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Boussinesq equation, elasticity, beams, plates

Two-dimensional Boussinesq equation in a disc and anisotropic Sobolev spaces

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Abstract

The two-dimensional damped Boussinesq equation with a forcing term is considered in a unit disc. It governs forced, small, nonlinear oscillations of a thin elastic membrane in the presence of viscosity. The eigenfunction expansion method is used for constructing global-in-time solutions of the initial-boundary-value problem in question. Specially designed anisotropic Sobolev spaces are introduced in order to reflect the effect of nonlinear smoothing in the angular coordinate. Existence and uniqueness in these spaces are proved on the basis of the construction. ***To cite this article: V. Varlamov, C. R. Mecanique 335 (2007).***

Résumé

L'équation de Boussinesq à deux dimensions dans un disque et les espaces de Sobolev anisotropes. L'équation de Boussinesq amortie à deux dimensions avec terme de forçage est considérée dans un disque unité. Elle gouverne les petites oscillations non linéaires, forcées d'une membrane élastique fine en présence de viscosité. La méthode de développement en fonctions propres est utilisée pour la construction des solutions globales en temps de problème mixte considéré. Des espaces anisotropes de Sobolev spécialement conçus, sont introduits pour démontrer l'effet de régularité non linéaire dans la coordonnée angulaire. L'existence et l'unicité dans ces espaces sont prouvées sur la base de cette construction. ***Pour citer cet article : V. Varlamov, C. R. Mecanique 335 (2007).***

Keywords: Vibrations; Boussinesq equation; Disc; Anisotropic Sobolev spaces

Mots-clés : Vibrations ; Équation de Boussinesq ; Disque ; Espaces de Sobolev anisotropes

1. Introduction

The Boussinesq equation is well known in the context of describing small nonlinear oscillations of elastic beams. It can be deduced from the system of equations originally proposed by J. Boussinesq in the papers [1,2] and can be written in the form

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta (u^2)_{xx} \tag{1}$$

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where $\alpha = const > 0$ is the dispersion parameter depending on the compression and rigidity characteristics of the material, $\beta = const \in \mathbf{R}$ is the coefficient controlling nonlinearity, $u(x,t)$ is the vertical deflection, and the quadratic nonlinearity $(u^2)_{xx}$ accounts for the curvature of the bending beam. Eq. (1) is known as the "good" Boussinesq equation. It has a counterpart in the theory of long wave propagation on the surface of shallow water, namely (1) with $\alpha = const < 0$ [3]. The latter received the name of the "bad" Boussinesq equation due to its linear instability.

There exists an extensive literature on the spatially 1D Boussinesq equation and its generalization, when $(u^2)_{xx}$ is replaced by $(f(u))_{xx}$ (see [4–11] and the references therein). However, the 2D version of this equation was much less studied. A 2D "bad" Boussinesq equation

$$u_{tt} = u_{xxxx} + u_{xx} + u_{yy} + 3(u^2)_{xx} \tag{2}$$

was proposed in [12] to describe the propagation of surface gravity waves and in particular the head-on collision of oblique waves. In this model the main wave propagation takes place in the x-direction with weak transverse effects in the y-direction. An initial-value problem for the 2D "bad" Boussinesq equation with a potential,

$$u_{tt} = 3\Delta^2 u + \Delta u - 12\big(V(x) * \Delta\big(|u|^\lambda u\big)\big), \quad x \in \mathbf{R}^2,\ t > 0$$

was considered in the paper [13]. For small initial data, higher-order nonlinearity, $\lambda > 8$, and sufficiently smooth potential $V(x)$ it was shown that a global-in-time solution exists and satisfies the estimate $\|u(\cdot,t)\|_{L_\infty} \leqslant c(1+t)^{-7}$.

In many practical situations damping effects are comparable in strength to nonlinear and dispersive ones and in such cases (1) requires a dissipative term [14]. A convenient item responsible for internal friction is $-2bu_{txx}$ with $b = const > 0$ (see, e.g., [15]). After adding it to the left-hand side of (1) it takes the form

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx} \tag{3}$$

A 2D version of (3) (sometimes called a damped plate equation [16]) is

$$u_{tt} - 2b\Delta u_t = -\alpha\Delta^2 u + \Delta u + \beta\Delta(u^2) \tag{4}$$

where $\alpha, b = const > 0$, $\Delta^2 = \Delta\Delta$ and $u(x,y,t)$ is a vertical deflection. It appears in the context of modeling small nonlinear oscillations of elastic membranes in the presence of viscosity [16].

Initial-boundary-value problems for (3) in 1D, 2D and 3D with small initial data were considered in [17–19], and the global-in-time mild solutions were constructed in the Sobolev spaces $H^s(\Omega)$ (where $s < 1$ for 2D and $s < 3/2$ for 3D). The method of eigenfunction expansions was used for constructing solutions of nonlinear problems. It is important to emphasize an essential difference of this approach from Galerkin's method (see [9] for comparison). The latter is based on projection onto a finite-dimensional space of eigenvectors and the use of a subsequence converging to the solution in question. In this way existence and regularity are established, but solutions are not constructed. The method proposed in [17–19] uses projection onto an infinite-dimensional space of eigenvectors and the proof of convergence of the corresponding series in an appropriate function space. As a result, solutions are constructed, and solvability follows from the construction. In the present work for the first time the main function space is adapted for the circular geometry. It is defined as anisotropic Sobolev space $\mathcal{H}^{s,\alpha}(\Omega)$ endowed with the norm (13). Here s is the usual Sobolev index and α is responsible for additional smoothness in the angular coordinate θ. These spaces should not be confused with anisotropic Sobolev spaces of [20]. In $\mathcal{H}^{s,\alpha}(\Omega)$ regularity of solutions can be raised due to the smoothing effect with respect to θ. In the current paper, instead of restrictive assumptions on the initial data (see [18]) it is assumed only that the source term belongs to $L_2(\Omega)$ with respect to spatial coordinates. The use of new special functions, convolutions of Rayleigh functions with respect to the Bessel index, introduced in [21,22], allows us to construct mild solutions in $\mathcal{H}^{s,\alpha}(\Omega)$ with $s + \alpha \leqslant 2 - \varepsilon$ and $s < 3/2$ for any $\varepsilon > 0$.

For a given Bessel function $J_\nu(x)$ Rayleigh functions are defined by the series (see, e.g., [23], p. 502)

$$\sigma_l(\nu) = \sum_{n=1}^{\infty} \lambda_{\nu,n}^{-2l}$$

where $l = 1, 2, \ldots$; $\lambda_{\nu,n}$ is the zero of the Bessel function and n is the number of the zeros. These functions have been used in solving classical linear problems of vibrating drumheads, heat conduction in cylinders and Fraunhofer diffraction through circular apertures. Convolutions of Rayleigh functions with respect to the Bessel index,

$$R_{i,j}(m) = \sum_{k=-\infty}^{\infty} \sigma_i\big(|m-k|\big)\sigma_j\big(|k|\big), \quad m \in \mathbb{Z} \tag{5}$$

were introduced in the papers [21,22] in order to treat quadratic nonlinearities for semi-linear equations in circular domains. They have representations

$$R_{i,j}(m) = \sum_{k=0}^{|m|} \sigma_i\big(|m|-k\big)\sigma_j(k) + \sum_{k=1}^{\infty} \sigma_i\big(|m|+k\big)\sigma_j(k) + \sum_{k=1}^{\infty} \sigma_i(k)\sigma_j\big(|m|+k\big)$$

The use of these special functions allows one to reveal the smoothing effect due to periodicity in θ. The convolution structure of $R_{i,j}(m)$ is a consequence of orthogonality of the angular eigenfunctions $\{\mathrm{e}^{\mathrm{i}m\theta}\}_{m=-\infty}^{\infty}$.

2. Problem statement and preliminaries

Denote by Ω a disk of a unit radius and put the origin of the coordinate system in its center, so that in polar coordinates $\Omega = \{(r,\theta)\colon r<1,\ \theta\in[-\pi,\pi]\}$. Let $\partial\Omega$ denote its boundary, i.e. $\partial\Omega = \{(r,\theta)\colon r=1,\ \theta\in[-\pi,\pi]\}$. Consider the following initial-boundary-value problem for a real function u:

$$\begin{aligned}
&u_{tt} - 2b\Delta u_t = -\alpha\Delta^2 u + \Delta u + \beta\Delta(u^2) + f, \quad (r,\theta)\in\Omega, \quad t>0\\
&u(r,\theta,0) = u_t(r,\theta,0) = 0, \quad (r,\theta)\in\Omega\\
&u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \quad t>0\\
&u(r,\theta+2\pi,t) = u(r,\theta,t), \quad (r,\theta)\in\Omega,\ t>0\\
&\big|u(0,\theta,t)\big| < \infty
\end{aligned} \tag{6}$$

where α, $b = const > 0$, $\beta = const \in \mathbf{R}$ and $f = f(r,\theta,t)$ is a real function. The above boundary conditions correspond to a simply supported boundary (see [24,25]). We restrict our attention to the case of small damping $\alpha > b^2$ which is the most interesting one from both mathematical and physical points of view. It corresponds to existence of an infinite number of damped oscillations. For $b^2 > \alpha > 0$ the analysis is even simpler, but aperiodic processes play the main role. Therefore this case is less interesting. Considering inhomogeneous initial conditions is not a problem and can be done along the lines of [18] with improvements offered in the current work but would make our formulas somewhat unwieldy.

According to the method proposed in [17–19], we seek solutions of the problem (6) in the form of a series

$$u(r,\theta,t) = \sum_{m,n} \hat{u}_{m,n}(t)\chi_{m,n}(r,\theta) \tag{7}$$

Here and in the sequel $\sum_{m,n}$ stands for $\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}$ and $\chi_{m,n}(r,\theta)$ are the eigenfunctions of the Laplace operator in a disc, namely nontrivial solutions of the boundary-value-problem

$$\begin{aligned}
&\Delta\chi = -\Lambda\chi, \quad (r,\theta)\in\Omega\\
&\chi|_{\partial\Omega} = 0\\
&\chi(r,\theta+2\pi) = \chi(r,\theta), \quad (r,\theta)\in\Omega\\
&\big|\chi(0,\theta)\big| < \infty
\end{aligned} \tag{8}$$

corresponding to the eigenvalues $\Lambda_{m,n} = \lambda_{m,n}^2$. They are

$$\chi_{m,n}(r,\theta) = J_m(\lambda_{m,n} r)\mathrm{e}^{\mathrm{i}m\theta}, \quad m\in\mathbf{Z},\ n\in\mathbf{N}$$

where $J_m(x)$ are Bessel functions of index m and $\lambda_{m,n}$ are their positive zeros numbered in the order of increasing magnitudes ($n = 1, 2, \ldots,$ is the number of the zero). In this way we satisfy the boundary and periodicity conditions in (6). Notice that for $m\in\mathbf{N}$:

$$J_{-m}(z) = (-1)^m J_m(z) \tag{9}$$

Introduce the complex space $L_2(\Omega)$ endowed with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Then, according to [26], p. 219, the following relations hold:

$$\|\chi_{m,n}\|^2 = 2\pi\,\big\|J_m(\lambda_{m,n}\cdot)\big\|^2_{L_2(0,1)} = \int_0^1 rJ_m^2(\lambda_{m,n}r)\,\mathrm{d}r = \frac{1}{2}J_{m+1}^2(\lambda_{m,n})$$

For sufficiently large $q > 0$ there exist such positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$ that (see [26], p. 219)

$$\frac{\mathcal{C}_1}{q} \leqslant \left\| J_m(q\cdot) \right\|^2 \leqslant \frac{\mathcal{C}_2}{q} \tag{10}$$

For bounded m, large positive zeros of $J_m(z)$ have the following asymptotic expansion (McMahon's expansion, see [27], p. 247):

$$\lambda_{m,n} = \upsilon_{m,n} + \mathrm{O}\left(\frac{1}{\upsilon_{m,n}}\right), \quad \upsilon_{m,n} = \left(m + 2n - \frac{1}{2}\right)\frac{\pi}{2} \text{ for } n \to +\infty \tag{11}$$

We shall use $L_2(\Omega)$-based Sobolev spaces $H^s(\Omega)$, $s \in \mathbf{R}$, equipped with the norm

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_s = \sum_{m,n} \lambda^{2s}_{m,n} |\hat{u}_{m,n}|^2 \|\chi_{m,n}\|^2$$

where $\hat{u}_{m,n} = \|\chi_{m,n}\|^{-2}\langle u, \chi_{m,n}\rangle$ are the complex Fourier–Bessel coefficients of the function u and $\lambda_{m,n} > 0$ for all $m \in \mathbf{Z}$, $n \in \mathbf{N}$. Let $H^s_0(\Omega)$ with $s \geqslant 0$ be the completion of the space $C^\infty_0(\Omega)$ endowed with the inner product $\langle\cdot,\cdot\rangle_s$ and the corresponding norm $\|\cdot\|_s$. For $s = 0$ the subindex is omitted in the notation of the inner product and the norm. The space $L_\infty(\mathbb{R}^+, H^s(\Omega))$ is equipped with the norm

$$\|u\|_{L_\infty(\mathbb{R}^+, H^s(\Omega))} = \operatorname*{ess\,sup}_{t \geqslant 0} \left\|u(\cdot, t)\right\|_s$$

Definition 2.1. For any $s, \alpha \in \mathbf{R}$ *anisotropic Sobolev space* $\mathcal{H}^{s,\alpha}(\Omega)$ is defined as a closure of the space $C^\infty(\Omega)$ in the norm

$$\|u\|^2_{s,\alpha} = \sum_{m,n} \left(1 + |m|^2\right)^\alpha \lambda^{2s}_{m,n} |\hat{u}_{m,n}|^2 \|\chi_{m,n}\|^2 \tag{12}$$

The representation (12) can also be written in the operator form

$$\|u\|^2_{s,\alpha} = \left\|(I - \partial^2_\theta)^\alpha (I - \Delta)^s u\right\|^2 \tag{13}$$

Denote by $\mathcal{H}^{s,\alpha}_0(\Omega)$ a closure of $C^\infty_0(\Omega)$ in the norm (12).

3. Main results

Integration of Eq. (6) with respect to t leads to a nonlinear integral equation which serves for the definition of a mild solution of the problem (6). Denote by A the operator $-\Delta$ defined on sufficiently smooth functions $\chi(r, \theta)$ satisfying the conditions (8) and let

$$\omega(A) = \left(\mu A^2 + A\right)^{1/2}, \quad \text{where } \mu = \alpha - b^2 > 0$$

Definition 3.1. A function $u(t)$ is called a *mild solution* of the problem (6) if it satisfies the integral equation

$$u(t) = \int_0^t \exp\left[-b(t-\tau)A\right]\left(\omega(A)\right)^{-1} \sin\left(\omega(A)(t-\tau)\right) f(\tau)\,\mathrm{d}\tau - \beta \int_0^t \exp\left[-b(t-\tau)A\right]\left(\omega(A)\right)^{-1} \sin\left(\omega(A)(t-\tau)\right) A u^2(\tau)\,\mathrm{d}\tau$$

in the Banach space $C_b(\mathbb{R}^+, \mathcal{H}^{s,\alpha}_0(\Omega))$. Here the operator function $\exp(-btA)$ and the nonlinear term $Au^2(t)$ are understood via eigenfunction expansions (see below (23) and (26)).

Our main result is the following statement.

Theorem 3.2. *If* $\alpha > b^2$, $f \in L_\infty(\mathbb{R}^+, L_2(\Omega))$ *and* $\|f\|_{L_\infty(\mathbb{R}^+,L_2(\Omega))} \leqslant 3b\sqrt{2\pi\mathcal{C}_1\mu}(2\pi^2|\beta|\tilde{C})^{-1}$, *where* $\mathcal{C}_1$ *and* $\tilde{C}$ *are the constants from* (10) *and* (17)*, respectively, then there exists a mild solution of the problem* (6) $u \in C_b(\mathbb{R}^+, \mathcal{H}_0^{s,\alpha}(\Omega))$ *with* $s+\alpha \leqslant 2-\varepsilon$ *and* $s < 3/2$ *for any* $\varepsilon > 0$. *For* $s+\alpha \geqslant -2+\varepsilon$ *and* $s \geqslant -3/2$ *this solution is unique. It can be represented in the form*

$$u(r,\theta,t) = \sum_{m,n} \hat{u}_{m,n}(t) J_m(\lambda_{m,n} r) \mathrm{e}^{\mathrm{i}m\theta} \tag{14}$$

where the coefficients $\hat{u}_{m,n}(t)$ *are defined by* (27) *and* (20) *and convergence is understood in the sense of* $\mathcal{H}^{s,\alpha}(\Omega)$.

The proof of the theorem is approached through a series of lemmas.

Lemma 3.3. *If* $u \in L_2(\Omega)$, *then*

$$|\hat{u}_{m,n}| \leqslant \frac{\|u\|\sqrt{\lambda_{m,n}}}{\sqrt{2\pi\mathcal{C}_1}} \tag{15}$$

where $\mathcal{C}_1$ *is the constant from the estimate* (10).

Proof. Follows from the definition of Fourier–Bessel coefficients and the Cauchy–Schwartz inequality. □

The analysis to follow is essentially based on the expansion of the nonlinearity u^2 into the eigenfunction series. This involves the use of projection coefficients (see (23))

$$b(m,n;p,q,k,s) = \frac{\langle \chi_{p,q}\cdot\chi_{k,s}, \chi_{m,n}\rangle}{\|\chi_{m,n}\|^2} = \frac{\int_0^1 J_m(\lambda_{m,n}r)J_p(\lambda_{p,q}r)J_l(\lambda_{l,s}r)r\,\mathrm{d}r}{\|J_m(\lambda_{m,n}\cdot)\|^2}. \tag{16}$$

Lemma 3.4. *The following inequality holds for all* $m, p, k \in \mathbf{Z}$ *and* $n, k, s \in \mathbf{N}$

$$\left|b(m,n;p,q,k,s)\right| \leqslant \tilde{C}\sqrt{\frac{\lambda_{m,n}}{\lambda_{p,q}\lambda_{k,s}}} \tag{17}$$

where the constant $\tilde{C}$ *is independent of* m, n, p, q, k *and* s.

Proof. See [18]. □

The next statement is a slight modification of Lemma 4.4 of [19].

Lemma 3.5. *For any fixed integers* $n \geqslant 1$, *any integers* $m, p, k \geqslant 0$; $q, s \geqslant 1$ *and* $\lambda_{p,q}, \lambda_{k,s} \to \infty$ *there exists a constant* C *independent of* m, n, p, q, k, s *such that the following inequalities hold*:

$$\left|b(m,n;p,q,k,s)\right| \leqslant C_2\sqrt{\lambda_{m,n}} \begin{cases} \lambda_{p,q}^{-3/2}\lambda_{k,s}^{-1/2}, & \lambda_{p,q} > \lambda_{k,s} \\ \lambda_{p,q}^{-1/2}\lambda_{k,s}^{-3/2}, & \lambda_{p,q} < \lambda_{k,s} \\ \lambda_{p,q}^{-1}, & \lambda_{p,q} = \lambda_{k,s} \end{cases}$$

The following lemma is a summary of asymptotics obtained in [21] and [22].

Lemma 3.6. The functions $R_{i,j}(m)$, $i, j = 1, 2$, *defined by* (5) *are bounded for* $m \in \mathbf{Z} \cup \{0\}$ *and have the following asymptotic expansions for* $|m| \to \infty$:

$$\begin{aligned} R_{1,1}(m) &\sim \frac{1}{4}\left[\frac{\ln|m|}{|m|} + \frac{2\gamma-1}{2|m|} - \frac{\ln|m|}{|m|^2} + \mathrm{O}\left(\frac{1}{|m|^2}\right)\right] \\ R_{2,1}(m) &\sim \frac{1}{2^6}\left[\left(\frac{\pi^2}{3}-\frac{5}{2}\right)\frac{1}{|m|} + \left(\frac{\pi^2}{3}-\frac{5}{2}\right)\frac{1}{|m|^2} + \mathrm{O}\left(\frac{\ln|m|}{|m|^3}\right)\right] \\ R_{2,2}(m) &\sim \frac{1}{2^8}\left[\left(\frac{\pi^2}{3}-\frac{5}{2}\right)\frac{1}{|m|^3} + \left(\frac{\pi^2}{3}-\frac{5}{2}\right)\frac{1}{|m|^4} + \mathrm{O}\left(\frac{\ln|m|}{|m|^5}\right)\right] \end{aligned} \tag{18}$$

Set

$$\hat{v}^{(0)}_{m,n}(t) = \int_0^t \mathrm{e}^{-b\lambda^2_{m,n}(t-\tau)} \frac{\sin[\omega_{m,n}(t-\tau)]}{\omega_{m,n}} \hat{f}_{m,n}(\tau)\,\mathrm{d}\tau \tag{19}$$

$$\hat{v}^{(N)}_{m,n}(t) = -\frac{\beta\lambda^2_{m,n}}{\omega_{m,n}} \int_0^t \mathrm{e}^{-b\lambda^2_{m,n}(t-\tau)} \sin\bigl[\omega_{m,n}(t-\tau)\bigr] Q^{(N)}_{m,n}\bigl(\hat{v}(\tau)\bigr)\,\mathrm{d}\tau \quad \text{for } N \geqslant 1 \tag{20}$$

where

$$\omega_{m,n} = \lambda_{m,n}\sqrt{\mu\lambda^2_{m,n} + 1}$$

$$Q^{(N)}_{m,n}\bigl(\hat{v}(t)\bigr) = \sum_{\substack{p,q,k,s:\\ p+k=m}} b(m,n;\,p,q,k,s) \sum_{j=1}^{N} \hat{v}^{(j-1)}_{p,q}(t)\hat{v}^{(N-j)}_{k,s}(t)$$

and the notation $\sum_{\substack{p,q,k,s:\\ p+k=m}} = \sum_{\substack{p,k\in\mathbf{Z};\ q,s\geqslant 1;\\ p+k=m}}$ *has been used.*

Lemma 3.7. *Assume that $f \in L_\infty(\mathbb{R}^+, L_2(\Omega))$ for $m \in \mathbf{Z}, n \geqslant 1$. Then the functions $\hat{v}^{(N)}_{m,n}(t)$ satisfy the following inequalities for $m \in \mathbb{Z}$, $n \geqslant 1$ and $t > 0$:*

$$\bigl|\hat{v}^{(0)}_{m,n}(t)\bigr| \leqslant \frac{c_0}{\lambda^{7/2}_{m,n}}$$

$$\bigl|\hat{v}^{(1)}_{m,n}(t)\bigr| \leqslant \frac{c_0c_1}{\lambda^{3/2}_{m,n}} R_{2,2}(m)$$

$$\bigl|\hat{v}^{(2)}_{m,n}(t)\bigr| \leqslant \frac{c_0c_1^2}{\lambda^{3/2}_{m,n}} R_{2,1}(m)$$

$$\bigl|\hat{v}^{(N)}_{m,n}(t)\bigr| \leqslant \frac{c_0c_1^N}{(N+1)^2\lambda^{3/2}_{m,n}} R_{1,1}(m) \quad \text{for } N \geqslant 3 \tag{21}$$

where

$$c_0 = \frac{\|f\|_{L_\infty(\mathbb{R}^+, L_2(\Omega))}}{b\sqrt{2\pi\mu\mathcal{C}_1}}, \qquad c_1 = \frac{2\pi^2|\beta|c_0\tilde{C}}{3\sqrt{\mu}b} \tag{22}$$

Proof. By Lemma 3.3, $|\hat{f}_{m,n}(t)| \leqslant \|f\|_{L_\infty(\mathbb{R}^+, L_2(\Omega))}(2\pi\mathcal{C}_1)^{-1/2}\sqrt{\lambda_{m,n}}$ for all $t > 0$. Therefore (19) implies that

$$\bigl|\hat{v}^{(0)}_{m,n}(t)\bigr| \leqslant c_0\sqrt{\lambda_{m,n}}\mathrm{e}^{-b\lambda^2_{m,n}t} \int_0^t \mathrm{e}^{b\lambda^2_{m,n}\tau} \left|\frac{\sin(\omega_{m,n}(t-\tau))}{\omega_{m,n}}\right| \mathrm{d}\tau \leqslant \frac{c_0}{\lambda^{7/2}_{m,n}}$$

For $N = 1$ applying Lemma 3.3 we can write the following chain of inequalities:

$$\bigl|\hat{v}^{(1)}_{mn}(t)\bigr| \leqslant \frac{|\beta|\lambda^2_{m,n}}{\omega_{m,n}} \int_0^t \mathrm{e}^{-b\lambda^2_{m,n}(t-\tau)} \sum_{\substack{p,q,k,s:\\ p+k=m}} \bigl|b(m,n;\,p,q,k,s)\bigr|\bigl|\hat{v}^{(0)}_{p,q}(\tau)\bigr|\bigl|\hat{v}^{(0)}_{k,s}(\tau)\bigr|\,\mathrm{d}\tau$$

$$\leqslant \frac{|\beta|\sqrt{\lambda_{m,n}}}{b\sqrt{\mu}} \sum_{\substack{p,q,k,s:\\ p+k=m}} \frac{1}{\lambda^4_{p,q}\lambda^4_{k,s}} \leqslant \frac{c_0c_1}{\lambda^{3/2}_{m,n}} R_{2,2}(m)$$

where the function $R_{2,2}(m)$ is defined by (5). For $N = 2$ applying Lemma 3.4 we can write that

$$\left|\hat{v}_{mn}^{(2)}(t)\right| \leqslant \frac{2|\beta|\lambda_{m,n}^2}{\omega_{m,n}} \int_0^t e^{-b\lambda_{m,n}^2(t-\tau)} \sum_{\substack{p,q,k,s:\\ p+k=m}} \left|b(m,n;\,p,q,k,s)\right|\left|\hat{v}_{p,q}^{(0)}(\tau)\right|\left|\hat{v}_{k,s}^{(1)}(\tau)\right| \mathrm{d}\tau$$
$$\leqslant c_0 c_1 \frac{\sqrt{\lambda_{m,n}}}{\lambda_{m,n}^2} \sum_{\substack{p,q,k,s:\\ p+k=m}} \frac{R_{2,2}(k)}{\lambda_{p,q}^4 \lambda_{k,s}^2} \leqslant \frac{c_0 c_1^2}{\lambda_{m,n}^{3/2}} R_{2,1}(m)$$

Next, we use the induction on the number N. Assume that the inequalities (21) hold for all $\hat{v}_{mn}^{(l)}(t)$ with $2 \leqslant l \leqslant N-1$ and prove that they are true for $l = N$. First, we notice that uniformly with respect to N

$$\sum_{j=1}^{N} \frac{1}{j^2(N+1-j)^2} \leqslant \frac{2}{(N+1)^2} \sum_{j=1}^{N} \left[\frac{1}{j^2} + \frac{1}{(N+1-j)^2}\right] \leqslant \frac{4}{(N+1)^2} \sum_{1}^{\infty} \frac{1}{j^2} = \frac{2\pi^2}{3(N+1)^2}$$

Then we have

$$\left|\hat{v}_{mn}^{(N)}(t)\right| \leqslant \frac{|\beta|}{\sqrt{\mu}\lambda_{mn}^{3/2}} \frac{2}{(N+1)^2} \sum_{j=1}^{N} c_1^{j-1} c_1^{N-j} \left[\frac{1}{j^2} + \frac{1}{(N+1-j)^2}\right] \sum_{\substack{p,q,k,s:\\ p+k=m}} \frac{1}{\lambda_{pq}^2 \lambda_{ks}^2} \leqslant \frac{c_0 c_1^N}{\lambda_{m,n}^{3/2}} R_{1,1}(m)$$

The proof is complete. □

Now we can prove Theorem 3.2.

3.1. Proof of Theorem 3.2

3.1.1. Existence and construction of solutions

Expanding the nonlinearity u^2 into the eigenfunction series we can write that

$$u^2 = \sum_{m,n} \widehat{u^2}_{m,n}(t) \chi_{m,n}(r,\theta)$$

where the Fourier–Bessel coefficients of u^2 are

$$\widehat{u^2}_{m,n}(t) = \frac{\langle u^2(t), \chi_{m,n}\rangle}{\|\chi_{m,n}\|^2} = \frac{1}{\|\chi_{m,n}\|^2} \left\langle \sum_{p,q} \hat{u}_{p,q}(t)\chi_{p,q} \cdot \sum_{k,s} \hat{u}_{k,s}(t)\chi_{k,s},\, \chi_{m,n} \right\rangle$$
$$= \sum_{\substack{p,q,k,s:\\ p+k=m}} b(m,n;\,p,q,k,s)\hat{u}_{p,q}(t)\hat{u}_{k,s}(t) \tag{23}$$

and the coefficients $b(m,n;\,p,q,k,s)$ are defined by (16). Here we have used the orthogonality relation

$$\int_{-\pi}^{\pi} e^{i(p+k-m)\theta}\,\mathrm{d}\theta = \begin{cases} 2\pi, & p+k=m \\ 0, & p+k \neq m \end{cases}$$

Expanding the source term into the eigenfunction series

$$f(r,\theta,t) = \sum_{m,n} \hat{f}_{m,n}(t)\chi_{m,n}(r,\theta) \tag{24}$$

where the coefficients $\hat{f}_{m,n}(t) = \|\chi_{m,n}\|^{-2}\langle f(t), \chi_{m,n}\rangle$, and substituting (7), (23) and (24) into (6) we obtain the following initial-value problem for the coefficients $\hat{u}_{m,n}(t)$ with $m \in \mathbf{Z}$, $n \in \mathbf{N}$

$$\hat{u}_{m,n}''(t) + 2b\lambda_{m,n}^2 \hat{u}_{m,n}'(t) + (\alpha\lambda_{m,n}^4 + \lambda_{m,n}^2)\hat{u}_{m,n}(t) = -\beta\lambda_{m,n}^2 \widehat{u^2}_{m,n}(t) + \hat{f}_{m,n}(t), \quad t > 0$$
$$\hat{u}_{m,n}(0) = \hat{u}_{m,n}'(0) = 0 \tag{25}$$

Integrating (25) with respect to t we reduce it to a nonlinear integral equation

$$\hat{u}_{m,n}(t) = \frac{1}{\omega_{m,n}} \int_0^t \mathrm{e}^{-b\lambda_{m,n}^2(t-\tau)} \sin\bigl(\omega_{m,n}(t-\tau)\bigr) \hat{f}_{m,n}(\tau)\,\mathrm{d}\tau - \frac{\beta\lambda_{m,n}^2}{\omega_{m,n}} \int_0^t \mathrm{e}^{-b\lambda_{m,n}^2(t-\tau)} \sin\bigl(\omega_{m,n}(t-\tau)\bigr) \widehat{u^2}_{m,n}(\tau)\,\mathrm{d}\tau \tag{26}$$

We seek its solutions in the form

$$\hat{u}_{m,n}(t) = \sum_{N=0}^{\infty} \hat{v}_{m,n}^{(N)}(t) \tag{27}$$

Substituting (27) into Eq. (26) we obtain the recursion formulas (19) and (20) for $\hat{v}_{mn}^{(N)}(t)$. In order to guarantee the absolute and uniform convergence of the series (27) we choose $c_1 < 1$ in (22). It implies that $\|f\|_{L_\infty(\mathbb{R}^+, H^s(\Omega))} \leqslant 3b\sqrt{2\pi \mathcal{C}_1 \mu}(2\pi^2|\beta|\tilde{C})^{-1}$, where $\tilde{C}$ and $\mathcal{C}_1$ are defined by (17) and (10), respectively. Thus, the solution in question is constructed in the form (7), (27), (19) and (20).

In view of (27), Lemmas 3.6 and 3.7 we have uniformly with respect to $t \geqslant 0$

$$\bigl|\hat{u}_{m,n}(t)\bigr| \leqslant \sum_{N=0}^{\infty} \bigl|\hat{v}_{m,n}^{(N)}(t)\bigr| \leqslant c_0\left[\frac{1}{\lambda_{m,n}^{7/2}} + \frac{c_1 R_{2,2}(m)}{\lambda_{m,n}^{3/2}} + \frac{c_1^2 R_{2,1}(m)}{\lambda_{m,n}^{3/2}} + \frac{R_{1,1}(m)}{\lambda_{m,n}^{3/2}} \sum_{N=0}^{\infty} \frac{c_1^N}{(N+1)^2}\right] \tag{28}$$

Therefore, uniformly with respect to $t \geqslant 0$

$$\bigl|\hat{u}_{m,n}(t)\bigr| \leqslant \frac{cR_{1,1}(m)}{\lambda_{m,n}^{3/2}} \tag{29}$$

Recalling the definition of the norm in $\mathcal{H}^{s,\alpha}(\Omega)$ (12) and comparing the remainder of the corresponding series,

$$\sum_{m=M_0}^{\infty} \sum_{n=N_0}^{\infty} \bigl(1+|m|^2\bigr)^{\alpha} \lambda_{m,n}^{2s} \bigl|\hat{u}_{m,n}(t)\bigr|^2 \|\chi_{m,n}\|^2$$

with the integral

$$\int_{M_0}^{\infty} \frac{(1+m^2)^{\alpha} \ln^2 m}{m^2} \int_{N_0}^{\infty} \frac{\mathrm{d}n}{\lambda_{m,n}^{4-2s}} \sim \int_{M_0}^{\infty} \frac{(1+m^2)^{\alpha} \ln^2 m}{m^2} \int_{N_0}^{\infty} \frac{\mathrm{d}n}{(m+2n)^{4-2s}}$$

we deduce that uniform convergence of the series (12) takes place for $s < 3/2$ and $s + \alpha \leqslant 2 - \varepsilon$ with $\varepsilon > 0$. Therefore, the constructed solution belongs to $C_b(\mathbb{R}^+, \mathcal{H}_0^{s,\alpha}(\Omega))$ for $s < 3/2$ and $s + \alpha \leqslant 2 - \varepsilon$.

3.1.2. Uniqueness

Assume that there exist two mild solutions $u^{(1)}$ and $u^{(2)}$ from the space $C_b(\mathbb{R}^+, \mathcal{H}_0^{s,\alpha}(\Omega))$ with $s < 3/2$ and $s + \alpha \leqslant 2 - \varepsilon$. Then each of them can be expanded into the series (14), where the coefficients $u_{m,n}^{(1)}$ and $u_{m,n}^{(2)}$ satisfy the integral equation (26) and the estimate (29). Setting $w = u^{(1)} - u^{(2)}$ we expand it into the series of the type of (7) and get

$$w = \sum_{m,n} \hat{w}_{m,n}(t) \chi_{m,n}(r,\theta)$$

$$\hat{w}_{m,n}(t) = -\frac{\beta\lambda_{m,n}^2}{\omega_{m,n}} \int_0^t \mathrm{e}^{-b\lambda_{m,n}^2(t-\tau)} \sin\bigl(\omega_{m,n}(t-\tau)\bigr)\bigl[\widehat{(u^{(1)})^2}_{m,n}(\tau) - \widehat{(u^{(2)})^2}_{m,n}(\tau)\bigr]\,\mathrm{d}\tau$$

Subdividing the sum representing the term in brackets into three we can write that

$$\widehat{(u^{(1)})^2}_{m,n}(t)-\widehat{(u^{(2)})^2}_{m,n}(t)=S_1+S_2+S_3$$
$$=\left(\sum_{\substack{p,q,k,s:\ p+k=m;\\ \lambda_{p,q}>\lambda_{k,s}}}+\sum_{\substack{p,q,k,s:\ p+k=m;\\ \lambda_{p,q}<\lambda_{k,s}}}+\sum_{\substack{p,q,k,s:\ p+k=m;\\ \lambda_{p,q}=\lambda_{k,s}}}\right)b(m,n;p,q,k,s)\big[\hat{u}^{(1)}_{p,q}(t)\hat{w}_{k,s}(t)+\hat{w}_{p,q}(t)\hat{u}^{(2)}_{k,s}(t)\big]$$

Taking the sum S_1 as an example we choose sufficiently small $\delta>0$, apply Lemma 3.5, (10) and the Cauchy–Schwartz inequality and deduce that

$$S_1\leqslant C\sqrt{\lambda_{m,n}}\sum_{\substack{p,q,k,s:\ p+k=m;\\ \lambda_{p,q}>\lambda_{k,s}}}\frac{|\hat{u}^{(1)}_{p,q}(t)|}{\lambda^{3/2}_{p,q}}\frac{|\hat{w}_{k,s}(t)|}{\lambda^{1/2}_{k,s}}$$
$$\leqslant C\sqrt{\lambda_{m,n}}\sum_{\substack{q,k,s:\ p+k=m;\\ \lambda_{p,q}>\lambda_{k,s}}}\frac{q^{(1+\delta)/2}|\hat{u}^{(1)}_{p,q}(t)|}{\lambda^{3/2}_{m-k,q}\lambda^{\rho}_{k,s}(1+|k|)^{\alpha}}\cdot\frac{|\hat{w}_{k,s}(t)|\lambda^{\rho}_{k,s}(1+|k|)^{\alpha}}{\lambda^{1/2}_{k,s}q^{(1+\delta)/2}}$$
$$\leqslant C\sqrt{\lambda_{m,n}}\left(\sum_{\substack{p,q,k,s:\\ p+k=m}}\frac{R^2_{1,1}(p)}{\lambda^{2(\rho+2)-\delta_1}_{k,s}(1+|k|)^{\alpha}}\cdot\frac{q^{1+\delta}}{\lambda^{2+\delta_1}_{p,q}}\right)^{1/2}\left(\sum_{q=1}^{\infty}\frac{1}{q^{1+\delta}}\right)^{1/2}$$
$$\times\left(\sum_{k,s}\big(1+|k|\big)^{2\alpha}\lambda^{2\rho}_{k,s}\big|\hat{w}_{k,s}(t)\big|^2\big\|\chi_{k,s}\big\|^2\right)^{1/2}$$

Here we have taken $\delta_1>\delta>0$ and have dropped the condition $\lambda_{p,q}>\lambda_{k,s}$. Due to the symmetry properties $R_{1,1}(m)=R_{1,1}(|m|)$ and $\lambda_{-m,n}=\lambda_{m,n}$ we can consider the case $m\geqslant 0$ without loss of generality. The sum $\sum_{p,q,k,s:\ p+k=m}$ can be rewritten in the following way:

$$\sum_{k=-\infty}^{\infty}\sum_{q,s=1}^{\infty}\frac{R^2_{1,1}(m-k)}{\lambda^{2(\rho+2)-\delta_1}_{k,s}(1+|k|)^{2\alpha}}\cdot\frac{q^{1+\delta}}{\lambda^{2+\delta_1}_{m-k,q}}$$
$$=\sum_{q,s=1}^{\infty}\left[\sum_{k=0}^{m}\frac{R^2_{1,1}(m-k)}{\lambda^{2(\rho+2)-\delta_1}_{k,s}(1+k)^{2\alpha}}\cdot\frac{q^{1+\delta}}{\lambda^{2+\delta_1}_{m-k,q}}+\sum_{k=1}^{\infty}\frac{R^2_{1,1}(m+k)}{\lambda^{2(\rho+2)-\delta_1}_{k,s}(1+k)^{2\alpha}}\cdot\frac{q^{1+\delta}}{\lambda^{2+\delta_1}_{m+k,q}}\right.$$
$$\left.+\sum_{k=1}^{\infty}\frac{R^2_{1,1}(k)}{\lambda^{2(\rho+2)-\delta_1}_{k,s}(1+k+m)^{2\alpha}}\cdot\frac{q^{1+\delta}}{\lambda^{2+\delta_1}_{m+k,q}}\right]$$

Consider the first sum on the right-hand side of the last formula. In order to establish the uniform convergence of this series, we consider its remainder and use the asymptotics (11). Taking sufficiently large K_0, S_0, $Q_0=const>0$ and comparing this remainder with the integral

$$\int_{K_0}^{\infty}\frac{\ln^2(m+k)\,\mathrm{d}k}{(m+k)^2(1+k)^{2\alpha}}\int_{S_0}^{\infty}\frac{\mathrm{d}s}{\lambda^{2(\rho+2)-\delta_1}_{k,s}}\int_{Q_0}^{\infty}\frac{q^{1+\delta}}{\lambda^{2+\delta_1}_{m+k,q}}\,\mathrm{d}q$$
$$\sim\int_{K_0}^{\infty}\frac{\ln^2(m+k)\,\mathrm{d}k}{(m+k)^2(1+k)^{2\alpha}}\int_{S_0}^{\infty}\frac{\mathrm{d}s}{(k+2s)^{2(\rho+2)-\delta_1}}\int_{Q_0}^{\infty}\frac{q^{1+\delta}}{(m+k+2q)^{2+\delta_1}}\,\mathrm{d}q$$

It converges for $\rho+\alpha\geqslant -2+\varepsilon$ and $\rho\geqslant -3/2+\varepsilon$, where $\varepsilon>0$ is arbitrarily small. The second condition is imposed by convergence of the inner integral in s. Similar considerations allow one to establish convergence of the other two series under the same restrictions. Therefore, we have

$$S_1\leqslant C\sqrt{\lambda_{m,n}}\,\big\|w(t)\big\|_{\rho,\alpha}$$

The sums S_2 and S_3 can be estimated in a similar way, and the same inequality holds as a result.

Thus, we deduce that

$$\left|\hat{w}_{m,n}(t)\right|^2 \leqslant C\lambda_{m,n}\left(\int_0^t \mathrm{e}^{-b\lambda_{m,n}^2(t-\tau)}\left\|w(\tau)\right\|_{\rho,\alpha}\,\mathrm{d}\tau\right)^2$$

Multiplying both sides of the last inequality by $(1+|m|^2)^\alpha\lambda_{m,n}^{2\rho}\|\chi_{m,n}\|^2$ and summing the result with respect to m and n we deduce that for some $T>0$, $t\in[0,T]$ and $\rho<3/2$

$$\sup_{t\in[0,T]}\left\|w(t)\right\|_{\rho,\alpha}^2 \leqslant C\Phi(T)\Big(\sup_{t\in[0,T]}\left\|w(t)\right\|_{\rho,\alpha}\Big)^2$$

where

$$\Phi(t)=\sum_{m,n}\big(1+|m|^2\big)^\alpha\lambda_{m,n}^{2\rho}\frac{\left(1-\mathrm{e}^{-b\lambda_{m,n}^2 t}\right)^2}{\lambda_{m,n}^4}$$

Notice that $\Phi(t)$ is a nondecreasing continuous function on $[0,T]$ and $\Phi(0)=0$. Therefore

$$\Big(\sup_{t\in[0,T]}\left\|w(t)\right\|_{\rho,\alpha}\Big)^2 \leqslant C(T)\Big(\sup_{t\in[0,T]}\left\|w(t)\right\|_{\rho,\alpha}\Big)^2$$

where the constant $C(T)=C\Phi(T)$ can be made less than one by the appropriate choice of T. This contradiction allows one to prove uniqueness for the interval $[0,T]$. Then by standard arguments we extend this result for all $t\geqslant 0$.

4. Conclusions

In conclusion we would like to point out physical applications of the mathematical study presented above. The current work is of interest in the context of modeling nonlinear oscillations of a circular elastic membrane under the influence of an acoustic field. A circular geometry is a natural choice for ground sensors registering incident acoustic waves (see e.g., [28] and the references therein), and the boundary conditions in (6) correspond to a simply supported boundary. The source term $f(r,\theta,t)$ represents the given acoustic pressure on the surface of the membrane. The eigenfunction expansion (14) allows one to compute the vertical deflection distribution. This formula can be used for establishing long-time behavior and for numerical simulations since the algorithm shows excellent convergence properties (see [29]).

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