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Stochastic approach to size effect in quasi-brittle materials

Jean-Baptiste Colliat [a], Martin Hautefeuille [a,b], Adnan Ibrahimbegovic [a,*], Hermann G. Matthies [b]

[a] ENS-Cachan, LMT-Cachan, 61, avenue du président Wilson, 94235 Cachan cedex, France
[b] TU Braunschweig, Institut fűr Wissenschaftliches Rechnen, 38100 Braunschweig, Germany

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Abstract

In this Note we present a stochastic approach to model size effects in quasi-brittle materials structures. Contrary to Weibull's theory, the key ingredient is the use of correlated random fields in order to describe the material properties. Thus, a stochastic problem has to be solved that we handle using Monte Carlo method. The numerical results show the capability to retrieve size effects in a range between the two classical bounds which are Continuum Damage Mechanics and Linear fracture Mechanics. ***To cite this article: J.-B. Colliat et al., C. R. Mecanique 335 (2007).***

Résumé

Approche stochastique aux effets d'échelle pour les matériaux quasi fragile. Dans ce papier nous présentons une approche stochastique pour modéliser les effets d'échelle des matériaux quasi fragiles. L'ingrédient clef de cette approche réside dans l'utilisation des champs corrélés pour les propriétés matériaux, principale différence par rapport à la théorie de Weibull. Ainsi, un problème stochastique se pose et peut être résolue par la méthode de Monte Carlo. Les résultats obtenus montrent les capacités de ce modèle à retrouver les effets d'échelle compris entre les deux bornes que représentent la mécanique de l'endommagement et la mécanique de la rupture. ***Pour citer cet article : J.-B. Colliat et al., C. R. Mecanique 335 (2007).***

Keywords: Computational solid mechanics; Karhunen–Loève expansion; Quasi-brittle materials; Size effect

Mots-clés : Mécanique des solides numérique ; Décomposition de Karhunen–Loève ; Matériaux quasi fragiles ; Effet d'échelle

Version française abrégée

Dans cette Note, nous proposons une approche probabiliste basée sur l'utilisation de champs aléatoires spatialement corrélés pour décrire les effets d'échelle rencontrés expérimentalement dans la rupture des structures du génie civil. Les matériaux composant ces structures présentent un comportement quasi fragile, brutalement adoucissant après

* Corresponding author.
E-mail addresses: ai@lmt.ens-cachan.fr (A. Ibrahimbegovic), wire@tu-bs.de (H.G. Matthies).

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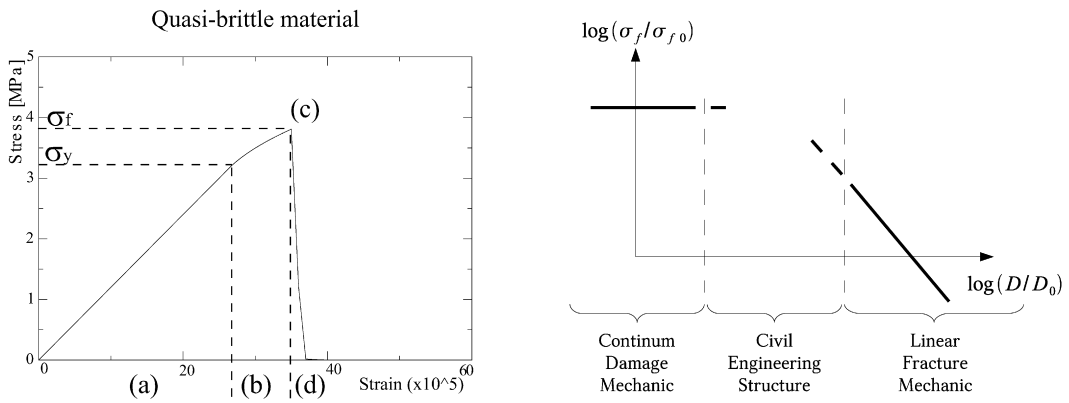
une phase d'écrouissage positif, induisant pour celles-ci une dépendance de leurs charges ultimes à leurs tailles. Les structure composées de matériaux purement fragiles, sans phase durcissante, présentent elles aussi un effet d'échelle qui peut être très pertinemment décrit par la mécanique de la rupture et la théorie de Weibull, basée sur des champs aléatoires non-corrélés. Un modèle de covariance exponentielle simple reliant cette corrélation spatiale à un unique paramètre, une longueur de corrélation et un modèle mécanique traitant efficacement la phase adoucissante sans dépendance au maillage et sans utilisation d'autre longueur caractéristique, permettent de mettre en évidence sur un simple exemple unidimmensionnel les différences existant entre la théorie de Weibull et l'utilisation de champs corrélés. Grâce à la décomposition de Karhunen–Loève, il est possible de résoudre un problème stochastique de traction simple sur trois barres de tailles différentes par des méthodes de Monte Carlo. Ces méthodes permettent de calculer des grandeurs statistiques de l'effort ultime de chacune de ces trois barres comme les fractiles de barres rompues. Le report du logarithme de ces différents fractiles en fonction du logarithme de la taille des barres, diagramme classique des effets d'échelle, montre qu'une relation non linéaire entre ces deux quantités peut être trouvées par cette approche, établissant ainsi un intermédiaire entre la mécanique de l'endommagement et la mécanique de la rupture.

1. Introduction

In this Note, we address the problem of modeling the size effect encountered in quasi-brittle materials structures within a stochastic framework. In engineering, stochastic approach provides a very good basis for dealing with building materials such as soil, concrete, mortar and other geomaterials to take into account the intrinsic randomness of their being heterogeneous. Such materials have a particular mechanical behavior. They are known as quasi-brittle materials which can be seen as a sub-category of softening materials (see [1] or [2]). This behavior can be described with four material parameters in 1D context: the Young modulus E, the yield stress σ_y which induces micro-cracking and the failure stress σ_f which induces macro-cracking after the sudden coalescence of the micro-cracks leading to a softening behavior (see Fig. 1(a)). The last parameter is the fracture energy G_f [J m^{-2}] which represents the amount of energy necessary to create and open a macro-crack. Several theories exist on how to model the failure in quasi-brittle materials and most of them link the micro-cracks coalescence phenomena to size effect, a dependency on the size of a structure to its failure load. The aim of all those theories is to hook up together the continuum damage mechanic (CDM) where the failure stress does not depend on the size of the structure to the linear fracture mechanic (LFM) where size effect appears naturally as the logarithm of the failure stress depends linearly on the logarithm of the size of the structure (see Fig. 1(b)). It can be experimentally demonstrated that even if purely brittle materials follow LFM, quasi-brittle materials do not and follow a nonlinear relationship between the two previous logarithms of the failure stress and the size of the structure. These materials exhibit a different size effect than the one encountered for purely brittle materials.

An extensive literature exists on that topic, from the early studies of W. Weibull (see [3]) dealing with chains built from independent purely brittle links (theory of the weakest link) to retrieve LFM, to the current two theories of



(a) Tensile test behavior

(b) Validity domain of mechanical theories

Fig. 1. Quasi-brittle material model (a) and size effect (b).

Z.P. Bažant on the one side and of A. Carpenteri on another. The first one tends to describe size effect as a deterministic theory of strength redistribution in a fracture process zone (FPZ) which size is proportional to a characteristic length leads to energetic dissipation. At some level, the micro-cracks coalescence that induces both heterogeneous behavior and some kind of localization, is strongly intricate to size effect. Hence, a good mechanical model that describes the fracture of quasi-brittle materials should be able to retrieve size effect. Recently, Z.P. Bažant has developed a new theory as an amalgamion of this previous theory with the Weibull's one leading to the so called energetic-statistical size effect (see [4]). Another theory melting a nonlocal model and a stochastic approach has been developed by K. Sab (see [5]). On the other hand, Carpenteri's theory is based on the study of quasi-brittle materials seen as materials with a fractal microstructure (see [6]).

Our goal in this Note is to propose an alternative solution to bridge together CDM and LFM (see Fig. 1(b)) in order to accurately and physically model size effect in quasi-brittle materials. In the following, we will show that a way to achieve our goal is to consider material properties as correlated fields.

2. Random fields as a tool to model quasi-brittle materials size effect

Dealing with softening materials and the Finite Elements Method is quite a hard task because of mesh dependency (e.g. see [2]). Among several methods which exist to eliminate this mesh dependency, one can find the cohesive or fictitious crack model (see [7]), the nonlocal approach (see [8]) and the strong discontinuity models (see [1]) which consists in surface dissipation. The latter seems nowadays to play the dominant role, and for that reason we select it (see [9]) as the most general among the available models. This model might also be coupled with diffuse (volume) plasticity to describe the volume dissipation due to the homogeneous micro-cracking which takes place in the fracture process zone (FPZ). The strong discontinuity part enables to describe the micro-cracks coalescence which leads to fracture and the softening behavior without any mesh dependency. The reader might refer to an extensive literature about the strong discontinuity models. Among them are [12,11] and [10].

2.1. Correlated random fields and the Karhunen–Loève expansion

In this study, we consider the elastic yield stress σ_y and the gap e_f between the maximum stress σ_f and the elastic yield stress σ_y of a quasi-brittle material as the main source of uncertainty. Young modulus and fracture energy are assumed to be deterministic quantities. Following Weibull theory, we would have chosen to model this set of material properties by two uncorrelated random fields—an infinite set of random variables indexed by the geometrical domain (see [13])—modeling a white noise along the geometrical domain. Here, the key point, and thus the difference w.r.t Weibull's theory, is that we have chosen to take into account correlated random fields. To be more precise, we assume that σ_y and e_f can be modeled by stochastically homogeneous random fields. For these fields, the spatial correlation is invariant by translation and can be described, following the second-order analysis (see [14]), by their marginal distribution and their covariance function. Since Both σ_y and e_f are positive and supposed to have a finite known variance and assuming that our solution—here the displacement along the bar—is of the second order, the maximum entropy theory (see [15,16]), leads to lognormally distributed marginal distribution. Without loss of generality, it is convenient to consider that these two random fields are defined as nonlinear transformations of Gaussian random fields γ_1 and γ_2 fully described by their expected values and their covariance functions written as,

$$\mathrm{Cov}_{\gamma_i}(x,y) = V_i \exp\left(-\frac{\|x-y\|_1}{L_c}\right) \tag{1}$$

where V_i is the marginal distribution variance and L_c is the correlation length. This correlation length is the key ingredient dealing with the description of the two random fields σ_y and $e_f = \sigma_f - \sigma_y$. The nonlinear transformation leading to lognormally distributed random fields can be written:

$$\sigma_y(x,\omega) = \exp\bigl(\gamma_1(x,\omega)\bigr) \tag{2}$$

Applying this nonlinear transformation to the covariance functions of σ_y and e_f leads to (see [17])

$$\mathrm{Cov}_{\sigma_y} = \mathrm{e}^{2\bar{\gamma}_1+V_1} \exp\bigl(\mathrm{Cov}_{\gamma_1}(x,y) - 1\bigr) \tag{3}$$

Using (1) and (3) leads to the parameters that should be given to both of the underlying Gaussian random fields γ_1 and γ_2 (see Table 1).

Table 1
Parameters of both material properties random fields and related underlying Gaussian ones

	γ_1	σ_y	γ_2	e_f
expectation [MPa]	1.59	5	−1.5	0.5
variance [MPa]^2	0.039	1	1.61	1
correlation length [m]	0.2	0.1	0.365	0.1

An effective computational representation of the random fields is needed. The latter can be provided by the Karhunen–Loève expansion (see [13]), which is basically an orthonormal projection in $L_2(\mathcal{D})$ onto the eigenvector basis which are obtained by solving the Fredholm eigenproblem of the second kind:

$$\int_{\mathcal{D}} \mathrm{Cov}_\gamma(x, y)\Phi_i(x)\,\mathrm{d}x = \rho_i \Phi_i(y), \quad y \in \mathcal{D} \tag{4}$$

The actual solution of the eigenvalue problem for any domain $\mathcal{D}$ can be obtained using finite elements techniques, for example with the finite element code FEAP (see [20]). Such problem always remains well posed since the covariance function is symmetric and positive semi-definite, which implies positive and decreasing ordered eigenvalues $\{\rho_i\}_{i=1,\infty}$ and a set of eigenfunctions $\{\Phi_i(x)\}_{i=1,\infty}$ which is a complete and orthonormal basis of $L_2(\mathcal{D})$. This yields to the spectral decomposition,

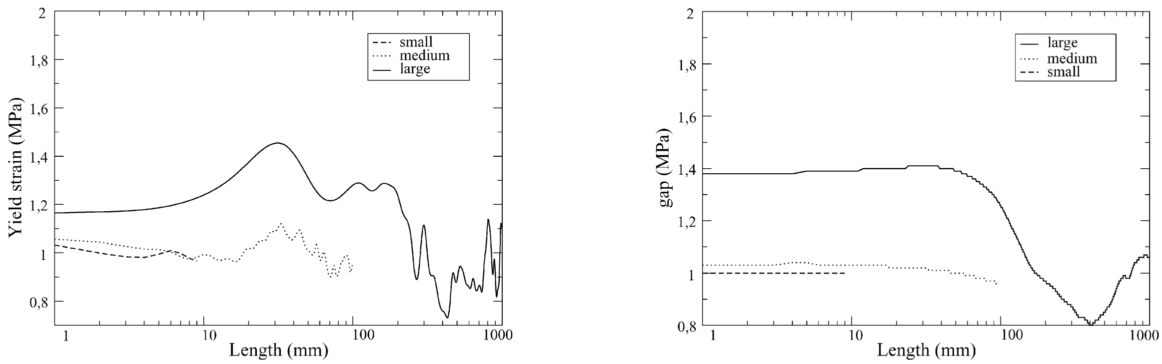
$$\gamma(x, \omega) = \bar{\gamma}(x) + \sum_{i=1}^{\infty} \sqrt{\rho_i}\, \Phi_i(x)\xi_i(\omega) \tag{5}$$

and a way to synthesize two Gaussian random fields $\gamma_{i=1,2}$ and consequently σ_y and e_f through relations (2). In Eq. (5), the $\xi_i(\omega)$ are uncorrelated Gaussian random variables (with unit variance and zero mean) and thus stochastically independent. Using a truncated form of (5)—50 modes for a good accuracy (see [19])—it is possible to generate a realization in two steps: first computing the truncated sum (5) using the Karhunen–Loève eigenmodes and a Gaussian random vector of independent coefficients. Then using (2), this Gaussian realization is transformed into a lognormal one.

2.2. Stochastic integration

In the first part of this section, we have described all the ingredients of our model, both from the mechanic framework, and from the stochastic one. In order to show that correlated random fields are the key point to model quasi-brittle materials size effect, we use a very simple 1D example of truss tensile test under displacement control. Although this example leads to an homogeneous stress field along the bar, it is relevant because directly comparable to Weibull's theory. Three lengths are treated—0.01 m with 10 elements of length 0.001 m, 0.1 m with 100 elements of length 0.001 m, 1 m truss with 100 elements of length 0.01 m—with a correlation length remaining constant and equal to 0.01 m. These three cases are called respectively small, medium and large. For the medium case, the bar is of the same size as the correlation length, contrary to the small case—the bar is ten times shorter than the correlation length—and the large one.

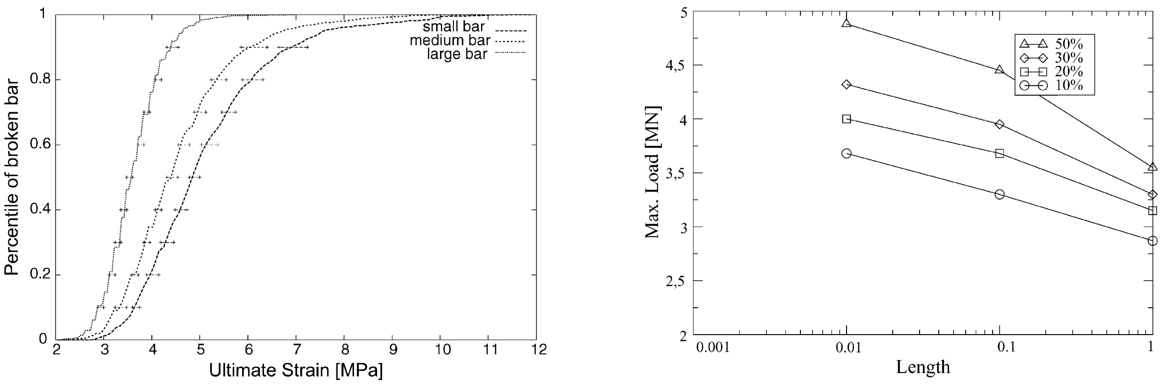
When solving for the response of a stochastic system, usually the main goal is the computation of response statistics, for example the expected value. To that end, we employ the so-called Monte Carlo method (other more efficient methods are currently under development, see [18]) which requires to solve many deterministic systems, each of them being built with realizations $\sigma_y(\cdot, \omega_i)$ and $e_f(\cdot,\omega_i)$ of the random fields σ_y and e_f. The normalized fluctuating parts of one σ_y and e_f realization for each of these bars are shown on Fig. 2. One can see that the larger the bar is with respect to the correlation length the more fluctuating the random fields are. Thus, the random fields corresponding to the longest bar are more likely to have lower bounds, leading to a weaker structure. Although such simple analysis might be done for every particular realization, the stochastic integration is needed in order to compute each bar strength statistics. The Monte Carlo method is performed with 10 000 integration points for each length using the *Platon* environment (see [21] for software engineering aspects) linked with the finite elements code FEAP by using the *Components Template Library* (see [22]). The numerical results are presented in the next section.



(a) Yield stress

(b) Gap between ultimate and yield stresses

Fig. 2. Normalized realizations of random fields σ_y and e_f.



(a) Cumulative Density Function with 99% error bars

(b) Different percentile for each size

Fig. 3. Cumulative distribution of the maximum load obtained after Monte Carlo simulation.

3. Numerical results and analysis

The maximum load cumulative density functions of these three bars are presented in Fig. 3(a). One can see that for a given percentile of broken bars, for example 10%, the smaller the bar is, the bigger is its ultimate stress, respectively from 3.68 MPa for the small truss, to 3.3 MPa for the medium one and 2.87 MPa for the largest (see Fig. 3(b)). In other words, the strength of the structure is linked to its size according to the correlation length scale. The larger is the structure comparing to the correlation length, the weaker is the structure. Hence, this stochastic way of modeling quasi-brittle failure reveals the size effect through the correlation length. Fig. 3(a) shows the 99% confidence interval. For each case, none of these error bars is overlapping, thus proving that the Monte Carlo simulation is accurate enough.

It is worth noting that the correlation length appearing in the probabilistic framework plays an equivalent role as the characteristic length in nonlocal models. To be more precise, introducing this length leads to settle a scale and allows to model size effects. It also can be linked to the size of a fracture process zone (FPZ) where micro-cracking occurs before macro-cracking, due to the coalescence of micro-cracks. The more the size of this FPZ prevails on the global size of the structure (which is the case for the small bar), the more similar to the Continuum Damage Mechanics (CDM) the structure's behavior is. On the other hand, if the size of the FPZ is neglectable with respect to the size of the structure (for the large bar), the FPZ has no influence on the global behavior and the macro-crack occurs according to Linear Fracture Mechanics (LFM). Moreover, Weibull's theory is then retrieved as each element is stochastically independent from the others. Fig. 3(b) shows that the probabilistic approach using correlated random fields is capable to link these two limit behaviors LFM and CDM as in Fig. 1.(a). Contrary to Weibull's size effect theory, the key point lies in using correlated random fields.

4. Conclusion

Size effect is a major issue in modeling quasi-brittle materials structures failure and usually two limit cases are considered, Continuum Damage Mechanic and Linear Fracture Mechanic as in Weibull's theory. In between, several authors have proposed size effects laws corresponding to different kind of structures and loading paths, or tried to model this particular feature. One attempt consists in using nonlocal models and retrieve size effect through their characteristic lengths, although this length has no physical basement. In this work, we show how to link CDM and LFM by using strong discontinuity models combined with correlated random fields.

Using correlated random fields through their Karhunen–Loève expansion in a simple 1D context leads to a stochastic problem which has been solved by Monte Carlo simulation. The results show that the correlation length appearing in the random fields covariance functions is acting like a length scale. This particular feature leads to the possibility to retrieve a size effect which is valid between the two limit cases of CDM and LFM. This method might be viewed as an extension of Weibull's theory which can be retrieved in our case considering uncorrelated random field – where the correlation length is null.

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