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Joseph Boussinesq, a Scientist of bygone days and present times

Joseph Boussinesq (1842–1929): a pioneer of mechanical modelling at the end of the 19th Century

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Researchers in Mechanics who have never heard the name of Joseph Boussinesq are few. Boussinesq first taught at the Faculty of Sciences in Lille for about fifteen years, then, after being elected to the Academy of Sciences in Paris, taught there for over forty years. From his work, both in Lille and in the Academy of Sciences, he left a great many publications, touching on a variety of subjects, as proof of his many talents. Oddly enough, however, Boussinesq is almost always known to the world of scientific research through only one facet of his many talents. He is little known in France outside some specialized circles where his subjects of research are studied.

For those who have some knowledge in the history of mechanical science, it certainly comes as a surprise to see such a phenomenon. Here is a scientist of international renown, who wrote equations, 'Boussinesq's equations', came up with new hypotheses, 'Boussinesq's hypotheses', formulae, 'Boussinesq's formulae' etc. and yet today seems to have vanished from all scientific memory. Boussinesq's name, however, is far from being forgotten. While his area is too specialised to be mentioned in a common dictionary he did have an entry, however, signed by H. Berger and C. Ballot [1] in the *Dictionary of Scientific Biography*. The town of Saint-André de Sangonis, his birthplace, had a commemoration in his honour on 20 April 1996. The town of Montpellier named a street after him. There is a portrait of Boussinesq at the Lille University of Sciences and Technology (USTL), as seen by a mathematician (M. Parreau [2]). During the last decade three papers about mechanics also commemorated Boussinesq, the scientist: one (W.H. Hager and F. Ræmy [3]) deals with his works on *turbulence*, the two others (R. Zeytounian [4,5]) deal with the *Approximation*. Finally in 2004, the Netherlands created the *Dutch Boussinesq Center for Hydrology* under the authority of the *Koninklijke Nederlandse Akademie van Wetenschappe* (KNAW: Royal Netherlands Academy of Arts and Sciences). Since 2005 the Center has been organizing a series of '*Boussinesq Lectures*' given by scientists of world renown dealing with hydrology and soil mechanics.

Boussinesq was clearly one of the most underrated scientists that USTL has ever had. It is hoped that a more complete portrait paying homage to his incredibly diverse talents, will at last help restore the rank he deserves.

It is often said that mechanics is the most physical field of mathematics, and, conversely, that it is the most mathematical field of physics. This double definition emphasizes one of the identity problems of this science, claimed by both communities, that of mathematics and of physics. Mathematicians see in mechanics a concrete field of application for the often abstract theories they formulate, or they analyse mechanical phenomena to draw concepts out of them in order to reach the same abstraction. An example of this mathematical way of looking at mechanics is the portrait of Boussinesq which M. Parreau made in Lille, and in which he remarks that Boussinesq is "*no longer well known nowadays*". Indeed it is most likely the case among Mathematicians. Is it so surprising?

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Then again, while physicists generally admit that mechanics is the mother of physics, nowadays they only see in it one of the sciences used to identify the properties of matter. Electricity, optics are other sciences which they study with the same perseverance and, at least, the same enthusiasm. They recognize that, unlike other physical sciences, mechanics only needs few assumptions for its modelling, which allows a far-reaching mathematical treatment.

It can be argued, however, that, since Archimedes and until the 19th century, mechanics has constantly been related to either one of those two sciences. Through the creation of engineering schools, which, roughly speaking, date back to the beginning of the 19th century, a third community was created, that of the engineering sciences. Originally – let us say in the Middle Ages – engineers belonged to the military, and built city fortifications. Whatever their change of orientation afterwards (bridges, engines, civil engineering etc.) it is clear that the first science that they relied on was mechanics. In short, since the 19th century, mechanics has always rested on three pillars: *Mathematics, Physics, Engineering Sciences*.[1] Even if it offers mechanics a certain stability, it does not mean that it makes it more readable or perceptible. Let us take the argument further: joining mechanics to engineering sciences might lead the scientific community to deny a more fundamental part to that science.

When Joseph Boussinesq (see below) was entering the scientific world, mechanics was in a process of defining its basic governing equations. It is what Boussinesq would be doing throughout his life. He happened to start his career in Lille (just like many tenured Professors, his geographic origins were not very important in his career) where his works led him to the Academy of Sciences. He then went to Paris, where he made his most outstanding discoveries.

1. The training years: his modest origins

Joseph Valentin Boussinesq was born on 13 March 1842 in Saint-André de Sangonis, a small town in Hérault between Montpellier and Lodève in the South of France. His father was a farmer and his mother the daughter of an industrialist. During this period, France was becoming industrialized, and the times were changing. Joseph's parents wanted him to study, at a time when schooling was not compulsory, so he started primary school, followed a secondary education under the rule of an uncle, a priest, who, recognizing his nephew's talents, taught him Latin and Greek. Boussinesq finally studied at the University, by working as a housemaster at the *Lycée* of Montpellier. At the age of 19, he got a Bachelor's Degree in mathematics at the Faculty of Sciences in the same town. Nothing precautious here, his mother having died in 1857, he went on to study in Montpellier against his father's will, who had wanted him to take over the family farm.

With his mathematics degree, Boussinesq started teaching in various small towns in southern France: Agde (1862), Le Vigan (1865), Gap (1866). In Agde he read Gabriel Lamé (1785–1870).[2] There he found the inspiration for a first piece of work on the impact of a water jet on an elastic plate. He managed to have it presented (by Lamé himself) at the Academy of Sciences. He has caught the bug and began a thesis, which he defended in Paris on 13 May 1867. Lamé most likely influenced the subject chosen for his thesis – "*Propagation of heat in homogeneous media*" [B1].[3] In parallel with his dissertation, he worked on linear elasticity: his work did not go unnoticed by the Academy of Sciences and particularly by Adhémar Barré de Saint-Venant (1797–1886) who started a correspondence with

[1] There is another debate on the very existence of the 'engineering sciences', as a part of the scientific community considers that the engineering sciences must be present in each discipline, but do not constitute a particular community. Let us admit that these sciences, whatever their object, contain a *rigorous scientific approach* pertaining to mathematics and a perspective *mandatorily oriented towards concrete applications* with a technological purport. One may say that these sciences are the first which 'reconciled' the *sciences* and the *technique*, which were nevertheless termed 'enemy sisters' by Pierre Thuillier [51].

[2] Lamé was, according to Albert de Lapparent [52], a "*geometer who taught Physics*". Between 1852 and 1861 he published at least five books, two of which Boussinesq must have come across: *Théorie mathématique de l'Elasticité* [53], *Leçons sur la théorie analytique de la chaleur* [54] (the latter quoted by Boussinesq in his thesis).

[3] Although Lamé was not among the jury for Boussinesq's thesis (in 1867 he was retired and the jury panel was composed of Joseph Serret, Joseph Bertrand and Charles-Auguste Briot), his influence was unmistakable in the way Boussinesq treated the subject: the thesis was almost entirely centred on geometrical aspects of the problem. Saint-Venant, who may have supervised the thesis while it was proceeding, was not among the jury either (is it because he was not a tenured Professor at the University?). Boussinesq gave a lengthy explanation ([B28], p. 267–268) on the subject of his thesis, which was not his "first choice". In fact, following his early works, Boussinesq first wrote a dissertation on optics (a *Mechanical theory of the light*). Because of his supervisor's untimely death (Emile Verdet, 1824–1866) it could not be presented. With a new supervisor whom he did not choose, Boussinesq then presented a "patched" piece of work, which was certainly finished faster and which better suited his supervisor's personality (as Boussinesq himself said thirty-six years later). This work was accepted, and eventually became his thesis. The work that had been initially undertaken for his thesis was shortly published [B2] in 1868, and then much later in an expanded version [B28].

Boussinesq, and hinted that an academic career might be possible. From then on he became Boussinesq's strongest supporter. To start with, the Academy of Sciences awarded Boussinesq the Poncelet prize (1872). Following Saint-Venant's advice, Boussinesq also obtained a Bachelor's degree in physics[4] (1872). In 1873 he was finally appointed Professor of '*Differential and Integral Calculus*' at the Faculty of Sciences in Lille.

Let us briefly determine the state of mechanics in science in the 1860s: the mechanics of rigid solids was more or less settled at the end of the 18th century (roughly: Lagrange, 1788). The mechanics of incompressible inviscid fluids was also rather well known thanks to the works of Daniel Bernoulli (1727), of d'Alembert (1755), of Euler (1759), of Lagrange (1781) and of Laplace (1784). Of course, some problems remained unexplained, like d'Alembert's paradox, but these were rare.

In 1822 Claude-Louis Navier (1785–1836) tried the first dynamical modelling of continua (elastics solids and viscous fluids) [6], using a complex intermolecular description of the frictional shear stresses in a continuous medium. Unfortunately, his model was both too basic and partially incorrect, so he only succeeded in infuriating his former master Siméon Denis Poisson (1781–1840). After Navier's death (1836), Saint-Venant, his former pupil, indeed, demonstrated [7] what Navier had found: Saint-Venant's work just started the mechanical approach of continua, which lasted more than a century and a half. In 1845 George Gabriel Stokes (1819–1903) demonstrated again Navier's equations [8] with a somewhat different approach[5] (see Anderson [9]). These equations are now called the *Navier–Stokes equations*.

In Mechanics of continua, the works of Poisson, Lamé, Navier, Young (1773–1829), Saint-Venant were authoritative. They had more or less made linear elasticity clear, and identified plastic behaviours. At the same time some engineers were working at developing the study of elastic bodies with a simplified geometrical description (beams, plates, etc.). When Boussinesq started taking an interest in these questions, the relationship between the simplified and fully three-dimensional approaches was not very clear. Some researchers were also working on media which were, physically, more complex, for instance thermoelasticity (which explains young Boussinesq's thesis), but also on porous media, powders, etc. In these cases the formulation was more hesitant.

In another field of investigation, the notion of heat was gradually introduced in the world of the physicists. Watt's discovery of the steam-engine (the condenser, 1769), was the detonator and, then, the catalyst: it became, indeed, a necessity to master this notion in order to be able to develop the new machine. Without going into too much detail about how the two principles of thermodynamics became universal, let us say that by 1860 they were more or less known, but were too recent and, therefore, still fragile. In 1847 Hermann Helmholtz (1821–1894) concluded the first principle, by formulating for the first time [10] the 'principle of conservation of the energy'. By contrast the scientific world could feel the power of the second principle, but still could not keep it under control. With that principle they made an amazing discovery: thermodynamics concerned not only the steam-engine, but physics in its entirety.

In parallel with these theoreticians, throughout the 19th century, a crowd of experimental physicists were discovering new properties of continuous media (fluids and solids) which the 18th century scientists had not been able to explain, or which they had too quickly dismissed as unjustified 'measures errors', as was often the case. German physicist Gotthilf Hagen (1797–1884), and, almost at the same time, French physician Jean-Léonard Poiseuille (1797–1869) discovered the 'Hagen–Poiseuille's law' [11]. Without waiting for Osborne Reynolds's experiments Hagen identified the existence of two kinds of viscous flows [12] as early as 1839. In fact he was not alone: Henry Darcy [13,14] in 1857 and his pupil Henri-Emile Bazin [15] in 1865[6] made experiments to confirm (or invalidate)

[4] At that time the only degree recognized by the State was the *Licence ès Sciences* (Bachelor's Degree in Sciences) which Boussinesq already had. However, several specialties existed: Mathematics (which was Boussinesq's), and Physics, which he was asked to take in order to be appointed Professor of Mechanics (note that it showed how ambiguous the status of Mechanics was). In order to prepare his degree in Physics, Boussinesq was forced to take a sabbatical year in 1872. In 1890, the *Licence de Mathématiques* and the *Licence de Physique* were separate degrees: Maurice Couette (1858–1943), also a mathematics graduate had to have them both to be appointed Professor at the Catholic University of Angers under the same conditions as Boussinesq's in Lille in 1872.

[5] In fact, Stokes originally wanted to justify a formula $\tau = \mu\,\mathrm{d}V/\mathrm{d}n$ proposed by Newton (see J.D. Anderson, [9]), providing the shear stress exerted by a viscous fluid moving around a solid body. The Navier–Stokes equations, indeed, justified this formula. One may dwell on these equations of Navier–Stokes, which ignore the name of the only person they should not have forgotten: Saint-Venant.

[6] The works of Henry Darcy (1803–1858, famous for Darcy's law on porous media and his experiments on fluid flows in pipes) made him a forerunner (but there are many of them) both for the notion of boundary-layer and for the notion of turbulence. Unfortunately, his work was stopped by his death in 1858, and published in 1865 by his former Assistant, Jules Dupuit (1804–1866) and his former pupil Henri-Emile Bazin (1829–1917).

the Navier–Stokes equations, and very quickly distinguished the difference between laminar flows and 'tumultuous' flows. In the meantime Adolf Fick (1829–1901), an Austrian physiologist and acoustician, discovered [16] the law of diffusion of the components in an heterogeneous fluid: 'Fick's law'. All these laws were established experimentally, very often in a good deal of confusion, and, sometimes, they only discovered a new version of what another law had previously demonstrated for another phenomenon.

What, then, were the problems that the scientific community had to deal with? Today, a hundred and forty years later, we are more or less able to identify them. Within the frame of macroscopic physics, which characterizes the *Mechanics of continua*, those problems were threefold: *rheology* (thermodynamics was still full of mysteries, and it lasted until the middle of 20th century); *statistics* (the problem of the distinction between the deterministic and nondeterministic descriptions had not yet come up); *asymptotics*: given the progress of the analysis in the 19th century, it is probably this field that the scientific community knew best. However these different categories of problems were confused as to which was which. In spite of that, breakthroughs were achieved in the three directions in the second part of the century.[7]

2. The Lille period: a disciple of Saint-Venant ... but also of Descartes and Galileo

It is very difficult to follow Boussinesq's scientific itinerary in detail. As Emile Picard said in an obituary [17] in 1933, Boussinesq's work concerned itself with such a vast field of physics that one is still impressed by such a polyvalence. To quote Picard "*Boussinesq brought an eminent contribution in all fields of mathematical physics, with the exception of electromagnetism*". The titles of his papers are enough to evoke his many talents, including his more philosophical ([B13,B15]) or philosophico-historical [B38] works.

When in all of Boussinesq's studies one looks for a guideline, one always notices older experimental results of theoretical works from authors who were his contemporaries. He analyzed their works with the utmost care in the case, for instance, of Rankine's works [18] or Bazin's results [15]. Boussinesq was extremely meticulous about that. Therefore he was truly a disciple of Descartes, and even if he was no experimenter, he remained a disciple of Galileo. However, he did not initiate new discoveries: his works always dealt with *mechanical modelling*. Moreover, because he had such a rich and creative mind, he often mingled subjects, which were apparently distinct or even very far apart, making his writings somewhat difficult to follow at times.

2.1. The problem of static stresses in soils

When Boussinesq arrived in Lille, he was fascinated by Saint-Venant. Saint-Venant, a Professor of civil engineering at the *École des ponts et chaussées*, introduced Boussinesq to another of his pupils: Alfred Flamant (1839–1914), a civil engineer with whom Boussinesq would later collaborate.[8] For the time, probably encouraged by Saint-Venant, he grappled with problems of soil mechanics. He tried to estimate the stresses in pulverulent masses (the kinematical and dynamic description of such media are partially that of a fluid and partially that of a solid), in order to find applications to civil engineering. Saint-Venant wanted to improve the results [18] that the Scotsman William John Rankine (1820–1872) had obtained in 1856, and which Saint-Venant had found too restrictive. In order to construct a model of soil, Boussinesq introduced, in an elastic model, a 'pulverulent' correction. That correction consisted in mainly introducing, in the constitutive law of the medium, Lamé's coefficients depending on the pressure in the soil. Boussinesq's models were very often 'corrections' of other models, which were representative of his scientific approach. That approach was different from the one that Rankine had adopted before him. Rankine had obtained valid equations only when the soil was close to breaking (for instance when a zero pressure on the external side of a heap of sand made it collapse). Furthermore, unable to solve the equations he had obtained, he would make further assumptions only to get solutions analytically;[9] Boussinesq was quick to reduce this hypothesis before proposing

[7] Note that we have said nothing, here, about many fields of Mechanics such as the Mechanics of compressible flows, which developed with thermodynamics, and which led to the Mechanics of flight: these fields were outside Boussinesq's area of study, and it is impossible to make a full description of Mechanics here.

[8] In several of Flamant's works on the Mechanics of soils ([55]) as well as on Fluid Mechanics ([56], nonexhaustive references), this collaboration can be recognized. The '*Boussinesq problem*' is also called, today, the '*Boussinesq–Flamant problem*' (see, for instance, Dai Zhou and Bo Jin [57]).

[9] Rankine's assumption consisted in postulating that some function $g(x, y)$ intervening in the equations could be written $g(x, y) = \phi(x) + \psi(y)$. Boussinesq analyzed this assumption and its consequences very precisely ([B10] p. 157–173) and concluded that it must be discarded.

his correction. He then obtained solutions for the problem of a heap of sand in an arbitrary position, as opposed to Rankine's limit case. Even better, he was surprised to see another case of breaking take place because of an excess of pressure, corresponding to a crushed massif. Applications of this problem to civil engineering are obvious: the results were published in many papers (e.g. [B10]). The name of Boussinesq started to be widely known.

2.2. Turbulent flows (first phase)

This is one of the first hydrodynamical problems which baffled Boussinesq; he had been interested in fluids since he had been in Gap (maybe while he was strolling along mountain torrents: their turbulent flows are quite aptly said to be … torrential!). In order to understand the questions at hand, let us remember that, in 1868, the Navier–Stokes equations (1845) were still quite recent. As no experiment was decisive enough, those equations were still being discussed: before Couette's experiments (1890), their main validation was Poiseuille's experiment, which only validated Navier–Stokes for rather slow flows. As he was aware of Darcy's experimental results obtained in 1857, and, then, Bazin's in 1865, which confirmed the 'tumultuous' flows described by Hagen, Boussinesq [B3] already attempted to give a description of a constitutive law relating stress and deformation rate, taking the whirlpool agitation of the environment into account. In fact he thought that the Navier–Stokes description might be incomplete, or that some terms were missing in their formulae, and those terms could only appear in sufficiently fast flows [B11]. He tried again in 1878 [B12]: he was still unaware of Reynolds's works (which started to be published only in 1883) but his contribution is especially valuable because, as Hager and Ræmy [3] remarked, "*he identified the basic problem*". So when he formulated his hypothesis for the first time, his contribution in 1897 (see below) was both stronger and much more fruitful. Let us say straight away that it was a mistake to try to find a phenomenological law for a problem which was not rheological. Today we know why it is impossible to solve this problem in the way Boussinesq had phrased it; this statistical phenomenon has to lean on a nondeterministic description, which must keep some consistence: two mistakes, caused, as we said earlier, by an insufficient analysis of the problems.

2.3. Surface waves and Boussinesq's equation

When he tackled this problem, Boussinesq trod on much firmer ground, studying at first surface waves in canals which the Scottish scientist J.S. Russell (1808–1882) had observed in 1834. Russell had recorded his observations very precisely, describing, in particular, a wave of infinite wavelength, which he called a 'solitary wave'. Stokes, followed by Airy (1801–1892), then Kelvin (1824–1907), had already tried to find a theoretical solution for this problem. Nevertheless their solutions were unsatisfactory: none of these scientists correctly took into account the 'shallow water' effect, which, as we know today, characterizes such problems.

Should we see Saint-Venant's influence in the choice of that topic? It is not unlikely. In 1871 Saint-Venant [19] obtained simplified equations (called today the '*Saint-Venant's equations*') to describe shallow water flows. However, these equations did not describe the phenomenon observed by Russell. One of the first problems which Boussinesq tackled after writing his thesis was the explanation of this phenomenon. Combining both the influence of the small amplitude of waves observed and that of the shallow depth of the canal in relation to its horizontal extension, he obtained the following equation, called today the 'Boussinesq equation':

$$u_{tt} - u_{xx} = \alpha u_{xxxx} + \beta\left(u^2\right)_{xx} \tag{1}$$

where u denotes vertical change of the free surface, and x horizontal distance along the canal. In this work (published in several papers from 1871 to 1877, [B4,B6,B7,B11]) Boussinesq went much further than Saint-Venant. Indeed Saint-Venant's equations alone were not suitable to obtain Eq. (1) (the α coefficient was not there). Boussinesq, however, not only derived the equation, he also gave solutions, one of them actually referring to the solitary wave. Boussinesq's equation and its solution were found again soon after [20] by Rayleigh (1842–1919). Almost twenty years later (1895) a Dutch scientist, Diederik Johannes Korteweg (1848–1941) and his pupil Gustav de Vries (1866–1934) found the 'Korteweg–De Vries equation' [21] and its solution: that equation roughly corresponds to a simple wave itself associated to Boussinesq's equation. Korteweg and De Vries obtained it using a direct approach and without noticing how closely related the two problems were.[10] Boussinesq also showed that the same Eq. (1) holds for longitudinal vibrations of elastic bars for slightly nonlinear motions. The two interfering parameters are then the amplitude of the motion and the relative thickness of the bar.

When he tackled the problem of the surface waves in shallow water, Boussinesq defined the scientific method which guided him during most of his discoveries. With a good deal of intuition relying on the study of experimental data, which were sometimes very basic (see above), he extracted relevant parameters from the equations of the phenomena he studied. Those parameters governed the behaviour of the solutions given by the experiment. Boussinesq underlined peculiar cases when several of these parameters became competing. Most of the time these parameters were not identified by a formal, or, shall we say, asymptotic method. He determined his choice by comparing the various numerical values obtained by the experimentalist. In Eq. (1), for example, the left-hand side represents a wave operator governing propagation phenomena. The first term of the right-hand side represents the modification due to the fact that the depth is 'not exactly small' (coefficient α). It is the *predominance* of that term compared to the wave operator in Stokes' model, which a priori prevented him from obtaining the solitary wave behaviour. On the contrary, it is its *absence* that prevented Saint-Venant from obtaining the same result: thus Boussinesq was clearly between the two models. Nevertheless, that middle-of-the-way position is still not enough; the presence of the second perturbation term in the right-hand side (coefficient β) was necessary, meaning that the solitary wave can only take place in waves of not quite infinitesimal amplitude. If all terms of the equation were taken into account at the same time, it meant that the three effects were competing.

Unaware of William Froude's works (1810–1879) on similitude, which were about to be published, Boussinesq did not formally justify his equation. For his analysis to be complete in the contemporary meaning, it lacked the notion of 'slow time' which should have been included in his study. In that sense, we can say that he was one of the forerunners of *asymptotic modelling*.

2.4. The BBO equation and the 'historical term' of Basset–Boussinesq

Boussinesq was now working on fluids. Still intrigued by viscous behaviour patterns he tackled a problem that Stokes had already considered, but that he solved only in the stationary case[11]: the motion of a solid body within a viscous fluid at rest. The problem was to estimate the forces on the body, due to the non-uniformity of the flow. Boussinesq (1885) published a modest *Note* at the Academy of Sciences [B18] in which he suggested a formula based on Stokes' approximation, but where he added a corrective term, as he was used to doing. That formula was rediscovered three years later [22] by the British scientist Alfred Basset (1854–1930), and the force observed by Boussinesq and Basset became known in Britain as the 'Basset force'. Boussinesq's corrective term, which is, essentially, unsteady, is also called '*the historical term of Basset–Boussinesq*'. Much later, in [23], the Swedish scientist Carl Oseen (1879–1944) will give a third set of terms for the equation, consistent with the *Oseen Approximation*, which he himself had found in order to solve Stokes' paradox. This equation, which governs the motion of particles floating in air, became known as the '*BBO equation*'. It will regular reappear in the preoccupations of hydraulic engineers and researchers (Faxen [24], Corrsin and Lumley [25], Gatignol [26], ...), and is still nowadays the subject of many scientific studies (see, for instance, Coimbra and Rangel [27] and the references herein).

[10] A version of the Korteweg–De Vries equation may be found in Boussinesq's book [B11], but Korteweg and De Vries did not seem have read this book in 1895. In fact, the mathematical relation between the two – the Korteweg–De Vries equation and Boussinesq's – only appears if one looks for approximate solutions of Boussinesq's equation. This property does not seem to be perceived very quickly: the first book I found which talks about the status of simple wave in the Korteweg–De Vries equation with respect to Boussinesq's (it describes one of its Riemann's invariants) is G.B. Whitham's book '*Linear and nonlinear waves*' [58], published almost a hundred years after these papers. Then the same observation is made by other authors (for instance J.W. Miles [59,60]). However, it is not noticed in the book of Drazin and Johnson [61], who nevertheless observe that the Boussinesq equation allows one to depict two solitary waves travelling together in opposite senses, while, of course, such motions cannot be depicted using Korteweg–De Vries equation.

Although the relation between their equations certainly puzzled Boussinesq (and probably Korteweg also), which can be understood from Auguste Boulanger's book [40], it must be pointed out that the theory of characteristics, which gives the explanation, only appeared in Fluid Mechanics in 1887 with Hugoniot's works [36]. Unfortunately, the untimely death of this scientist the same year came too soon for him to illustrate it.

[11] This is Stokes' flow around a sphere, and G.G. Stokes found the solution [62]. In the same piece of work Stokes came up with the '*Stokes' paradox*', which was solved later by C.W. Oseen [23], and, then, mathematically deciphered by I. Proudman and J.R.A. Pearson [63].

Boussinesq was still studying that equation; in 1903 in his book *Théorie analytique de la chaleur* [B28], he published a detailed study, then went on further [B35,B36,B42], to study the fall of water droplets in air by taking into account the influence of capillary forces.

2.5. The method of potential, the 'Boussinesq problem' and the vibrations of bars

The 'method of potential', which is actually simple when examined today, represented in a certain way the pinnacle of Boussinesq's works in Lille. From 1879 onwards (his first paper about that topics was a *Note* at the Academy of Sciences [B16], but other *Notes* published in 1878 had prepared this work) he looked over all the problems he had studied so far. He looked for potential functions, say ϕ, in the equations he had derived previously (such functions are not so difficult to find: for example, the stress function of a solid in static torsion is one of them) and first proposed a method to solve a problem governed by the Laplace equation $\Delta\phi = 0$, in the case where ϕ has spherical symmetry and a prescribed value in a small domain surrounding the singularity; one might say today that he determined the solution obtained for a Dirac point mass. The derivative with respect to one variable (say z) defined another solution, itself defined in a semi-infinite domain, and nil on the plane boundary, except at the singularity. Finally, integrating that solution when the source covers a (two- or three-dimensional) domain with a variable intensity, he obtained a general formula; to put it in contemporary terms, he had just realised a convolution.

After finding a solution of Laplace's equation, Boussinesq realised that it might not be the only equation that his method could be applied to. He then considered the biharmonic equation

$$\Delta\Delta\phi = 0 \tag{2}$$

the solution of which he could obtain first by solving $\Delta\psi = 0$, and then $\Delta\phi = \psi$. The method could therefore be easily generalized, and Boussinesq actually did it: in fact, he generalized it to the equation $(\Delta)^n\phi = 0$. Such equations are found in many elasticity problems: for example Laplace's equation is found in the torsion of bars, Eq. (2) is satisfied by the components of the displacements. However, the most famous problem Boussinesq solved was the so-called 'bodkin problem' (how to determine the stresses in a semi-infinite elastic body when a perpendicular force is applied on the horizontal surface), known today as the '*Boussinesq problem*'.

Finally Boussinesq was interested in the equation

$$\phi_{tt} = -\phi_{xxxx} \tag{3}$$

which describes transversal vibrations of beams; but that equation is also a linearized version of Eq. (1) written in a moving frame. A fundamental solution may be considered as a solution of the sudden immersion of a solid body in water. Hence the problem of surface waves got a wide array of solutions which did not appear in the 1877 papers.

All these results were published in another book [B19], which showed the way the author worked. In order to announce the outcome of his studies and to make them credible, he first gave a rather abundant collection of *Notes aux Comptes-Rendus de l'Académie des Sciences*. For the reader they constituted a summary of the final work. He then published a bulky dissertation (here: 722 pages) in which he presented not only his work in detail, but also how his ideas progressed. He often digressed on subjects that unintentionally turned out to be, in the end much more than appendixes. In this monograph [B19], following the main subject of the study (318 pages), he dealt with other topics in a 404-page-long 'appendix' dealing with the equations of vibrations of plates and of bars (which are solutions of the same Eq. (3)), and, finally, Boussinesq's equations of surface waves on liquids, which he had derived in the nonlinear case in 1877. It was his first important work on elasticity (after minor works, see [B5] and [B14]) and so on, see [B33]: his contribution to that field was later given a deserved recognition by his successors (e.g. Timoshenko [28]). When we read his works today, the progress of his thought is easily understandable, since it is guided by the structure of the solutions he looked for. But one cannot help to remark that separate papers might have been just as good: one about his mathematical problem, the others about applications.[12]

[12] And what can be said about the titles, which are sometimes many lines long! Again for the monograph [B17] the full title is: "*Application of the potentials to the study of equilibrium and of the motion of elastic solids, mainly to the calculation of the deformations and pressures generated in those solids by any efforts exerted on a small part of their surface or their interior*". A subtitle follows: "*Dissertation followed by extended notes on various points of mathematical Physics and analysis*".

The BBO equation and the potential solutions were Boussinesq's last scientific works in Lille, as during the session that took place on 18 January 1886, the Academy of Sciences elected Boussinesq to Eugene Rolland's chair (Rolland had died in 1885). To comply with the rules existing at the time, because of his election, he had to move to Paris. Boussinesq therefore requested and obtained a transfer[13] the same year, and was appointed *Professor of Physical and Experimental Mechanics* at the Faculty of Sciences of Paris.

3. Paris and the Academy of Sciences

Boussinesq was no dashing young man when he arrived in Paris at the beginning of 1886. He was a modest-looking man. Saint-Venant, his former mentor, was not there to greet him because, by an unlucky stroke of fate, he who supported Boussinesq with so much perseverance from the start of his career in Agde up to the Academy of Sciences, had just died, on 6 January, at the age of 89, twelve days before his pupil's election. Upon arrival among his new colleagues, Boussinesq proceeded to write his former mentor's obituary, in collaboration with Alfred Flamant, Saint-Venant's second favorite disciple. Such events contributed to his further withdrawing into himself; his only distraction was his research and he started working even more relentlessly.

Boussinesq remained Professor of Physical and Experimental Mechanics at the Sorbonne for ten years, until he inherited the more prestigious chair of *Mathematical Physics and Theory of Probabilities*, a position which he kept until his retirement in 1918 at the age of 76. As a Professor he wrote several pedagogical books (for instance [B20, B21] and others).

Boussinesq, a researcher, gave his full measure when he joined the Academy. First, he continued the research he had been doing in the field of turbulence in Lille. His ideas had obviously matured, as the scientific community, in general, had progressed in the past ten years. Maybe he tried to take up his early works when he started studying heat? At any rate, it is in this field that he made his greatest scientific contribution, when he formulated the 'Approximation'. Let us consider these two problems.

3.1. Turbulent flows (second phase): Boussinesq's hypothesis

Between 1876 and 1897, two fundamental discoveries were made in the field of incompressible Fluid Mechanics: the first (1883) was made by English scientist Osborne Reynolds (1842–1912). It was published in the *Philosophical Transactions of the Royal Society* [29], and was followed by several papers of lesser importance ([30], for example). The second was, in 1890, Maurice Couette's experiment [31]: it came as a new validation of the Navier–Stokes equations, while also confirming the existence of turbulent flows.[14]

Reynolds' experiment is well known, and mentioned in and every textbook. Perhaps Reynolds was lucky to find coherent structures in his experiments when he was exploring the turbulent field of flows. His great innovation was to decompose the variables in averaged values and fluctuations in the modelling of the flow. He showed how average values are governed by the Navier–Stokes equations, under the condition that another term coming from the fluctuations is added to the average stress-deformation law: that term is of tensorial nature and is called, today, the *Reynolds' turbulent stress tensor*. On the other hand, he had no way to model that tensor. He had just put forth the fundamental problem of turbulence, which is the "*problem of closure*".

Boussinesq took up the research he had been doing in Lille, and started reading Bazin's experimental papers[15] very attentively. He started by adopting (although he did not say much about how he changed the writing) Reynolds' de-

[13] Let us not believe that that operation happened with haste: Boussinesq had applied to be a Member of the Academy since ... 1872(!). He was writing detailed résumés to that effect since 1880 [B17]. Preparing for that, he had been in charge of the course of Physical and Experimental Mechanics at the Faculty of Sciences of Paris in 1885 since the beginning of the academic year.

[14] As pointed out by F. Schmitt [64], it seems that the expression 'turbulent flow' was coined by Kelvin [65]. Boussinesq spoke of 'tumultuous flow', while Reynolds spoke of 'sinuous flow'. The word '*turbulent*' comes from the Latin word '*turba*' (whirlpool), which reminds us that turbulence is associated with a property of the *vorticity* of the motion. This property is also one of the shear flows.

[15] In this respect, Boussinesq's papers [B24] is full of information concerning the justification of the hypotheses formulated by their author: Boussinesq relied very heavily on the measurements Bazin had made [15]. Henri-Emile Bazin had the reputation of being one of the most precise and reliable experimenters of his time, and this reputation was needed to justify the use that Boussinesq later made of those results.

composition into an average component and a fluctuation.[16] Examining later Bazin's measurements, he found out that the central question of turbulence lie in the neighbourhood of walls, or, as we say nowadays, in the area of maximum shear of the flow. Boussinesq studied this area carefully. He studied it in the cases given by Bazin's measurements: first the uniform flow in a canal with a rectangular section, then in a canal with a circular section, and, finally, for a flow with a sheared profile [B24].

Taking up the underlying idea of his 1872 paper [B6], Boussinesq attempted to model the contribution of turbulent terms with a friction coefficient, say ε, which integrated the contributions of the fluctuations in the average flow equation. Hence he formulated the equation

$$\rho g J + \varepsilon\, \mathrm{d}^2u/\mathrm{d}z^2 = 0 \tag{4}$$

where u denoted the average longitudinal velocity (along the x-axis) and J the slope of the piezometric line (equal to the quantity $\partial/\partial x\{p/(\rho g)\}$), which is constant if the average flow is uniform. The problem was to estimate ε. Qualitative considerations and comparisons with Bazin's results made him adopt hypotheses on the parameters determining that coefficient: for example, in a canal with a rectangular section, the coefficient was assumed proportional to the depth h of the canal, and to the speed u_0 of the flow on the wall. Scaling with ρg he wrote

$$\varepsilon = \rho g A^* h u_0 \tag{5}$$

In the case of a pipe with a circular section, observing that the rate of turbulence is stronger near the axis than on the periphery,[17] he chose

$$\varepsilon = \rho g A^*(D/4)u_0(R/r) \tag{6}$$

where D denoted the diameter of the pipe, r/R the scaled radial distance. The remaining problem was, now, to evaluate A^*: in both cases (5) and (6), Boussinesq reduced this calculation back to that of another coefficient, and then determined the values of the coefficient A^* by calibrating his numerical values on Bazin's data.

Boussinesq's hypothesis brought his author considerable credit, especially when later on Bazin [32], experimentally confirmed Boussinesq's formulae. His formulae led to a velocity profile proportional to $(r/R)^3$ in circular pipes, and Bazin found the same proportion. That hypothesis was, later on, taken up by Ludwig Prandtl (1875–1953), who improved it, and proposed a variant (known today as '*Prandtl's mixing length*' [33]) valid in boundary-layer flows. In formulae like (5) and (6) Prandtl could then get rid of the reference to a 'u_0 *speed at the wall*', which was incompatible with the no-slip condition at the wall surface.[18] The no-slip condition had been described (in 1898) by Hele-Shaw's experiments [34]. As early as 1904 Prandtl himself [35] had proposed the hypothesis of a laminar boundary layer separating a flow of speed u_0 (outside the boundary layer) from the wall.

3.2. An analytical theory of heat: the Boussinesq Approximation

Did Boussinesq want to take up his early works or just to keep himself up-to-date? At any rate, between 1901 and 1903 Boussinesq published two volumes as thick as usual [B26,B28], more than a thousand pages on a subject that had been often covered before him: the modelling of heat diffusion. But the studies on heat that we know by Fourier (1768–1830), Péclet (1793–1857), Clausius (1822–1888), Lamé etc. mainly dealt with static diffusion of heat in solids. Clearly Boussinesq's book did not aim at a scientific revolution: in the first volume [B26], he only wrote a synthesis of what had been done before him. However, in the second volume [B28], 198 pages of which really stuck to the point, he wandered into truly unexplored grounds: hydrodynamic convection.[19]

[16] It seems that Reynolds' model gave Boussinesq the spark he needed to formulate the 'constitutive law' that he had an intuition about in 1872 (apparition of terms that were not in the flow that were *fast enough* (in fact: *unsteady enough*)), but that he had no coherent way to calculate. However, nowhere in the 1897 dissertation did Boussinesq explicitly quote Reynolds: with a look at the 1872 and the 1877 dissertations that he himself had published on the subject, it can possibly argued that in his opinion, he should be given a part of the paternity of Reynolds' decomposition. This question is left to the historians of Science.

[17] The detailed calculations leading to formulæ (5) and (6) may be found in [B24]. The use of the Vaschy–Buckingham theorem would probably lead to the same results, but that theorem was only discovered in 1914.

[18] Prandtl [33] was the one who coined the phrase '*Boussinesq's hypothesis*'. For Boussinesq, the non-zero velocity at the wall depended on a roughness coefficient at the wall, which Boussinesq linked to a dry Coulomb friction exerted on the fluid by the wall.

[19] Boussinesq explained how his book progressively took another path (some of the changes were made by correcting the printed proofs of the book!) in the preface of the second volume. As usual, his book was followed by two appendixes: the first one (68 pages long) was called "*On the*

The nonstatic diffusion of heat and its effects in fluids were indeed becoming popular: in 1887 Pierre-Henri Hugoniot (1851–1887) gave his analytical theory of compressible fluids [36], and in 1889 Ernst Mach (1838–1916) realized the first *schlieren*[20] in nozzles [37]. But above all Henri Bénard (1874–1939) drew the attention of the scientific community to the effects of heating heavy liquids. He published (1900) an experimental study [38], in which he pointed out very curious convection effects, called nowadays *Bénard's cells*. Those cells are prismatic in shape with a horizontal hexagonal section, and only appear in the medium when the vertical temperature gradient reaches a certain value (now called *instability threshold*). Unfortunately Bénard, although a brilliant physicist, was unable to explain the phenomenon.[21]

In his 1903 volume Boussinesq very ingenuously formulated for the first time the conditions in which the famous 'Approximation' applied: "*One still had to observe that in most heat-induced motions of our heavy fluids, the volumes or densities are approximately conserved, although the corresponding variation of the weight of the unit of volume is actually the cause of the phenomena we are studying. One possibility stems from there: neglecting the variations of the density where they are not multiplied by the gravity g, while conserving its product by the gravity in the calculations*".[22] It is hard to say if Boussinesq's ambition was to model Bénard's problem. He only added that the question of the integration of the equations was "no longer unapproachable". However, he undoubtedly laid his finger on the particularity of this problem: if he had used nondimensional variables as we do today, and if he had introduced the characteristic numbers he had to use, he would have noticed that the *Froude number* (i.e. gravity effects) was competing with the *Mach number* (which defines compressibility). In fact, still unaware of dimensional analysis, he was unable to see that he had just identified a mathematical singularity. Moreover, just like the problem of surface waves, that competing effect was not sufficient to define the Approximation: there was a second condition that Boussinesq phrased as follows: "*The weight ρg of the unit of volume [...] will have approximately decreased of the quantity $\rho g \alpha \theta$, as if a small antagonistic [i.e. ascensional] force proportional to the temperature difference θ, but in the upward direction, had added itself to the initial or normal weight of the unit of volume*" (p. 174: θ denotes, in that sentence, the difference of temperature from its value at the equilibrium state).

Both phrases make up the so-called *Boussinesq Approximation* today. When first published, the Approximation went relatively unnoticed: its essential goal was to obtain the solutions of equations more easily which would have been very difficult to get without it. However, Boussinesq was in the habit of formulating simplified equations based on observations and experimental realism. So at first the Approximation was just a scientific oddity; it remained so until Rayleigh tried to model [39] the convective motions that Bénard had found in his experiments, and decided to use the Approximation to that effect (1916). Although Rayleigh made a minor mistake in the definition of his problem, he got results which were more or less the same as Bénard's[23]: under the antagonistic influences of heating and viscosity,

Resistance opposed to the small Motions of an indefinite Fluid by a Solid immersed in that Fluid". This appendix was the developed version of [B18]. The second appendix was 360 pages long, and dealt with the *Mechanical Theory of Light Waves*: it was actually the dissertation he had prepared for his thesis, and that he had never published, see note 3.

20 Schlieren (from the old-German word '*schliere*', *ray*) are photographical representations of propagation rays (Mach lines) in a supersonic flow (hence, of a compressible fluid), visualized through at the application of a property of varying fluid refraction index with respect to the speed (and, hence, with respect to the direction of the ray). By visualizing those rays, Mach validated Hugoniot's theory.

21 About Bénard's experiment, it is of interest to read the excellent paper of J.E. Wesfreid [66]. Bénard was not only incapable of explaining his results (we easily admit it, nowadays), but his examiners had not thought much more than him, as we may see from their judgement passed on his work: "[*a thesis which*] *did not bring significant results, by its further developments, to our knowledge*...".

22 Foreword for [B23], p. vii: that passage was quite rightly underlined by Zeytounian [4,5]. However, there are also two or three sentences preceding it in the foreword: Boussinesq explains why he had formulated his Approximation, how inspiring the works of Fourier and Poisson had been, and where his own formulæ should be in relation to these works. Let us quote in particular the following extract (p. vi): "*As to the fluids, where the visible motions can be very far-reaching, even under the influence of feeble causes, the characteristic equation of their temperatures, to add to the ordinary equations of hydrodynamics, was given first by Fourier (in a posthumous paper) in a form that was, all in all, good enough for the approachable questions, but that his immortal author inadvertently complicated somewhat needlessly. Poisson found it in its reduced and correct form*". Boussinesq gives details further on (p. 158–159) on the legitimacy of the linear dependence of the variations of ρ with respect to those of θ, and again pays tribute to Poisson for the practical justification of that hypothesis. Boussinesq claims to have taken it in Poisson's book "*Théorie mathématique de la chaleur*" [67]. Note that nowhere there is a reference to Oberbeck's (1846–1900) work [68] on the same subject.

23 We know nowadays (see for instance S. Chandrasekhar [69]) that the convection Bénard had actually observed mixed superficial tension effects and convection effects, which Rayleigh did not take into account in his modelling. We now call *Rayleigh–Bénard convection* the convection that Rayleigh modelled whereas Bénard's experiment showed a convection which is called *Bénard–Marangoni convection*. It must be noticed that the Bénard–Marangoni convection (and many others afterwards) might never have been modelled if the Rayleigh–Bénard convection had not been modelled first.

the fluid only starts moving when a parameter materializing the temperature gradient (and called today the *Rayleigh number*) reaches a certain threshold. Therefore Rayleigh's model promoted the Boussinesq Approximation.

When Boussinesq formulated his Approximation did he realise how far his idea would go? It is impossible to know; on the one hand, when he formulated it, he was only trying out the method he had already devised for other problems. On the other hand, he meticulously positioned it in relation to other scientists' analyses – which he evidently admired and respected (see note 22) – in a way that showed that he was aware of the importance it might gain in the future. At any rate, the Approximation gave the opportunity to explore a very large field of applications. Such cases are not rare in other areas of physics.

The scientific community gradually realised the importance of what Bénard, Boussinesq and Rayleigh had found with joint efforts. Indeed Bénard's cells are forms to be found in nature in a great number of circumstances: the convection of a liquid, meteorological flows, dynamo effects, and even sand-drying cells in the desert. These phenomena were studied mainly during the second half of the 20th century, which explains why the Approximation became so popular during that period.

4. The last years of a solitary scientist

Boussinesq did not slacken the pace of his publications until his retirement in 1918. His research (see [B29,B30, B31,B32,B33,B34,B35,B36,B37,B38,B39,B40,B41,B42], the list is not exhaustive) dealt with varied subjects: in addition to the problems he had been studying ever since his younger years ([B29,B32,B39,B40]) he looked at new topics: longitudinal waves in elastic tubes [B33], flow in a spillway [B22,B31], capillary effects [B35,B36,B42], or water filtered through sand [B37]. He indulged in philosophical comments [B38]. He also published his Sorbonne lectures, which can be compared to modern 'handouts' and manuals. Following this example, Auguste Boulanger published two volumes entitled "*Hydraulique générale*" [40], in which he stated that the purpose of the book was a pedagogical synthesis aimed at engineers describing Boussinesq's discoveries in fluid mechanics since 1871.[24]

One should mention two research subjects which occupied Boussinesq throughout his entire life: the first concerns the study of *light propagation* partially published in many short papers (e.g. [B2,B8,B23,B25,B27,B41]), and partially published as an appendix in the book [B28] (see also note 3). In these studies Boussinesq attempted to obtain a *mechanical modelling* of the medium (the 'ether') which was considered as an elastic medium. Unfortunately another approach (initiated by de Broglie) later partially ruined Boussinesq's discoveries in that area. The second topics dealt with some philosophical aspects of sciences: in many papers Boussinesq raised questions about the deep basis of mechanical principles ([B9,B34,B38]). One cannot ignore similar questions raised by Duhem, concerning both thermodynamics and mechanics.

As I said earlier, Boussinesq retired from University in 1918. At that time, he was very lonely. Married three times, with no children, he had separated from his third wife nine years before.[25] Flamant had died in 1914, Bazin in 1917, and there is no way of knowing if he had any other close friends. Withdrawn by nature, he became more and more solitary. In his obituary in 1933, his colleague Emile Picard only talked about Boussinesq's last years as an academic. However, even academically, Picard stressed how introverted his colleague had been. One cannot help but wonder what were his relationships with his colleagues at the Academy, what kind of exchanges did he have with Henri Poincaré for instance,[26] who, although twelve years his junior, passed away in 1912. Picard pointed out, however, that there was no bitterness in Boussinesq's solitude. Although shy and withdrawn, he never judged people severely. Every

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[24] Auguste Boulanger (1866–1923) was not strictly speaking a former student of Boussinesq. At this time he was Professor of Mechanics at the Faculty of Sciences of Lille, and later (1914) he became Professor at the *Conservatoire National des Arts et Métiers*. His book *Hydraulique générale* was published by Doin in 1909 in the "*Bibliothèque de Mécanique appliquée et Génie civil*", a collection directed by Maurice d'Ocagne. The purpose of this collection was to give engineers access to the most recent results in the field of technological sciences. Boulanger's book is a detailed explanation of more than sixty of Boussinesq's papers dealing only with Fluid Mechanics, published between 1867 and 1909. These papers mainly dealt with surface waves, turbulence, and motions including viscosity: the Approximation was too recent, and not sufficiently understood at that time, to be included.

[25] Boussinesq successively married Jeanne Giscard de la Roque (1867), who died in 1894; Claire Onfroy de Véretz (1895), who died in 1905; and Jeanne Le Bouteiller (1906) from whom he separated in 1909. None of them gave Boussinesq any children.

[26] Some letters between the two scientists can be found in the archives of the Academy, for example a letter from Poincaré (1891) asking Boussinesq for his opinion concerning a thesis which he thought was wrong. However, he did not want to turn it down without a colleague's advice.

day, almost until the end, he would come to the library at the Academy and sit at the same table. That was his only link with the outside world.

Boussinesq died in Paris on 19 February 1929. Except for some testimonies about his solitary life, not much more is known about his last years. His whole life, either private or professional, remains somewhat shrouded in mystery.

5. Boussinesq's modern legacy

A hundred years later, let us try to understand what remains today of the formulæ and equations mentioned earlier.

The main aspect of Boussinesq's talent is unquestionably his finesse and extraordinary intuition. During the 19th century many experiments about numerous and various subjects were performed, and he always managed to pick out their essential points in order to interpret them and deduce consistent modelling patterns, although he sometimes had to make hypotheses he was not able to justify. Unfortunately, because he was so reserved, he never gained the reputation nor the influence he deserved. However, just like Duhem, his main interests were in macroscopic physics, at a time when the great problems of physics were veering towards exploration of the atom. Therefore, just like Duhem, he sank into oblivion in the years following his death. Only in the last three or four decades of the 20th century did fundamental researchers find renewed interest in macroscopic physics. The development of new materials, the improved knowledge of dynamic behaviour patterns in media with complex rheology, etc.[27] helped restore these scientists to their deserved place.

The Approximation is obviously the most important part of Boussinesq's legacy. It remained a mystery during the first part of the 20th century (the researchers who were studying Bénard's flows were trying to understand how hexagonal and not rectangular cells or even rolls could take form). Hence it might have remained an oddity, as I said earlier, if it had not had the same destiny as Bénard's convection. That convection had been progressively acknowledged as one of the motive forces of a phenomenon that can be found in nature in the most varied circumstances (see, for instance, Lesieur's book [41]). Meteorologists for instance (Meteorology became a science around 1925) had recognized the convective nature of many common meteorological phenomena: for example, clouds forming in the atmosphere. Explaining such a phenomenon as Bénard's convection required analysing the equations more rigorously than Rayleigh or Boussinesq did. The deep foundations of these equations had to be studied, and were found in their asymptotic formulation,[28] stricter than the mere numerical evaluations which had inspired Boussinesq. From the 1970s onwards, explanations have been found: they showed, on the one hand, where to find 'non-Boussinesq cases' (as we say nowadays) and, on the other hand, what these cases were (for example, the *anelastic approximation* is one of them). There also were indirect effects of the asymptotic analysis: other fields of applications for the Approximation were discovered, such as the internal gravity waves.[29] More generally, there is perhaps one last reason for the revival of a 'Boussinesq effect' in the scientific world today, as we are becoming more concerned with protecting our environmental heritage. The modelling of global warming[30] also has to do with Boussinesq's approximation.

The conditions for the Approximation to be valid were not the only conclusions found in those works: Bénard convection was recognized as one of the simplest models to describe hydrodynamic instability phenomena, the convection instability leading to the formation of horizontal rolls, and Bénard cells turning out to be degenerate rolls by accumulation of nonlinearities. Such cells can also degenerate into more and more unstable patterns, becoming progressively turbulent. In short, Boussinesq's Approximation is a necessary prerequisite to depict the so-called '*tran-*

[27] One example is the double diffusive behaviour, which governs the diffusion of pollutants in environmental fluids like the oceans or the atmosphere.

[28] The asymptotic justifications of Boussinesq's Approximation are especially due to a strong French school based in Lille (Zeytounian [70], Bois [71]): these papers were both followed by many others.

[29] Even better things were found: the mathematical study of the Approximation was done in the 1970s. When it was mathematically formulated, it simultaneously showed, however, that the writing that Rayleigh and his successors did (including Chandrasekhar!) was not asymptotically coherent. Saint-Venant (see Picard [17]) was always blaming Boussinesq for a lack of mathematical precision in his demonstrations, which was due to Boussinesq's tendency to foresee a result beyond the formalism that led there. If Boussinesq or Rayleigh had been more inspired by a better mastery of the asymptotic tools used, the Approximation might not have been formulated, let alone used, because its lack of precision in the 1903 formula disturbed a lot their successors of the first half of the 20th century. For more than fifty years, the only justification of the Approximation was its concordance with the experimental results it was compared to.

[30] Among those phenomena are the warming due to the excess of carbon dioxide in the air; the consequences of the melding of the ice caps in the oceans; the disappearance of the oceanic currents like the Gulf stream, ...

sition to turbulence' through Bénard convection. Let us dream for a while: at the very beginning of the 21st century, the problem of the validity of Navier–Stokes equations is not yet solved.[31] The first step toward bringing a solution to this problem would probably be to solve the question of the transition to turbulence. Hence one can clearly see the involuntary importance that Boussinesq's modest approximation took.

Closely related to the validity of the Navier–Stokes equations is the problem of turbulent closure. *Boussinesq's hypothesis* (also called the *mixing length closure*) is today in all books dealing with turbulent fluid mechanics. In fact, one must admit that it is interesting because it is simple. Prandtl himself presented this hypothesis in 1925, but as he noticed the importance of the shear when the turbulence appeared, he proposed a closure which took into account the shear near the boundaries. Its advantage was to respect the condition of nil velocity on the boundaries. At any rate, considering the progress made in the study of turbulence from 1941 onwards with the works [42] of Kolmogorov (1903–1987), Prandtl's closure as well as Boussinesq's are interesting mainly because they can give quick estimations of turbulent solutions.[32] However, research in turbulence is now focused on other aspects: spectral methods, 'several-point' closures etc.

To conclude with fluid mechanics: I said earlier that, since the *BBO equation* helps explain the motion of particles floating in air, one should stress the importance it has taken today in the field of chemical engineering (settling and decanting, aerosols, transportation of dusty materials) and in the protection of the environment (takeoff, transportation and diffusion of polluting particles, behaviour of aerosols in the high atmosphere). In more concrete terms, the BBO problem is today at the origin of the validation of powerful techniques such as holographic velocimetry in fluid mechanics (see, for example, Dadi et al. [43]).

Let me say a few words on elasticity problems (bars and plates): they are not so popular in modern research because almost all of them are now considered to be classical problems, and so in textbooks there are references to *Boussinesq's tensor* (Germain [44]), *Boussinesq's solution* (Timoshenko and Goodier [45]) etc. Let us consider these 'classical problems' for a moment: most often they are modelled by linear equations, and their solutions help solve concrete problems linked to these equations. Modern theories (essentially that of *distributions* [46]) allow one to formulate them within a rigorous mathematical framework. We should thank, not only mathematicians for formalizing this study, but also scientists like Boussinesq who were more inspired by physics: their method, which was becoming universal to solve linear systems, gave mathematicians the idea to find its mathematical expression. This is a great example of a collaboration between *physics* and *mathematics* through *mechanics*.

One of the great qualities of Boussinesq's problem (the 'bodkin problem') is its simplicity: among the practical results that can be extracted is the relation between *the depth of the bodkin* when it is driven in (for a given load) and *Young's modulus* of the material punched by a concentrated force. It is therefore a practical way to measure Young's modulus. An improved version of this method is still used today (see Giroud [47]), especially in rock mechanics. It now takes many factors into account, such as the thickness of the bodkin, the locally plastic effects etc (see Henry et al. [48]).

Like Boussinesq hypothesis for turbulence, Boussinesq modelling of pulverulent media can be considered as a first approximation of solutions which researchers are trying to improve. In this area, however, fundamental research is still in its infancy and the advantage of Boussinesq's modelling is to give solutions which are qualitatively realistic. Boussinesq's modelling is used in civil engineering, at least to give leading order approximations that are then refined by successive iterations.

Nowadays we are lucky to have a remarkable *History of Mechanics* (1950) by René Dugas [49]. If one is interested to know how he imagines the future of mechanics and in what directions it is developing, one finds that Dugas is attracted to *relativistic mechanics* (then developed by Einstein) and *quantum mechanics* (developed by Louis de Broglie). Fifty years later with more hindsight, we are able to assess what the 20th century has meant for mechanics. What conclusions can we draw? Apart from these fundamental aspects of mathematical physics mentioned in the introduction (the 'three pillars' of mechanics) other facets had been discovered, although scientists could not even guess

[31] In 2000 the *Clay Mathematics Institute* (University of Princeton) offered a prize – after the example of Hilbert's 23 problems in 1900 – of one million dollars for seven problems which should interest the 21st century. Problem n° 6 (Ch. Fefferman [72]) concerns the "*Existence and regularity of the Navier–Stokes equations*".

[32] In fact this statement is not entirely true: because of their simplicity both Prandtl's closure and Boussinesq's closure are very popular for applications, mainly through their variant called the 'k–ε model'.

their existence half-way through the 20th century. A few examples are: rheology, aviation, hydrodynamic instability and turbulence.

Thanks to the discovery of thermodynamics of irreversible processes, *rheology* (again part of physics) made it possible to carry out the systematic exploration of new or more complex materials. Eugene Bingham (1878–1945) started its investigation in 1916 [50]. It evolved on a much more fundamental level with Lars Onsager (1903–1976) and then Ilia Prigogine (1917–2003). *Aviation* – the archetype of the engineering sciences – was developed between 1906 and 1950[33] under the impulse of scientists like Prandtl and his Göttingen school: Richard Blasius (1883–1970), Albert Betz (1885–1968), Jakob Ackeret (1898–1981), Adolf Busemann (1901–1986) etc. Last, but not least, *hydrodynamic instability*, half-way between physics and engineering science, is in some ways the link between the deterministic modelling of Navier–Stokes equations and the stochastic modelling of turbulence, and it reached its current level of development because it started with Boussinesq's modelling. One of its most spectacular applications was perhaps, in the field of astrophysics, when Subramanyan Chandrasekhar (1910–1998, Nobel prize for Physics in 1983 for the discovery of the so-called Chandrasekhar's mass), understood the life and death of stars. If, finally, Boussinesq does find his place midway between Poisson and Chandrasekhar, is it such a bad outcome after all?

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