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Unsteady flows, turbulent flows

On the history term of Boussinesq–Basset when the viscous fluid slips on the particle

Renée Gatignol

Laboratoire de modélisation en mécanique, Université Pierre et Marie Curie & CNRS, 4, place Jussieu, 75252 Paris cedex 05, France

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Abstract

Within the framework of the Stokes approximation, a method is proposed for calculating the drag and the torque acted on a rigid particle by an incompressible viscous fluid, when the fluid–particle boundary conditions are slip conditions. By using the Fourier Transform and a reciprocity formula, the drag and torque are deduced from these obtained for two simple vibration motions of the particle in a fluid at rest. The results are explicitly given in the case of a spherical particle. They are in agreement with the formulae known in various special cases. ***To cite this article: R. Gatignol, C. R. Mecanique 335 (2007).***

Résumé

Sur le terme d'histoire de Boussinesq–Basset quand le fluide visqueux glisse sur la particule. Dans le cadre de l'approximation de Stokes, une méthode est proposée pour calculer les efforts exercés sur une particule solide par un fluide visqueux incompressible, les conditions aux limites fluide–particule étant celles du glissement. En utilisant la Transformée de Fourier et une formule de réciprocité, les efforts sont déduits de ceux obtenus pour deux mouvements simples de vibration de la particule dans un fluide au repos. Les résultats sont explicités pour une particule sphérique. Ils sont en accord avec les expressions connues de la littérature. ***Pour citer cet article : R. Gatignol, C. R. Mecanique 335 (2007).***

Keywords: Fluid mechanics; Stokes equation; Drag; Torque; History term of Boussinesq–Basset

Mots-clés : Mécanique des fluides ; Equations de Stokes ; Traînée ; Couple ; Terme d'histoire de Boussinesq–Basset

1. Introduction

Suspensions, where the carrying fluid or solvent is a liquid or a gas and where the suspended phase is composed of solid particles, are the subject of very many both fundamental and applied studies since several decades. This is due to the very great number of applications, for example in the industrial engineering, the biological engineering, and the area of the environmental problems. Very often, the aim is the modeling of the bulk behavior of the homogeneous suspension in which we are interested. Then, it is necessary to accurately understand the fluid–particle interaction for an isolated particle moving in a fluid flow, while ignoring the presence of all other particles. It is with this idea that

E-mail address: Renee.Gatignol@upmc.fr.

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we are interested in the efforts exerted on a rigid solid particle P by a viscous fluid having a constant shear viscosity μ and a constant volumetric mass ρ. The unperturbed fluid motion is unsteady and non-uniform.

Historically, the first known work (1851) relates to a rigid sphere in uniform translation motion with the velocity $\mathbf{U}_0$ in a viscous incompressible fluid, at rest at infinity: The result is the famous formula of drag of Stokes $\mathbf{F} = -6\pi\mu a \mathbf{U}_0$, a being the radius of the sphere [1]. Boussinesq [2] in 1885, and then Basset [3] in 1888, extended that result to the case of a sphere which has a translation velocity depending on time. The new term which is obtained, very often called *historical term of Basset* should be known under the name *term of Boussinesq* or, at least, *term of Boussinesq–Basset* [4,5]. Here, it is interesting to note an approached formula obtained by Basset [3] for the torque exerted by the viscous fluid on a sphere which has a rotation velocity depending on time.

Concerning the efforts exerted by an incompressible viscous fluid, occupying all space, on a rigid and spherical particle, efforts calculated within the framework of the approximation of Stokes and a boundary condition of adherence of the fluid on the particle, a great number of generalizations followed. In 1983 [6], we have analyzed in a short historical review the most outstanding contributions. From now onwards we specify the notations. With respect to a Galilean frame $\Re^a$, $\mathbf{V}_p(t)$ denotes the velocity of a point O of the particle P, $\Omega_p(t)$ its instantaneous vector of rotation and, $\mathbf{V}_0^a(\mathbf{r}^a, t)$ the velocity field of the fluid in the absence of the particle. Of course, $\mathbf{r}^a$ and t are the position vector in the frame $\Re^a$ and the time. The efforts exerted by the fluid on P have $\mathbf{F}$ as resultant (drag force) and Γ as moment in the point O (torque). For a sphere, according to the assumptions on $\mathbf{V}_p(t)$, $\Omega_p(t)$ or $\mathbf{V}_0^a(\mathbf{r}^a, t)$, the authors have obtained expressions for $\mathbf{F}$, Γ or both [6]. Other generalizations have followed for non-spherical particles, for a compressible or micropolar carrying fluid, etc. as it is written in [7]. A considerable quantity of works concern the trajectories of the particles where the complex efforts $\mathbf{F}$ and Γ are taken into account [5,8–12], and for the most recent works one can see [11,12].

One method to calculate the efforts $\mathbf{F}$ and Γ, in the general case where the fluid flow is, in the absence of the particle P, a motion spatially not uniform and unsteady, and where P has also a non-stationary motion, has been proposed in 1983 [6]; the boundary condition fluid–particle was that of adherence. In this article, the previous work is generalized by replacing the adherence condition by a slip condition. This slip condition appears, in particular, in the micro-filters used to detect or capture fine particles in order to use or to limit environmental pollution (such as particle filters for the diesel engines). The gas flow charged with nano or micrometric particles takes place in systems of very small dimensions about some micrometers [13]. In these systems, the rarefaction of gas must be taken into account via the slip gas–particle which is about the mean free path of gas. It should be noted that the gas considered in this physical problem, is supposed, in our paper, to be an incompressible fluid, which constitutes a strong assumption. We emphasize the paper of Albano et al. [14], in which the boundary condition on the particle P is formulated by introducing a density of induced force. A formal solution is written by using a Green representation. Only the drag $\mathbf{F}$ in Fourier variable is given. We emphasize also the paper of Michaelides and Feng [15] in which the problem is solved, for only the drag $\mathbf{F}$, by using Laplace transform. Here the drag $\mathbf{F}$ and the torque Γ are calculated, in the general situation where, in the absence of P, the fluid has an unsteady and spatially non-uniform motion. This same problem was also solved in a conventional way by using the method proposed in [6] in the case of adherence fluid–particle, by Aggad in his work of thesis [7]. This article contains some of his results.

The equations describing the motion of the fluid around a particle P, not necessarily spherical, are written in a relative reference frame linked to the particle. With the assumption of very small Reynolds numbers, these equations become linear. One introduces the unperturbed situation where the particle is missing, the fluid occupying all space. The efforts on P are written as the sum of two contributions, one having its origin in the unperturbed situation, and the second in the perturbation (Section 2). Then the perturbation is analysed. By using the Fourier transform and by establishing a formula of reciprocity, we deduce the efforts in the unsteady complex situation from the efforts calculated in two simple cases. There are: Firstly the particle P has a rectilinear sinusoidal motion in a fluid at rest, and secondly the particle P has a rotational sinusoidal motion around an axis always in a fluid at rest (Section 3). The particular case of a spherical particle is then examined (Section 4) and the efforts exerted by the fluid on P are explicitly calculated when P has the two described vibration motions. Then by the inverse Fourier transform, it is possible to obtain the expressions of the required efforts. In Section 5, the final results are given and discussed. Lastly, in the conclusion, we insist on some recent works in connection with the historical term of Boussinesq–Basset, showing that the history term of Boussinesq–Basset is always of topicality.

2. Formulation of the problem

2.1. Governing equations

A viscous fluid with a constant volumetric mass ρ and a constant shear viscosity μ occupies the entire physical space. Its velocity in the Galilean frame $\Re^a$ is denoted by $\mathbf{V}^a(\mathbf{r}^a, t)$ and its pressure by p. As it has been previously said, the considered gas is seen as an incompressible fluid. Inside this fluid, there is a rigid particle P having an arbitrary shape with volume $\mathcal{V}$ and surface $\mathcal{S}$. We introduce a point O which is linked to the particle P, the position vector $\mathbf{R}$ of O in $\Re^a$, the velocity $\mathbf{V}_p(t)$ of O, the angular velocity $\Omega_p(t)$ of P and we put $\mathbf{r} = \mathbf{r}^a - \mathbf{R}$. In the following, we use the notations r^a and r for the modulus of $\mathbf{r}^a$ and $\mathbf{r}$. The Navier–Stokes equations for the fluid and the Newton law for P can be written as following:

$$\nabla_a \cdot \mathbf{V}^a = 0, \quad \frac{\partial \mathbf{V}^a}{\partial t} + (\mathbf{V}^a \cdot \nabla_a)\mathbf{V}^a = -\frac{1}{\rho}\nabla_a p + \nu \Delta_a \mathbf{V}^a + \mathbf{F}_e \tag{1}$$

$$[\mathcal{A}_p] = [\Sigma_f] + [\Sigma_e] \tag{2}$$

where $\mathbf{F}_e$ is the volumetric external force, where $\nu = \mu/\rho$ and where the label a is for the quantities and the operators related to the frame $\Re^a$. In Eq. (2), $[\mathcal{A}_p]$ denotes the wrench of dynamical quantities of the particle P (that is the resultant and the moment in the point O), $[\Sigma_e]$ those of the external forces applied to P, and $[\Sigma_f]$ those of the pressure and friction forces exerted by the fluid on P.

We add to Eqs. (1), a condition at infinity and the slip boundary condition on P:

$$(\mathbf{V}^a, p) \to (\mathbf{V}^a_\infty, p_\infty) \quad \text{when } r^a \to \infty \tag{3}$$

$$\mathbf{V}^a(\mathbf{r}^a, t) - \mathbf{V}_p(t) - \Omega_p(t) \wedge \mathbf{r} = 2\lambda(\mathbb{I}_d - \mathbf{nn}) \cdot \mathbb{D}_a(\mathbf{V}^a) \cdot \mathbf{n} \quad \text{on } \mathcal{S} \tag{4}$$

where the velocity $\mathbf{V}^a_\infty$ and the pressure p_∞ are given at the infinity, and where the relation (4) expresses the slip boundary condition. In Eq. (4), $\mathbf{n}$ is the outward normal unit vector to the surface $\mathcal{S}$, $\mathbf{nn}$ a dyadic tensor, $\mathbb{I}_d$ the unit tensor of order two and $\mathbb{D}_a(\mathbf{V}^a) = (1/2)(\nabla_a \mathbf{V}^a + (\nabla_a \mathbf{V}^a)^T)$ the shear rate tensor. The slip coefficient λ is the mean free path of the gas and is considered as a positive constant. Of course, $\lambda = 0$ corresponds to a no-slip condition. This condition (4), usual in the problems of rarefied gas dynamics [16], is present in the paper of Albano et al. [14] and also in the thesis of Aggad [7].

A frame $\Re$ linked to the particle P with its origin in O is introduced. The fluid velocity in $\Re$ is denoted by $\mathbf{V}(\mathbf{r}, t)$. We have: $\mathbf{V} = \mathbf{V}^a - \mathbf{V}_p - \Omega_p \wedge \mathbf{r}$. The fluid equations and the boundary conditions in the frame $\Re$ are as follows:

$$\nabla \cdot \mathbf{V} = 0, \quad \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{V} + \mathbf{F}_e - 2\Omega_p \wedge \mathbf{V} - \frac{\mathrm{d}\mathbf{V}_p}{\mathrm{d}t} - \frac{\mathrm{d}\Omega_p}{\mathrm{d}t} \wedge \mathbf{r} - \Omega_p \wedge (\Omega_p \wedge \mathbf{r}) \tag{5}$$

$$(\mathbf{V}, p) \to (\mathbf{V}^a_\infty - \mathbf{V}_p - \Omega_p \wedge \mathbf{r}, p_\infty) \quad \text{when } r \to \infty, \qquad \mathbf{G}(\mathbf{V}) = 0 \quad \text{on } \mathcal{S} \tag{6}$$

where, to simplify the notations, we put:

$$\mathbf{G}(\mathbf{u}) = \mathbf{u} - 2\lambda(\mathbb{I}_d - \mathbf{nn}) \cdot \mathbb{D}(\mathbf{u}) \cdot \mathbf{n} \quad \text{with } \mathbb{D}(\mathbf{u}) = (1/2)(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \tag{7}$$

The two problems (1), (3), (4) and (5), (6) are non-linear. In the first case, the non-linearity is due to the convective term in (1) and to the boundary condition (4) on $\mathcal{S}$ because $\mathcal{S}$ is moving. In the second case, the non-linearity is more clear because $\mathcal{S}$ is a surface at rest and the non-linear terms appear only in the equations (5), with the convective term $(\mathbf{V} \cdot \nabla)\mathbf{V}$, the Coriolis acceleration $2\Omega_p \wedge \mathbf{V}$ and the part $\Omega_p \wedge (\Omega_p \wedge \mathbf{r})$ of the entrainment acceleration.

2.2. Linearized equations

Let v_0 be a representative magnitude of the relative velocity $\mathbf{V}$ of the fluid and also of the velocities $\mathbf{V}_p$ and $\Omega_p \wedge \mathbf{r}$ in relation with the particle motion. Let τ_0 be the characteristic time of the variations of the fluid and particle velocities. The magnitude order of the variable $\mathbf{r}$ is supposed to be a. So, we have the following magnitude orders: v_0/τ_0 for $\partial\mathbf{V}/\partial t$, $\mathrm{d}\mathbf{V}_p/\mathrm{d}t$ and $(\mathrm{d}\Omega_p/\mathrm{d}t) \wedge \mathbf{r}$; $(\nu/a^2)v_0$ for $\nu\Delta\mathbf{V}$ and $(v_0/a)v_0$ for $2\Omega_p \wedge \mathbf{V}$ and $\Omega_p \wedge (\Omega_p \wedge \mathbf{r})$. As in the paper [6], we introduce the Reynolds number $R_e = (a^2/\nu)/(a/v_0)$ which is the ratio of the characteristic time of the viscous effects to the characteristic time of the convective terms, and the Strouhal number $S_r = (a/v_0)/\tau_0$ which is useful in this unsteady problem. We assume the Reynolds number R_e very small in order to neglect the non-linear terms in Eqs. (5). Moreover, the Strouhal number is such that $S_r R_e = \mathrm{O}(1)$ in order to keep the time derivatives in Eqs. (5). In other words, the unsteady phenomena are taken in account when the two characteristic times τ_0 and a^2/ν are of the same order. If τ_0 was large compared to a^2/ν, the partial time derivatives should be neglected. From now on, we assume $\tau_0 = \mathrm{O}(a^2/\nu)$ and $R_e \ll 1$ (i.e. the characteristic time a^2/ν of the viscous effects very small in front of the characteristic time a/v_0 of the convective terms). We write the unsteady Stokes equations for the field $(\mathbf{V}, p)$:

$$\nabla \cdot \mathbf{V} = 0, \quad \frac{\partial \mathbf{V}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{V} + \mathbf{F}_e - \frac{\mathrm{d}\mathbf{V}_p}{\mathrm{d}t} - \frac{\mathrm{d}\Omega_p}{\mathrm{d}t} \wedge \mathbf{r} \tag{8}$$

with in addition, the boundary conditions (6). As a matter of fact, we have linearized the acceleration terms too. The operator $\mathrm{d}/\mathrm{d}t$ being the material differentiation associated with the Galilean motion, we have:

$$\frac{\mathrm{d}\mathbf{V}^a(\mathbf{r}^a, t)}{\mathrm{d}t} \cong \frac{\partial \mathbf{V}^a(\mathbf{r}^a, t)}{\partial t} \cong \frac{\partial \mathbf{V}(\mathbf{r}, t)}{\partial t} + \frac{\mathrm{d}\mathbf{V}_p}{\mathrm{d}t} + \frac{\mathrm{d}\Omega_p}{\mathrm{d}t} \wedge \mathbf{r} \tag{9}$$

2.3. Preliminary expressions for the forces exerted by the fluid on P

Here we introduce the *unperturbed situation without the particle P*. The fluid velocity and the pressure in the frame $\Re$ are denoted by $\mathbf{V}_0$ and p_0. The field $(\mathbf{V}_0, p_0)$ is defined on the whole physical space. It is solution of the unsteady Stokes equations (8) and satisfies the same boundary conditions at infinity as the field $(\mathbf{V}, p)$. So we have:

$$\begin{cases} \nabla \cdot \mathbf{V}_0 = 0, \quad \frac{\partial \mathbf{V}_0}{\partial t} = -\frac{1}{\rho}\nabla p_0 + \nu \Delta \mathbf{V}_0 + \mathbf{F}_e - \frac{\mathrm{d}\mathbf{V}_p}{\mathrm{d}t} - \frac{\mathrm{d}\Omega_p}{\mathrm{d}t} \wedge \mathbf{r} \\ (\mathbf{V}_0, p_0) \to (\mathbf{V}^a_\infty - \mathbf{V}_p - \Omega_p \wedge \mathbf{r}, p_\infty) \quad \text{when } r \to \infty \end{cases} \tag{10}$$

Now we put: $\mathbf{W} = \mathbf{V} - \mathbf{V}_0$ and $q = p - p_0$. For the field $(\mathbf{W}, q)$, we have to solve the following problem:

$$\begin{cases} \nabla \cdot \mathbf{W} = 0, & \mathbf{G}(\mathbf{W}) = -\mathbf{G}(\mathbf{V}_0) \quad \text{on } \mathcal{S} \\ \frac{\partial \mathbf{W}}{\partial t} = -\frac{1}{\rho}\nabla q + \nu \Delta \mathbf{V}, & (\mathbf{W}, q) \to (0, 0) \quad \text{when } r \to \infty \end{cases} \tag{11}$$

The field $(\mathbf{W}, q)$ called *perturbation* corresponds to a fluid at rest at infinity and in which the particle P is moving with the velocity field $-\mathbf{V}_0$ on its external surface $\mathcal{S}$ and with a slip boundary condition.

For a force $\mathbf{f}$ applied in the point Q, the corresponding wrench is denoted $[Q, \mathbf{f}]$. So the wrench $[\Sigma_f]$ defined in Section 2.1 can be written as follows:

$$[\Sigma_f] = \int_{\mathcal{S}} \big[Q, \mathfrak{S}\big(\mathbf{V}^a, p\big) \cdot \mathbf{n}\big]\,\mathrm{d}\mathcal{S} = \int_{\mathcal{S}} \big[Q, \mathfrak{S}(\mathbf{V}, p) \cdot \mathbf{n}\big]\,\mathrm{d}\mathcal{S}$$

where Q is the running point on $\mathcal{S}$ and where $\mathfrak{S}$ is the stress tensor: $\mathfrak{S}(\mathbf{u}, p) = -p\mathbb{I}_d + 2\mu\mathbb{D}(\mathbf{u})$. The second equality is classical, the internal forces in a continuous medium do not depending on the frame in which the motion is observed. By introducing the perturbation $(\mathbf{W}, q)$, we have:

$$[\Sigma_f] = \int_{\mathcal{S}} \big[Q, \mathfrak{S}(\mathbf{V}_0, p_0) \cdot \mathbf{n}\big]\,\mathrm{d}\mathcal{S} + \int_{\mathcal{S}} \big[Q, \mathfrak{S}(\mathbf{W}, q) \cdot \mathbf{n}\big]\,\mathrm{d}\mathcal{S} \equiv [\Sigma_{\mathrm{I}}] + [\Sigma_{\mathrm{II}}] \tag{12}$$

The first term $[\Sigma_{\mathrm{I}}]$ comes from the unperturbed flow alone and the second $[\Sigma_{\mathrm{II}}]$ comes from the perturbation. In Eqs. (11) for $(\mathbf{W}, q)$, only the values of the velocity $\mathbf{V}_0$ and of its spatial derivatives on the surface $\mathcal{S}$ appear. Therefore in the final expression for $[\Sigma_{\mathrm{II}}]$, we must find only these values.

Now we give an expression for $[\Sigma_{\mathrm{I}}]$. As in the paper [6], the surface integral is transformed into a volume integral and by using Eqs. (8), we arrive to:

$$[\Sigma_{\mathrm{I}}] = \int_{\mathcal{V}} \left[M, \rho\left\{\frac{\partial \mathbf{V}_0}{\partial t} + \frac{\mathrm{d}\mathbf{V}_p}{\mathrm{d}t} + \frac{\mathrm{d}\Omega_p}{\mathrm{d}t} \wedge \mathbf{r} - \mathbf{F}_e\right\}\right]\mathrm{d}\mathcal{V}$$

where M is the running point in $\mathcal{V}$. Using (9), we express the final result in terms of the Galilean velocities:

$$[\Sigma_{\mathrm{I}}] = \int_{\mathcal{V}} \left[M, \rho\frac{\partial \mathbf{V}^a_0}{\partial t}\right]\mathrm{d}\mathcal{V} - \int_{\mathcal{V}} [M, \rho\mathbf{F}_e]\,\mathrm{d}\mathcal{V} \tag{13}$$

where $\mathbf{V}_0^a$ is the velocity with respect $\Re^a$ of the unperturbed flow. The first term in the right member of (13) is the wrench of the inertia forces of the so-called displaced fluid, and the second term is the wrench of the external forces applied to the displaced fluid. This last term corresponds to the Archimède forces in the case of gravity forces. Moreover, these terms do not depend on the slipping.

3. General considerations on the perturbation contribution

With the aim of calculating $[\Sigma_{\mathrm{II}}]$, it is necessary to study the problem (11) for $(\mathbf{W}, q)$. Following Mazur and Bedeaux [17] for a similar problem, a Fourier transform in the variable t is used. We put:

$$\hat{\varphi}(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(\mathbf{r}, t) \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t \tag{14}$$

and we obtain for the transformed field $(\widehat{\mathbf{W}}, \hat{q})$ the Problem I given farther in Section 3.2 (Eq. (19)). Of course, because $\mathcal{S}$ is a fixed surface in the frame $\Re$, we have:

$$\widehat{[\Sigma_{\mathrm{II}}]} = \int_{\mathcal{S}} \big[Q, \mathfrak{S}\big(\widehat{\mathbf{W}}, \hat{q}\big) \cdot \mathbf{n}\big]\, \mathrm{d}\mathcal{S} \tag{15}$$

3.1. Reciprocity formula

Let us consider the two following problems for the fields $(\mathbf{u}_k, p_k)$ with k equal to 1 or 2, which differ only by their boundary conditions on $\mathcal{S}$:

$$\begin{cases} \nabla \cdot \mathbf{u}_k = 0, & \mathbf{G}(\mathbf{u}_k) = \text{a known function on } \mathcal{S} \\ (\mathrm{i}\omega\rho - \mu\Delta)\mathbf{u}_k + \nabla p_k = 0, & (\mathbf{u}_k, p_k) \to (0,0) \text{ when } r \to \infty \end{cases} \tag{16}$$

By using the expression (7) for $\mathbf{G}(\mathbf{u})$ and those given in Section 2.3 for $\mathfrak{S}(\mathbf{u}, p)$, it is easy to prove that:

$$\mathbf{G}(\mathbf{u}_1) \cdot \mathfrak{S}(\mathbf{u}_2, p_2) \cdot \mathbf{n} - \mathbf{G}(\mathbf{u}_2) \cdot \mathfrak{S}(\mathbf{u}_1, p_1) \cdot \mathbf{n} = \mathbf{u}_1 \cdot \big[-p_2 \mathbb{I}_d + 2\mu \mathbb{D}(\mathbf{u}_2)\big] \cdot \mathbf{n} - \mathbf{u}_2 \cdot \big[-p_1 \mathbb{I}_d + 2\mu \mathbb{D}(\mathbf{u}_1)\big] \cdot \mathbf{n}$$

In other words, the right member does not depend on λ. Let Δ be a bounded domain of the space occupied by the fluid, limited by a surface Σ and let $\mathbf{n}$ be the outward unit normal to Σ. With the divergence theorem and also Eqs. (16), we have [6,7]:

$$\begin{aligned} &\int_{\Sigma} \mathbf{G}(\mathbf{u}_1) \cdot \mathfrak{S}(\mathbf{u}_2, p_2) \cdot \mathbf{n}\, \mathrm{d}\mathcal{S} - \int_{\Sigma} \mathbf{G}(\mathbf{u}_2) \cdot \mathfrak{S}(\mathbf{u}_1, p_1) \cdot \mathbf{n}\, \mathrm{d}\mathcal{S} \\ &\quad = \int_{\Delta} \big\{(\mathrm{i}\omega\rho\mathbf{u}_2 - \mu\Delta\mathbf{u}_2 + \nabla p_2) \cdot \mathbf{u}_1 - (\mathrm{i}\omega\rho\mathbf{u}_1 - \mu\Delta\mathbf{u}_1 + \nabla p_1) \cdot \mathbf{u}_2\big\}\, \mathrm{d}\mathcal{V} = 0 \end{aligned} \tag{17}$$

The fluid volume Δ is now a very large domain around the particle P, limited by a sphere Σ_R having a large radius R. The surface Σ is $\mathcal{S} \cup \Sigma_R$. We assume that the behavior at infinity of the solutions of the two problems (16) is such that: $\mathbf{u}_k = \mathrm{O}(R^{-1})$, $p_k = \mathrm{O}(R^{-1})$, $\mathbf{u}_k \cdot \mathbf{n} = \mathrm{O}(R^{-2})$. For a spherical particle and for two particular problems, those behaviors will be obtained in Section 3 (see (25) and (27)). Consequently, the part of the surface integral relative to Σ_R in Eq. (17) tends to zero when R tends to infinity. So we can write the following *reciprocity formula*:

$$\int_{\mathcal{S}} \mathbf{G}(\mathbf{U}_1) \cdot \mathfrak{S}(\mathbf{U}_2, p_2) \cdot \mathbf{n}\, \mathrm{d}\mathcal{S} = \int_{\mathcal{S}} \mathbf{G}(\mathbf{U}_2) \cdot \mathfrak{S}(\mathbf{U}_1, p_1) \cdot \mathbf{n}\, \mathrm{d}\mathcal{S} \tag{18}$$

This formula is known in the no-slip case [6], and is well known for the steady Stokes flows. It gives a global result: It will be not necessary to have the explicit solution for the field $(\mathbf{W}, q)$ in each point of the fluid, to obtain the drag and the torque exerted on the particle P by the fluid.

3.2. Result for $[\Sigma_{\mathrm{II}}]$

Now we consider $[\Sigma_{\mathrm{II}}]$. The field $(\widehat{\mathbf{W}}, \hat{q})$ previously introduced is solution of the problem:

$$\begin{cases} \nabla\cdot\widehat{\mathbf{W}} = 0, & \mathbf{G}(\widehat{\mathbf{W}}) = -\mathbf{G}(\widehat{\mathbf{V}}_0) \text{ on } \mathcal{S} \\ (\mathrm{i}\omega\rho - \mu\Delta)\widehat{\mathbf{W}} + \nabla\hat{q} = 0, & (\widehat{\mathbf{W}}, \hat{q}) \to (0,0) \text{ when } r\to\infty \end{cases} \tag{19}$$

This problem will be called Problem I. Now we introduce Problem II for the field $(\widetilde{\mathbf{W}}, \tilde{q})$:

$$\begin{cases} \nabla\cdot\widetilde{\mathbf{W}} = 0, & \mathbf{G}(\widetilde{\mathbf{W}}) = \mathbf{G}(\widetilde{\mathbf{W}}_0) \text{ on } \mathcal{S} \\ (\mathrm{i}\omega\rho - \mu\Delta)\widetilde{\mathbf{W}} + \nabla\tilde{q} = 0, & (\widetilde{\mathbf{W}}, \hat{q}) \to (0,0) \text{ when } r\to\infty \end{cases} \tag{20}$$

where $\mathbf{G}(\widetilde{\mathbf{W}}_0)$ is a given function defined on $\mathcal{S}$. Then, by using the reciprocity relation (18), one deduces:

$$\int_{\mathcal{S}} \mathbf{G}(\widetilde{\mathbf{W}}_0)\cdot\mathfrak{S}(\widehat{\mathbf{W}}, \hat{q})\cdot\mathbf{n}\,\mathrm{d}\mathcal{S} = -\int_{\mathcal{S}} \mathbf{G}(\widehat{\mathbf{V}}_0)\cdot\mathfrak{S}(\widetilde{\mathbf{W}}, \tilde{q})\cdot\mathbf{n}\,\mathrm{d}\mathcal{S} \tag{21}$$

Concerning the Problem II, the function $\mathbf{G}(\widetilde{\mathbf{W}}_0)$ can be selected as we want. It is possible to give to $\mathbf{G}(\widetilde{\mathbf{W}}_0)$ a simple expression in order to obtain an explicit solution of the Problem II, and also to conduct to simple expressions for the drag $\widehat{\mathbf{F}}_{\mathrm{II}}$ and the torque $\widehat{\Gamma}_{\mathrm{II}}$ defined as the resultant force and the moment at the point O of the wrench $[\widehat{\Sigma}_{\mathrm{II}}]$:

$$\widehat{\mathbf{F}}_{\mathrm{II}} = \int_{\mathcal{S}} \mathfrak{S}(\widehat{\mathbf{W}}, \hat{q})\cdot\mathbf{n}\,\mathrm{d}\mathcal{S}, \qquad \widehat{\Gamma}_{\mathrm{II}} = \int_{\mathcal{S}} \mathbf{r}\wedge\big(\mathfrak{S}(\widehat{\mathbf{W}}, \hat{q})\cdot\mathbf{n}\big)\,\mathrm{d}\mathcal{S} \tag{22}$$

First we take $\mathbf{G}(\widetilde{\mathbf{W}}_0) = \mathbf{k}$ and second $\mathbf{G}(\widetilde{\mathbf{W}}_0) = \mathbf{k}\wedge\mathbf{r}$ where $\mathbf{k}$ is a unit constant vector. We find:

$$\mathbf{k}\cdot\widehat{\mathbf{F}}_{\mathrm{II}} = \mathbf{k}\cdot\int_{\mathcal{S}} \mathfrak{S}(\widehat{\mathbf{W}}, \hat{q})\cdot\mathbf{n}\,\mathrm{d}\mathcal{S} = -\int_{\mathcal{S}} \mathbf{G}(\widehat{\mathbf{V}}_0)\cdot\mathfrak{S}\big(\widetilde{\mathbf{W}}, \tilde{q}; \mathbf{G}(\widetilde{\mathbf{W}})|_{\mathcal{S}} = \mathbf{k}\big)\cdot\mathbf{n}\,\mathrm{d}\mathcal{S} \tag{23}$$

$$\mathbf{k}\cdot\widehat{\Gamma}_{\mathrm{II}} = \mathbf{k}\cdot\int_{\mathcal{S}} \mathbf{r}\wedge\big(\mathfrak{S}(\widehat{\mathbf{W}}, \hat{q})\cdot\mathbf{n}\big)\,\mathrm{d}\mathcal{S} = -\int_{\mathcal{S}} \mathbf{G}(\widehat{\mathbf{V}}_0)\cdot\mathfrak{S}\big(\widetilde{\mathbf{W}}, \tilde{q}; \mathbf{G}(\widetilde{\mathbf{W}})|_{\mathcal{S}} = \mathbf{k}\wedge\mathbf{r}\big)\cdot\mathbf{n}\,\mathrm{d}\mathcal{S} \tag{24}$$

The two relations (23) and (24) give the projections on $\mathbf{k}$ of $\widehat{\mathbf{F}}_{\mathrm{II}}$ and $\widehat{\Gamma}_{\mathrm{II}}$ respectively. Of course the knowledge of $\mathfrak{S}(\widetilde{\mathbf{W}}, \tilde{q})$ on $\mathcal{S}$ is necessary, but the calculation of this tensor is easier than that of $\mathfrak{S}(\widehat{\mathbf{W}}, \hat{q})$ because the boundary condition on $\mathcal{S}$ is very simple in the Problem II. The Problem II with $\mathbf{G}(\widetilde{\mathbf{W}}_0) = \mathbf{k}$ on $\mathcal{S}$ corresponds to the vibration motion of the particle P in the $\mathbf{k}$ direction with the frequency ω in a fluid at rest at infinity and with the slip boundary condition on the particle P. Indeed let consider a particle P having the velocity $\mathbf{k}\mathrm{e}^{\mathrm{i}\omega t}$. It is easy to see that $\mathbf{G}(\mathbf{k}\mathrm{e}^{\mathrm{i}\omega t}) = \mathbf{k}\mathrm{e}^{\mathrm{i}\omega t}$ and that the solution $(\mathbf{W}, q)$ is given by the solution of Eqs. (11) with $-\mathbf{G}(\mathbf{V}_0)$ replaced by $\mathbf{k}\mathrm{e}^{\mathrm{i}\omega t}$. In the same manner, the Problem II with $\mathbf{G}(\widetilde{\mathbf{W}}_0) = \mathbf{k}\wedge\mathbf{r}$ corresponds to a rotational vibration of P around an axis $(O, \mathbf{k})$ with the frequency ω (that is the problem (11) with $-\mathbf{G}(\mathbf{V}_0)$ replaced by $\mathbf{k}\wedge\mathbf{r}\mathrm{e}^{\mathrm{i}\omega t}$). Therefore, we deduce the drag $\widehat{\mathbf{F}}_{\mathrm{II}}$ and the moment $\widehat{\Gamma}_{\mathrm{II}}$ in the general time dependent situation from the knowledge of the stress tensor in two simple cases.

4. Application to a spherical particle

4.1. Vibration motions of a spherical particle

In this section, the particle P is a sphere with the centre O and the radius a. At first, P has a vibration motion in the direction $\mathbf{k}$ as this is described in Section 3.2. This problem can explicitly be solved [7,14]. The usual spherical coordinates (r, θ, φ) referred to the axis $(O, \mathbf{k})$, θ being the angle between $\mathbf{k}$ and the radius vector $\mathbf{r}$, are introduced. The local frame in any point M of the space is denoted $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$. The solution of Eqs. (11) with $-\mathbf{G}(\mathbf{V}_0)$ replaced by $\mathbf{k}\mathrm{e}^{\mathrm{i}\omega t}$ is

$$\mathbf{W}(\mathbf{r}, t) = \widetilde{\mathbf{W}}(\mathbf{r}, \omega)\mathrm{e}^{\mathrm{i}\omega t}, \qquad q(\mathbf{r}, t) = \tilde{q}(\mathbf{r}, \omega)\mathrm{e}^{\mathrm{i}\omega t}$$

with $\widetilde{\mathbf{W}}(\mathbf{r}, \omega)$ and $\tilde{q}(\mathbf{r}, \omega)$ given by:

$$\widetilde{\mathbf{W}} = \frac{K\alpha e^{\alpha r}}{r}\sin\theta\mathbf{e}_\theta - \left(\frac{K(\alpha r-1)e^{\alpha r}}{\alpha r^3} + \frac{C}{r^3}\right)(2\cos\theta\mathbf{e}_r + \sin\theta\mathbf{e}_\theta), \qquad \tilde{q} = -\mu\frac{C\alpha^2}{r^2}\cos\theta$$

$$K = -\frac{3a}{2\alpha}\frac{1-\delta}{1-\delta\alpha a}e^{-\alpha a}, \qquad C = -\frac{a^3}{2}\left(1+\frac{3(1-\delta)}{\alpha^2 a^2}\frac{1-\delta\alpha}{1-\delta\alpha a}\right), \qquad \delta = \frac{\lambda}{a+3\lambda}, \qquad \alpha^2 = \frac{i\omega}{\nu} \tag{25}$$

where $\operatorname{Re}\alpha < 0$. The determination of the square root α must be made carefully: The Fourier variable ω is positive or negative and the proper choice for α is fixed by the condition that the solution decreases to zero at infinity. As a remark, the solution (25) with the constants K and C is also given in [6]. Of course, with $\lambda = 0$, i.e. $\delta = 0$, the constant values for K and C are these of the no-slip problem [6]. After some calculations we deduce the stress vector in each point of the sphere surface $\mathcal{S}$:

$$\mathfrak{S}\big(\widetilde{\mathbf{W}}, \tilde{q}; \mathbf{G}(\widetilde{\mathbf{W}})|_{\mathcal{S}} = \mathbf{k}\big)\cdot\mathbf{n} = -\mu\left(\frac{3}{2a}\frac{1-\alpha a}{1-\delta\alpha a}(1-3\delta)\right)\mathbf{k} - \mu\left(\frac{9}{a}\frac{1-\alpha a}{1-\delta\alpha a}\delta + \frac{\alpha^2 a}{2}\right)\cos\theta\mathbf{n} \tag{26}$$

Now, let be consider the angular vibration motion of the sphere P around its diameter $(O, \mathbf{k})$. The solution is also given in [7]. As before, we write the solution of Eqs. (11) with $-\mathbf{G}(\mathbf{V}_0)$ replaced by $\mathbf{k}\wedge\mathbf{r}e^{i\omega t}$ in spherical coordinates, and we deduce the stress vector in each point of the sphere surface $\mathcal{S}$. We have $\mathbf{W}(\mathbf{r},t) = \widetilde{\mathbf{W}}(\mathbf{r},\omega)e^{i\omega t}$, $q(\mathbf{r},t) = \tilde{q}(\mathbf{r},\omega)e^{i\omega t}$ with:

$$\widetilde{\mathbf{W}} = A\left(\frac{-1}{r^2} + \frac{\alpha}{r}\right)e^{\alpha r}\sin\theta\mathbf{e}_\varphi, \quad \tilde{q} = 0, \quad \text{with } A = \frac{(3\delta-1)a^3}{1-\alpha a+\delta\alpha^2 a^2}e^{-\alpha a} \tag{27}$$

$$\mathfrak{S}\big(\widetilde{\mathbf{W}}, \tilde{q}; \mathbf{G}(\widetilde{\mathbf{W}})|_{\mathcal{S}} = \mathbf{k}\wedge\mathbf{r}\big)\cdot\mathbf{n} = -\mu\frac{3(1-\alpha a)+\alpha^2 a^2}{1-\alpha a+\delta\alpha^2 a^2}(1-3\delta)\mathbf{k}\wedge\mathbf{n} \tag{28}$$

4.2. *Expressions for the drag $\mathbf{F}_{\mathrm{II}}$ and the moment Γ_{II}*

Now it is easy to obtain the quantities $\mathbf{F}_{\mathrm{II}}$ and the moment Γ_{II}, by putting the expressions (26) and (28) in (23) and (24). So:

$$\mathbf{k}\cdot\widehat{\mathbf{F}}_{\mathrm{II}} = \mu\int_{\mathcal{S}}\mathbf{G}(\widehat{\mathbf{V}}_0)\cdot\left\{\left(\frac{3}{2a}\frac{1-\alpha a}{1-\delta\alpha a}(1-3\delta)\right)\mathbf{k} + \left(\frac{9}{a}\frac{1-\alpha a}{1-\delta a\alpha}\delta + \frac{\alpha^2 a}{2}\right)\cos\theta\mathbf{n}\right\}d\mathcal{S}$$

$$\mathbf{k}\cdot\widehat{\Gamma}_{\mathrm{II}} = \mu\int_{\mathcal{S}}\mathbf{G}(\widehat{\mathbf{V}}_0)\cdot\left\{\frac{3(1-\alpha a)+\alpha^2 a^2}{1-\alpha a+\delta\alpha^2 a^2}(1-3\delta)\mathbf{k}\wedge\mathbf{n}\right\}d\mathcal{S} \tag{29}$$

By using the expression (7) for $\mathbf{G}(\mathbf{u})$, the divergence theorem and the property $\nabla\cdot\mathbf{u} = 0$, it is easy to prove the following relations:

- ◇ $\displaystyle\int_{\mathcal{S}}\mathbf{G}(\mathbf{u})\cdot(\cos\theta\mathbf{n})\,d\mathcal{S} = \frac{1}{a}\mathbf{k}\cdot\int_{\mathcal{V}}\mathbf{u}\,d\mathcal{V}$
- ◇ $\displaystyle\int_{\mathcal{S}}\mathbf{G}(\mathbf{u})\,d\mathcal{S} = \frac{a+4\lambda}{a}\int_{\mathcal{S}}\mathbf{u}\,d\mathcal{S} - \frac{6\lambda}{a^2}\int_{\mathcal{V}}\mathbf{u}\,d\mathcal{V} - \lambda\lim_{r\to a}\frac{\partial}{\partial r}\int_{\mathcal{S}(r)}\mathbf{u}\,d\mathcal{S}_r$
- ◇ $\displaystyle\int_{\mathcal{S}}\mathbf{G}(\mathbf{u})\wedge\mathbf{n}\,d\mathcal{S} = -\frac{a+4\lambda}{a^2}\int_{\mathcal{S}}\mathbf{r}\wedge\mathbf{u}\,d\mathcal{S} + \frac{\lambda}{a}\lim_{r\to a}\frac{\partial}{\partial r}\int_{\mathcal{S}(r)}\mathbf{r}\wedge\mathbf{u}\,d\mathcal{S}_r$

where $\mathcal{S}(r)$ is the surface of a sphere of centre O and of radius r, and where $d\mathcal{S}_r$ is the surface element on $\mathcal{S}(r)$. In the same way, let be introduce the volume $\mathcal{V}(r)$ of this sphere. The results are used to give new expressions for the right members of (29), where, of course, $\mathbf{u}$ is replaced by $\widehat{\mathbf{V}}_0$. With some algebraic calculus [7], and by writing (29) for any vector $\mathbf{k}$, we arrive to the final results for $\widehat{\mathbf{F}}_{\mathrm{II}}$ and $\widehat{\Gamma}_{\mathrm{II}}$:

$$\widehat{\mathbf{F}}_{\mathrm{II}} = \frac{\mu\alpha^2}{2}\int_{\mathcal{V}}\widehat{\mathbf{V}}_0\,d\mathcal{V} + \frac{3\mu}{2a}\frac{1-\alpha a}{1-\delta\alpha a}\left\{(1+\delta)\int_{\mathcal{S}}\widehat{\mathbf{V}}_0\,d\mathcal{S} - \delta a\lim_{r\to a}\frac{\partial}{\partial r}\int_{\mathcal{S}(r)}\widehat{\mathbf{V}}_0\,d\mathcal{S}_r\right\} \tag{30}$$

$$\widehat{\varGamma}_{\mathrm{II}} = \frac{\mu}{a}\frac{3(1-\alpha a)+\alpha^2 a^2}{1-\alpha a+\delta\alpha^2 a^2}\left\{(1+\delta)\int_{\mathcal{S}} \mathbf{r}\wedge\widehat{\mathbf{V}}_0\,\mathrm{d}\mathcal{S} - \delta a\lim_{r\to a}\frac{\partial}{\partial r}\int_{\mathcal{S}(r)} \mathbf{r}\wedge\widehat{\mathbf{V}}_0\,\mathrm{d}\mathcal{S}_r\right\} \tag{31}$$

These formulae (30) and (31) are simple. They give the Fourier transforms of $\mathbf{F}_{\mathrm{II}}$ and $\varGamma_{\mathrm{II}}$. The result for $\widehat{\mathbf{F}}_{\mathrm{II}}$ was also obtained by Albano et al. [14] by a different way based on the concept of induced forces.

Calculations are performed in the moving frame $\Re$ in which the surface $\mathcal{S}$ and the volume $\mathcal{V}$ are fixed. So, $\mathcal{S}$ and $\mathcal{V}$ do not depend on the Fourier variable ω and consequently, it is easy to make the inverse Fourier transform. By recalling the determination $\mathrm{Re}\,\alpha < 0$ for the square root α, and by using the formula given in the book of formulae [18] and rewritten here with our notations as follows:

$$\hat{\varphi}(\omega) = \frac{-1}{\alpha(1-\delta\alpha a)} \quad\Longleftrightarrow\quad \varphi(t) = \sqrt{\frac{2\pi\nu}{\tau_\delta}}\exp\left(\frac{t}{\tau_\delta}\right)\mathrm{erfc}\sqrt{\frac{t}{\tau_\delta}} \quad \text{for } t>0$$

$$\hat{\varphi}_\gamma(\omega) = \frac{1}{1-\delta\alpha a\gamma} \quad\Longleftrightarrow\quad \varphi_\gamma(t) = \frac{1}{\gamma}\sqrt{\frac{2}{\tau_\delta t}} - \frac{\sqrt{2\pi}}{\gamma^2\tau_\delta}\exp\left(\frac{t}{\gamma^2\tau_\delta}\right)\mathrm{erfc}\left(\frac{1}{\gamma}\sqrt{\frac{t}{\tau_\delta}}\right) \quad \text{for } t>0$$

with $\varphi(t)=\varphi_\gamma(t)=0$ for $t<0$. The parameter γ is real or complex and where $\tau_\delta = a^2\delta^2/\nu$ is a characteristic time depending on the slipping.

First, we pay attention to the inverse Fourier transform of $\mathbf{F}_{\mathrm{II}}$. We write:

$$\frac{1-\alpha a}{1-\delta\alpha a} = 1-(1-\delta)\frac{a\alpha^2}{\alpha(1-\delta\alpha a)}$$

We recall that $\alpha^2 = \mathrm{i}\omega/\nu$. It is easy to see that $\mu\alpha^2\widehat{\mathbf{V}}_0$ is the Fourier transform of the function $\rho(\partial/\partial t)\mathbf{V}_0$. With (30) and the properties of the convolution product, the following formula is obtained:

$$\begin{aligned}\mathbf{F}_{\mathrm{II}} = {} & \frac{\rho}{2}\int_{\mathcal{V}}\frac{\partial\mathbf{V}_0}{\partial t}\,\mathrm{d}\mathcal{V} + \frac{3\mu}{2a}\left\{(1+\delta)\int_{\mathcal{S}}\mathbf{V}_0\,\mathrm{d}\mathcal{S} - \delta a\lim_{r\to a}\frac{\partial}{\partial r}\int_{\mathcal{S}(r)}\mathbf{V}_0\,\mathrm{d}\mathcal{S}_r\right\}\\ & + \frac{3\rho}{2}\sqrt{\frac{1}{2\pi}}(1-\delta)\varphi(t)\star\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+\delta)\int_{\mathcal{S}}\mathbf{V}_0\,\mathrm{d}\mathcal{S} - \delta a\lim_{r\to a}\frac{\partial}{\partial r}\int_{\mathcal{S}(r)}\mathbf{V}_0\,\mathrm{d}\mathcal{S}_r\right\}\end{aligned} \tag{32}$$

Then, consider the expression (31) for the moment. Depending on the number of roots in α of the denominator $1-\alpha a+\delta\alpha^2a^2$, two cases must be considered. At first, $\delta\neq 1/4$ and then there are two roots (real or complex) respectively denoted by $1/(\delta a\gamma_1)$ and $1/(\delta a\gamma_2)$ with $2/\gamma_1 = 1+(1-4\delta)^{1/2}$ and $2/\gamma_2 = 1-(1-4\delta^{1/2})$ and where $(1-4\delta)^{1/2}$ is for $\sqrt{1-4\delta}$ if $\delta<1/4$ and for $\mathrm{i}\sqrt{4\delta-1}$ if $\delta>1/4$. It is easy to write:

$$\frac{3(1-\alpha a)+\alpha^2a^2}{1-\alpha a+\delta\alpha^2a^2} = 3+\frac{(1-3\delta)\delta a^2}{(1-4\delta)^{1/2}}\left(-\gamma_1\frac{\alpha^2}{1-\alpha\delta a\gamma_1}+\gamma_2\frac{\alpha^2}{1-\alpha\delta a\gamma_2}\right)$$

$$\psi(t) \equiv \frac{1-3\delta}{(1-4\delta)^{1/2}}\big(-\gamma_1\varphi_{\gamma_1}(t)+\gamma_2\varphi_{\gamma_2}(t)\big)$$

$$\begin{aligned}\varGamma_{\mathrm{II}} = {} & \frac{3\mu}{a}\left\{(1+\delta)\int_{\mathcal{S}}\mathbf{r}\wedge\mathbf{V}_0\,\mathrm{d}\mathcal{S} - \delta a\lim_{r\to a}\frac{\partial}{\partial r}\int_{\mathcal{S}(r)}\mathbf{r}\wedge\mathbf{V}_0\,\mathrm{d}\mathcal{S}_r\right\}\\ & + \rho a\psi(t)\star\left\{(1+\delta)\int_{\mathcal{S}}\mathbf{r}\wedge\frac{\partial\mathbf{V}_0}{\partial t}\,\mathrm{d}\mathcal{S} - \delta a\lim_{r\to a}\frac{\partial}{\partial r}\int_{\mathcal{S}(r)}\mathbf{r}\wedge\frac{\partial\mathbf{V}_0}{\partial t}\,\mathrm{d}\mathcal{S}_r\right\}\end{aligned} \tag{33}$$

Second, if $\delta = 1/4$ the denominator $1-\alpha a+\delta\alpha^2a^2$ has a double root equal to $\alpha a/2$. Then we have:

$$\big[3(1-\alpha a)+\alpha^2a^2\big](1-\alpha a/2)^{-2} = 3+\big(\alpha^2a^2/4\big)(1-\alpha a/2)^{-2}$$

In this case $\psi(t)$ is replaced by the inverse Fourier transform of $(1-\alpha a/2)^{-2}$. As a remark, the functions of which one has to take the inverse Fourier transforms are algebraic in α and regular in δ. Like consequence, the expressions of $\mathbf{F}_{\mathrm{II}}$ and $\varGamma_{\mathrm{II}}$ are continuous in δ and in particular in $\delta = 1/4$.

5. Drag and torque on a spherical particle

In the expressions (32) and (33), the velocity $\mathbf{V}_0$ appears only through some integrals. This velocity and also the Galilean velocity $\mathbf{V}_0^a$ depends on the spatial variable $\mathbf{r}$ and t. Generalizing an idea of Mazur and Bedeaux [16], we introduce some average quantities [6,7,14]:

$$\mathbf{V}_0^{aS}(r,t)=\frac{1}{4\pi r^2}\int\limits_{\mathcal{S}(r)}\mathbf{V}_0^a\,\mathrm{d}\mathcal{S}_r,\qquad \mathbf{V}_0^{aV}(r,t)=\frac{3}{4\pi r^3}\int\limits_{\mathcal{V}(r)}\mathbf{V}_0^a\,\mathrm{d}\mathcal{V}_r$$
$$\Omega_0^{aS}(r,t)=\frac{3}{8\pi r^4}\int\limits_{\mathcal{S}(r)}\mathbf{r}\wedge\mathbf{V}_0^a\,\mathrm{d}\mathcal{S}_r,\qquad \Omega_0^{aV}(r,t)=\frac{15}{8\pi r^5}\int\limits_{\mathcal{V}(r)}\mathbf{r}\wedge\mathbf{V}_0^a\,\mathrm{d}\mathcal{V}_r$$

If the radius r is equal to the radius of the particle a, these quantities are simply denoted by: $\mathbf{V}_0^{aS}$, $\mathbf{V}_0^{aV}$, Ω_0^{aS} and Ω_0^{aV}. Finally we can express the forces exerted by the fluid on the particle P in term of the Galilean velocities. Due to the linearization assumptions and to the relations (9), one arrives at the following final expressions for $\mathbf{F}_{\mathrm{II}}$ and Γ_{II}:

$$\mathbf{F}_{\mathrm{II}}=\frac{2\pi a^3\rho}{3}\left(\frac{\mathrm{d}\mathbf{V}_0^{aV}}{\mathrm{d}t}-\frac{\mathrm{d}\mathbf{V}_p}{\mathrm{d}t}\right)+6\pi\mu a\mathbf{R}_0^{aS}(t)+\frac{1}{\sqrt{2\pi}}6\pi a^2\rho(1-\delta)\varphi(t)\star\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{R}_0^{aS}(t)$$
$$\Gamma_{\mathrm{II}}=8\pi\mu a^3\mathbf{C}_0^{aS}(t)+\frac{8\pi a^5\rho}{3}\psi(t)\star\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{C}_0^{aS}(t)\tag{34}$$

where to reduce the writing of the expressions we have introduced the following notations:

$$\mathbf{R}_0^{aS}(t)\equiv(1-\delta)\big(\mathbf{V}_0^{aS}(t)-\mathbf{V}_p(t)\big)-\delta a\lim_{r\to a}\frac{\partial}{\partial r}\mathbf{V}_0^{aS}(r,t)$$
$$\mathbf{C}_0^{aS}(t)\equiv(1-3\delta)\big(\Omega_0^{aS}(t)-\Omega_p(t)\big)-\delta a\lim_{r\to a}\frac{\partial}{\partial r}\Omega_0^{aS}(r,t)$$

We come back to the initial problem for $[\Sigma_f]$: $[\Sigma_{\mathrm{I}}]$ is given in (13), and $[\Sigma_{\mathrm{II}}]$ has just given in (34). In the final results given later on, the labels a and 0 are omitted. But the fluid velocity which appears there must always be understood like the Galilean velocity of the *unperturbed flow*.

$$\mathbf{F}=6\pi\mu a\mathbf{R}^S(t)+\frac{2\pi a^3\rho}{3}\left(\frac{\mathrm{d}\mathbf{V}^V}{\mathrm{d}t}-\frac{\mathrm{d}\mathbf{V}_p}{\mathrm{d}t}\right)+\frac{4\pi a^3\rho}{3}\frac{\mathrm{d}\mathbf{V}^V}{\mathrm{d}t}$$
$$+6\pi\mu a\frac{1-\delta}{\delta}\int\limits_{-\infty}^{t}\exp\left(\frac{t-\theta}{\tau_\delta}\right)\mathrm{erfc}\left(\sqrt{\frac{t-\theta}{\tau_\delta}}\right)\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{R}^S(\theta)\,\mathrm{d}\theta-\int\limits_{\mathcal{V}}\rho\mathbf{F}_e\,\mathrm{d}\mathcal{V}\tag{35}$$
$$\Gamma=8\pi\mu a^3\mathbf{C}^S(t)+\frac{8\pi a^5\rho}{15}\frac{\mathrm{d}\Omega^V}{\mathrm{d}t}+\frac{8\pi a^5\rho}{3}\psi(t)\star\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{C}^S(t)-\int\limits_{\mathcal{V}}\rho(\mathbf{r}-\mathbf{R})\wedge\mathbf{F}_e\,\mathrm{d}\mathcal{V}\tag{36}$$

It is possible to prove that by making $\delta=0$ in (35) and (36), i.e. in the absence of slip, the formulae obtained for $\mathbf{F}$ and Γ are identical to those obtained in [6]. On the other hand, Michaelides and Feng [15] have treated the problem of the motion of a small sphere in an unsteady flow with a slip condition on the sphere surface. They have a formula for the part $\mathbf{F}_{\mathrm{II}}$ of the drag. Our result is identical to theirs if we take $\mathbf{V}_0^{aV}=\mathbf{V}_0^{aS}=\mathbf{V}_0^{aS}(r,t)=\mathbf{V}_0^a(\mathbf{R},t)$. Another remark is for the case where the fluid slips perfectly on the particle, that is λ tends to ∞ or δ tends to 1/3. Then $\psi(t)=0$. It is possible to show that the contribution due to the two first terms in the right member of (36) is null [7]. As consequence, $\Gamma=0$; This is a physically natural result. To finish, let us note that the drag force exerted by the fluid on P in the case of the slip condition is weaker than that is obtained in the case of adherence. This was shown in [7] where several examples are treated either analytically or numerically.

If we introduce the non-dimensionless time τ defined by $\tau=\nu t/a^2$, the expressions for $\mathbf{F}$ and Γ become simpler, but they are not given here. It is then easy to compare the orders of magnitude of the various terms appearing in (35) and (36) [6–8].

In the expressions (35) and (36) for **F** and Γ, there appear some integral or volume averaged velocities. We emphasize that **V** is the *unperturbed velocity* field which is generated by the external forces and which contains the effects of boundary conditions at infinity. We also emphasize that **F** and Γ depend on the unperturbed fluid and on the shape of the particle (here the radius) only.

There are two types of averages. Firstly, the rather simple averages, that one had in the situation without slip [6, 14], and which are related to the surface or the volume of the particle P. Secondly, the variation according to r of the average relating to a surface $\mathcal{S}(r)$. This term is only present in the case of slip: It takes into account the velocity variation in the radial direction. This is due to the slip condition which is expressed by using the stress vector on the particle surface.

In the expressions (35) and (36) of **F** and Γ, there are terms arising from the unperturbed flow as stated in Section 2.3: Firstly the inertia forces of the displaced fluid (the third term in (35) and the second in (36)), and secondly the term similar to the Archimède forces (the last in (35) and (36)). Let be notice that these terms do not depend on the slip parameter. On the other hand, there are terms arising from the perturbation: The Stokes drag and torque (the first terms in (35) and (36)), the so-called added mass term which is only present in **F** (the second term) because the particle P does not move fluid in a rotation motion and, the *history terms* i.e. the integral terms with the convolution products. These terms are present at the same time is present in **F** and in Γ. They are the so-called *terms of Boussinesq–Basset*.

6. Conclusion

In this article, we have given general expressions for **F** and Γ. They are complex, and it would now be necessary to analyze each term. In particular, the terms of Stokes in the case of slip contain additional terms compared to the traditional case of adherence. Let us note that these terms tend towards the traditional terms of Stokes when δ tends to 0.

To finish, we emphasize that there are many papers describing the motion of a particle P in a flow. Indeed, the last twenty years have seen a great number of studies which involve the Lagrangian dynamic simulation of particles [5,10–12]. The history term cannot be neglected in the unsteady phenomena where the characteristic time is of order of a^2/ν [8]. So, many studies have been interested in the integration of the motion equation of P when one takes into account the history term [7–12,19]. The unperturbed flow is given and is a shear flow in [7], a rotating flow in [11] or an eccentrically rotating flow in [12].

The Boussinesq–Basset forces have also an important effect when the particles are suspended in a turbulent flow. Many calculations in the literature relate the turbulence characteristics of particle motion to the turbulence characteristics of the fluid [20,21]. On these turbulence problems, recent papers are also very numerous (see [20]).

In conclusion, we can affirm that the history term of Boussinesq–Basset is always of topicality.

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