singularities of elastic fields at the point $\mathcal{O}$ was formulated in book [12] as an open problem. In [13] it is proved that the stresses and strains are bounded (see also [14] where a formal asymptotic analysis was performed).

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